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On the universality of the normal law and some connected problems useful in the chaotic and complex systems analysis

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Abstract. We demonstrate that reservoir computing can be utilized as an observer not only of chaotic temporal dynamics, such as those produced by the Rössler system, but also for two-dimensional dynamics generated by an optoelectronic system. Our optoelectronic experimental system consists of a spatial-light modulator with self-feedback that generates complex two-dimensional spatio-temporal patterns. The observer successfully cross-predicts the dynamics at all spatial locations based on observed time series from a selected subset of locations. The observer consists of reservoir computing subnetworks that receive input and predict local regions in space only, making the proposed observer resource efficient.
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1 Introduction

The ***Central Limit Problem*** in probability theory is *the problem of laws of convergence of sequences of sums of random variables.*

It appeared due to a new approach by Paul Lévy (Paul Lévy, French, born in 1886) of the ***classical limit problem***. Lévy formulated and solved the problem: *find the family of all possible limit laws of normed sums of independent and identically distributed random variables* (in brief *i.i.d.*). If these random variables have a finite second moment, the limit law, with classical norm, is normal. Therefore, Lévy was interested, firstly, with a new case that of infinite second moments, the first moments being finite or infinite. Thus, the question of all possible limit laws of normed sums with independent but not necessarily identically distributed random variables arises in a natural way.

Therefore, one can reformulate the central limit problem in the following way: *find the limit laws of sequences of sums of independent summands and also find conditions for convergence to a specific one.*

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Let us consider arbitrary random variables Y_n and set $X_{n1} = Y_n$ and $X_{nk} = 0$ *a.s.*, for $k > 1$ and every n. The sequence of laws becomes, in this case, the sequence $\mathcal{L}(Y_n)$ and, therefore, the family of possible limit laws contains any law $\mathcal{L}$ (denote $\mathcal{L}(Y_n) \equiv \mathcal{L}$). Hence, some restriction is necessary to be imposed.

Central Limit Problem. *The common feature is that the number of summands increases indefinitely, and the limit law remains the same if an arbitrary finite number of summands is omited.*

A natural restriction is the following (M. Loève): the summands X_{nk} are uniformly asymptotically negligible, that is to say $X_{nk} \xrightarrow{P} 0$ uniformly in k or, equivalently, for every $\varepsilon > 0$ arbitrarily small

$$\max_k P[\,|X_{nk}| \geq \varepsilon] \to 0.$$

A precise formulation of the central limit problem can be now given:

Central Limit Problem. Let

$$S_{nk_n} = \sum_{k=1}^{k_n} X_{nk}$$

be sums of uniformly asymptotically neglijable independent summands X_{nk}, with $k_n \to \infty$.

i *Find the family of all possible limit laws of these sums.*
ii *Find conditions for convergence to any specified law of these family.*

Limit theorems

We have already introduced the laws $\mathcal{L}$, so that we shall refer in this section, in short, to the basical laws of probability theory.

At the origin of the classical limit problem one finds three limit theorems and corresponding limit laws.

Let us denote by S_n the number of occurrences of an event A with constant probability p in n independent trials. Also it is assumed that $p + q = 1$ and both p and q are different to zero, which is referred to as *the Bernoulli case*. For X_k being the indicator of the event A in the k-th trial then,

$$S_n = \sum_{k=1}^{n} X_k, \quad n = 1, 2, \cdots$$

where the summands are *i.i.d.* indicators. Thus,

$$EX_k = p, \quad EX_k^2 = p$$

while the variance is

$$\sigma^2 X_k = p - p^2 = pq,$$

so that it results

$$ES_n = \sum_{k=1}^{n} EX_k = np$$

and

$$\sigma^2 S_n = \sum_{k=1}^{n} \sigma^2 X_k = npq.$$

The first limit theorem of probability theory is due to J. Bernoulli and it affirms that

$$\frac{S_n}{n} \xrightarrow{P} p$$

or, equivalently, *if* $\varepsilon > 0$ *is arbitrarily small*, then

$$\lim_{n\to\infty} P\left[\left|\frac{S_n}{n} - p\right| < \varepsilon\right] = 1.$$

Bernoulli obtained this theorem directly, by a difficult analysis of the symptotic behaviour of the binomial distribution

$$\mathbf{b}(k; n, p) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \cdots, n.$$

The second limit theorem was obtained firstly by Abraham DeMoivre in 1730, for $p = q = \frac{1}{2}$ and, then, this was generalized by Laplace for $0 < p < 1$. Laplace found also the following integral version of this theorem

$$P\left[\frac{S_n - np}{\sqrt{npq}} < x\right] \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy, \quad -\infty \le x \le +\infty.$$

which stands for the second limit theorem.

The third limit theorem was found by Poisson. He modified the Bernoulli case by considering that the probability $p = p_n$ depends upon the total number n of trials in such a way that, for $\lambda > 0$, to have $np_n \to \lambda$. If we write, for this case, X_{nk} and S_{nn}, instead of X_k and S_n, one observes that such a case corresponds to sequences of sums

$$S_{nn} = \sum_{k=1}^{n} X_{nk}, \quad n = 1, 2, \cdots,$$

where for every given n, the summands X_{nk} are *i.i.d.* indicators with

$$P[X_{nk} = 1] = \frac{\lambda}{n} + 0\left(\frac{1}{n}\right).$$

It is already known the result obtained by Poisson, which we write now in the following form

$$P[S_{nn} = k] \longrightarrow \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \cdots.$$

Jacques Bernoulli, Swiss 1654-1705. His works *Ars Conjectandi* was published only in 1713.
Abraham DeMoivre, English, 1667-1754.
Pierre Simon de Laplace, French, 1749-1827
Siméon Denis Poisson, French, 1781-1840.

Basical laws of probability theory

In the way presented above the following three laws of probability theory were born.

1. The degenerate law $\mathcal{L}(0)$ *of a random variable* degenerate at 0, the distribution function of which has one point of increse, only at $x = 0$, and whose characteristic function is reduced to 1.

2. The normal law $\mathcal{N}(0,1)$ *of a normal random variable* having the distribution function defined in the following way

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

and its characteristic function defined as

$$\begin{aligned} \varphi(t) &= \frac{1}{\sqrt{2\pi}} \int e^{itx-\frac{x^2}{2}} \, dx = \\ &= e^{-\frac{t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty-it}^{+\infty-it} e^{-\frac{z^2}{2}} \, dz = e^{-\frac{t^2}{2}}. \end{aligned}$$

3. The Poisson law $\mathcal{P}(\lambda)$ *of a Poisson random variable* which has the distribution function as follows

$$F(x) = e^{-\lambda} \sum_{k=0}^{|x|} \frac{\lambda^k}{k!}$$

and its characteristic function defined by the equality

$$\begin{aligned} \varphi(t) &= e^{-\lambda} \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k}{k!} = \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} = e^{\lambda(e^{it}-1)}. \end{aligned} \tag{1}$$

It is interesting that the first two laws played a fundamental role in the development of probability theory, while the Poisson law was isolated for a long time. Nevertheless, the Poisson law is recognized to be, in some sense, more fundamental for the central limit problem than the first two. [For more details see [13], [24], [3], [6], [7]].

But we shall consider below the second law, known as *the normal law*, and we shall refer to some aspects which emphasize its universal character.

Now some aspects regarding to the normal approximation of the binomial distribution and the integral limit theorem due to DeMoivre and Laplace are discussed firstly. Then, we shall refer, in short, to the problem of convergence to the normal distribution law.

With another occasion we shall refer to new aspects and applications.

2 Normal approximation of the binomial distribution

The Bernoulli - type experiment is defined as the one in which the following conditions are respected : (1) in each of succesive independent trials there are only two possible outcomes; (2) the probabilities will remain the same for all trials.

Let us denote by p and q these probabilities : p is the probability that the outcome is a *success* and q is the probability that the outcome is a *failure*. Obviously, p and q are nonnegative numbers and their sum is equal to the unity

$$p + q = 1.$$

Now if we denote a *success* by the letter R and a *failure* by the letter I, then the event *"from n trials result k successes and $n-k$ failures"* can be obtained in a number of ways as we can distribute k letters R in n places. Hence this event contains $\mathtt{C}_n^k$ points whence one finds that every point has the probability $p^k q^{n-k}$. Thus the following result is obtained

Theorem 1. *Let $P_n(k)$ be the probability to result k successes with probabilities p, and $n-k$, $0 \leq k \leq n$, failures with probabilities $q = 1-p$ from n Bernoulli trials. Then*

$$P_n(k) = \mathtt{C}_n^k \, p^k \, q^{n-k}. \tag{2}$$

Particularly, the probability to obtain no success is q^n. Then, the probability to obtain at least one success is $1 - q^n$.

Definition 1. The ***normal density function*** is defined by the equality

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}x^2}; \tag{3}$$

and its integral

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} \, dy \tag{4}$$

is called the ***normal distribution function***.

The graph of the normal density function $\psi(x)$ is a symmetric, bell-shaped curve, for the representation of which different measure units must be used on the two axes.

An well known result is given in the following lemma

Lemma 1. *The area of the domain bounded by the graph of the function $\psi(x)$ and the xx'- axis is equal to the unity, i. e.*

$$\int_{-\infty}^{+\infty} \psi(x) dx = 1. \tag{5}$$

By Definition 1 and Lemma 1 it results that $N(x)$ is continuous increasing from 0 to 1. The graph of $N(x)$ is an S - shaped curve with

$$N(-x) = 1 - N(x) \tag{6}$$

Now we come back to $P_n(k)$ in (2) for observing that for large values of n and k, calculation of the probabilities $P_n(k)$, involves considerable difficulties. So the necessity arises to derive an asymptotic formula that permit calculating these probabilities with a "sufficient degree of accuracy". Thus, the main step is to obtain an asymptotic formula for (2).

DeMoivre is the first which found, in 1730, such an asymptotic formula in the case of the Bernoulli scheme for $p = q = \frac{1}{2}$. Later this result was generalized by Laplace to the case of arbitrary $0 < p < 1$.

Thus, the following limit theorem is obtained.

Theorem 2. (***The DeMoivre - Laplace local limit theorem***). *If the probability of occurrence of some event A in n independent trials is constant and is equal to p $(0 < p < 1)$ then, the probability $P_n(k)$ that in each of the trials event A will occur exactly k times satisfies the relation*

$$P_n(k) : \frac{1}{\sqrt{2\pi npq}}\, e^{-\frac{x^2}{2}} \to 1 \tag{7}$$

as $n \to \infty$, uniformly in all k for which x lies in some finite interval, and verifies the equality

$$x = \frac{k - np}{\sqrt{npq}}. \tag{8}$$

[It is clear that x depends both on n and p and on k].

A reasonable question is, anyway, how good the probability $\mathbf{b}(k; n, p)$, approximated by DeMoivre case $p = q = \frac{1}{2}$, is. The next example answers to this question.

Example 1. Let us consider $p = q = \frac{1}{2}$. We choose for n specific values for which is possible to have $x_{nk} = 1$. For example, for n=25 or 100 or 400 or 1156 we have $x_{nk} = 1$ if k=15 ; 55 ; 210 ; 595 respectively.

By the local limit theorem of DeMoivre-Laplace we have

$$P_n(k) : \frac{1}{\sqrt{2\pi npq}}\, e^{-\frac{x_{nk}^2}{2}} \to 1$$

as $n \to \infty$. Now if we denote

$$Q_n = \frac{1}{\sqrt{2\pi npq}}\, e^{-\frac{x_{nk}^2}{2}} = \frac{\psi(x)}{\sqrt{npq}}.$$

for $p = q = \frac{1}{2}$ and $x_{nk} = 1$ we observe that the ratio $\frac{P_n(k)}{Q_n}$ should tend to unity and the difference $P_n(k) - Q_n$ should tend to zero, as $n \to \infty$. In the table 1 the calculation for n= 25 ; 100 ; 400 and 1156 is given.

It is emphasized again a surprisingly good approximation of $P_n(k)$.

n	$P_n(k)$	$\frac{\psi(x_{nk})}{\sqrt{npq}}$	$P_n(k) - Q_n$	$P_n(k) : Q_n$
25	0,09742	0,09676	0,00063	1,0065
100	0,04847	0,04839	0,00008	1,0030
400	0,024207	0,024194	0,000013	1,0004
1156	0,014236	0,014234	0,000002	1,0001

Having in view the function ψ in (3), we can conclude that for $P_n(k)$ the following approximation have been obtained

$$P_n(k) \approx \frac{1}{\sqrt{npq}} \psi(x_k). \tag{9}$$

[Many examples and comments are due to B. V. Gnedenko [6]. For more details see also [3], [13] [9], [1]].

3 A variant of DeMoivre-Laplace local limit theorem

Let ν_n be the number of successes in n Bernoulli trials. Each success is supposed to have the probability p. Then $P_n(k)$ is the probability of the event $\nu_n = k$.

Usually we are interested in the probability of the following event: *the number of all successes lies between two initially given limits, α and β*. For α and β being integer numbers, with $\alpha < \beta$ then, this event is defined by the relation $\alpha \leq \nu_n \leq \beta$. The corresponding probability of this is

$$P[\alpha \leq \nu_n \leq \beta] = P_n(\alpha) + P_n(\alpha + 1) + \cdots\cdots + P_n(\beta). \tag{10}$$

Since the above mentioned sum can have many terms, a direct evaluation is usually impossible. DeMoivre firstly, and then Laplace, realized that whenever n is large one can use succesfully the normal distribution function in order to obtain simple approximations of the probability (10). This fact is very important, and not only for numerical computation.

An elementary problem involving a scheme of independent trials consists in determining the probability $P_n(k)$ that in n trials an event A will occur k times, and that in the remaining $n - k$ trials the complementary event $\overline{A}$ will occur.

Let us denote $\delta_k = k - np$, and let us suppose that $n \to \infty$, $k \to \infty$ such that

$$\frac{\delta_k}{n} \to 0 \quad \text{and} \quad \frac{\delta_k^3}{n^2} \to 0.$$

Obviously, the last condition implies the first one and it is similar to

$$\frac{x_k^3}{\sqrt{n}} \to 0.$$

In this way the following result is obtained

Abraham DeMoivre (1667-1754), *The doctrine of chance*, 1718.
Pierre Simon de Laplace (1749-1827), *Théorie analytique des probabilités*, 1812.

Theorem 3. *Let us suppose that $n \to \infty$, $k \to \infty$ such that*

$$\frac{x_k^3}{\sqrt{n}} \to 0.$$

Then, (9) holds. In other words, there exist two constants A and B such that

$$\left| \frac{P_n(k)}{(npq)^{-\frac{1}{2}} \, \psi(x_k)} - 1 \right| < \frac{A}{n} + \frac{B \left| x_{k+1}^3 \right|}{\sqrt{n}}.$$

The theorem 3 leads directly to simple approximations for the sum (10). If

$$\frac{x_\alpha^3}{\sqrt{npq}} \to 0 \quad \text{and} \quad \frac{x_\beta^3}{\sqrt{npq}} \to 0 \tag{11}$$

then, (9) holds uniformly for all terms in (10) and, consequently,

$$P[\alpha \le S_n \le \beta] \approx \frac{1}{\sqrt{npq}} [\psi(x_\alpha) + \psi(x_{\alpha+1}) + \cdots + \psi(x_\beta)]. \tag{12}$$

On the right-hand side we have a Riemann sum which is an integral approximation.

This because

$$N\left(x_{k+\frac{1}{2}}\right) - N\left(x_{k-\frac{1}{2}}\right)$$

represents the area of a trapezoid with the basis

$$x_k - \frac{1}{2}\frac{1}{\sqrt{npq}} < x < x_k + \frac{1}{2}\frac{1}{\sqrt{npq}}$$

and which is upper bounded above by the tangent to the curve $y = \psi(x)$ at $x = x_k$. The rectangle's area with the same basis is $\dfrac{\psi(x_k)}{\sqrt{npq}}$.

An interesting step is to check how good this approximation is. Such problems have been considered by W. Feller. We shall emphasize below, in short, some aspects.

By the mean value theorem there is t_k such that

$$\begin{aligned} N(x_{k+1}) - N(x_{k-1}) &= \frac{\psi(t_k)}{\sqrt{npq}} \\ x_k - \frac{1}{2\sqrt{npq}} < t_k &< x_k + \frac{1}{2\sqrt{npq}}. \end{aligned} \tag{13}$$

Then,

$$\frac{\psi(x_k)}{\sqrt{npq}} = e^{\frac{1}{2}(t_k^2 - x_k^2)} \left[N\left(x_{k+\frac{1}{2}}\right) - N\left(x_{k-\frac{1}{2}}\right) \right].$$

Let now $\varepsilon > 0$ be arbitrarily choosen and suppose that the conditions (11) hold. Then, for any k, $\alpha \le k \le \beta$, and n sufficiently large, it results

$$\begin{aligned} \frac{1}{2}\left|t_k^2 - x_k^2\right| &= \frac{1}{2}\left|t_k + x_k\right| \cdot \left|t_k - x_k\right| < \\ &< \frac{1}{\sqrt{npq}} \left[|x_k| + \frac{1}{4\sqrt{npq}} \right] < \varepsilon \end{aligned}$$

which implies

$$e^{-\varepsilon}\left[N\left(x_{k+\frac{1}{2}}\right)-N\left(x_{k-\frac{1}{2}}\right)\right]<\frac{\psi(x_k)}{\sqrt{npq}}<$$
$$<e^{\varepsilon}\left[N\left(x_{k+\frac{1}{2}}\right)-N\left(x_{k-\frac{1}{2}}\right)\right]. \tag{14}$$

Now, taking the sum over k, will be found that the ratio between the right-hand side in (12) and

$$N\left(x_{\beta+\frac{1}{2}}\right)-N\left(x_{\alpha-\frac{1}{2}}\right)$$

tends to 1.

Thus, it is obtained a new variant of the DeMoivre-Laplace local limit theorem.

Theorem 4. *Let us take α and β such that the conditions (11) hold. Then*

$$P\left[\alpha\le S_n\le\beta\right]\approx N\left(x_{\beta+\frac{1}{2}}\right)-N\left(x_{\alpha-\frac{1}{2}}\right) \tag{15}$$

where

$$x_k=\frac{k-np}{\sqrt{npq}}.$$

In other words, the difference between the two sides in (15) tends to zero together with $\frac{x_\beta^3}{\sqrt{npq}}$ and $\frac{x_\alpha^3}{\sqrt{npq}}$.

As a consequence, (15) is also true for α and β restricted to values correponding to x_α and x_β in a given interval.

A simple case of this limit theorem can be obtained when the *reduced number of successes* defined by

$$S_n^*=\frac{S_n-np}{\sqrt{npq}} \tag{16}$$

is taken instead of S_n.

This means that we have to measure the deviation of S_n from np in $\sqrt{npq}$ units. Here np is referred to as the *mean* of S_n and $\sqrt{npq}$ is called its *standard deviation*.

Now $\alpha\le S_n\le\beta$ is equivalent to $x_\alpha\le S_n^*\le x_\beta$, while by (15) it is said that for arbitrary given $x_\alpha<x_\beta$ one gets

$$P[x_\alpha\le S_n^*\le x_\beta]\approx N\left(x_\beta+\frac{(npq)^{-\frac{1}{2}}}{2}\right)-$$
$$-N\left(x_\alpha-\frac{(npq)^{-\frac{1}{2}}}{2}\right). \tag{17}$$

Since $\frac{1}{\sqrt{npq}}\to 0$ as $n\to\infty$, the right-hand side tends to $N(x_\beta)-N(x_\alpha)$. We have thus the following corollary as a weak version of the theorem 4

Corollary 1. *For any fixed $a < b$*

$$P[a \leq S_n* \leq b] \to N(b) - N(a). \tag{18}$$

This is the classical form of Laplace's limit theorem.

If $\frac{(npq)^{-\frac{1}{2}}}{2}$ in (17) is excluded, automatically an error is introduced which tends to zero as $n \to \infty$. But it becomes considerable big when npq has a moderate value.

Regarding to (18), the main observation is that for large n, the left-hand side probability becomes independent on p. So we may compare the fluctuations in distinct series of simple Bernoulli trials using the standart units.

We emphasize that the approximation theorems and limits are valid only if the number n of trials is given, in advance, independently of the outcome of the trials. From historical point of view, (18) is the first limit theorem in the probability theory. But from the modern point of view it is only a very special case of the central limit theorem.

4 DeMoivre-Laplace integral limit theorem

DeMoivre-Laplace local limit theorem can be used to derive the *integral limit theorem* which is given below

Theorem 5. (***DeMoivre-Laplace integral theorem.***) *If μ is the number of occurrences of an event in n independent trials, in each of which the probability of the event is equal to p, ($0 < p < 1$) then, the following relation holds uniformly in a and b ($-\infty \leq a \leq b \leq +\infty$) as $n \longrightarrow \infty$*

$$P\left[a \leq \frac{\mu - np}{\sqrt{npq}} < b\right] \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{z^2}{2}}\, dz.$$

[For proof and more details and examples see [6]], [2], [14], [17], [18], [19].

Application

We remind that the *intuitive notion of probability* uses one assumption: *if in n identical trials the event A occurs ν times the ration $\frac{\nu}{n}$ is very closed to the probability p of the event A when n is sufficiently large.*

In this way, will be considered any *identical trials* as *Bernoulli trials* with the probability p for success. We denote the number of successes in n trials by S_n. Then, the average number of successes is $\frac{S_n}{n}$ and it should be near p.

It is known the following main result

$$P\left[\left|\frac{S_n}{n} - p\right| < \varepsilon\right] \to 1 \tag{19}$$

which can be stated as follows: *the probability that the average number of successes deviation from p is smaller than ε, in absolute value, tends to 1 for sufficiently large n.*

This is one statement of the *law of large numbers.* But in applications we should add to it a more precise estimation of the probability in (19). This will be obtained by the normal approximation of the binomial distribution. [Then (19) appears as a consequence of this].

The law (19) is referred to as the *classical law of large numbers.*

Now as an application of the theorem 5, we propose to estimate the probability of the inequality

$$\left|\frac{\mu}{n} - p\right| < \varepsilon$$

for $\varepsilon > 0$ arbitrarily choosen.

The probability of this inequality can be approximated as follows

$$P\left[\left|\frac{\mu}{n} - p\right| < \varepsilon\right] = P\left[-\varepsilon\sqrt{\frac{n}{pq}} < \frac{\mu - np}{\sqrt{npq}} < \varepsilon\sqrt{\frac{n}{pq}}\right].$$

But by the theorem 5 we have

$$\lim_{n\to\infty} P\left[-\varepsilon\sqrt{\frac{n}{pq}} < \frac{\mu - np}{\sqrt{npq}} < \varepsilon\sqrt{\frac{n}{pq}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}\,dz = 1$$

from which it results

$$\lim_{n\to\infty} P\left[\left|\frac{\mu}{n} - p\right| < \varepsilon\right] = 1$$

which is just *the law of large numbers.*

5 Convergence to normal distribution law

The DeMoivre-Laplace integral limit theorem is a good basis for a large variety of problems of fundamental importance both to the theory of probability itself and to its multiplicity of applications in the natural sciences, technology, economic sciences, even in the process of transmission of information or in computer science.

Now a very important problem which was solved especially by the remarkable contributions of Lyapunov, Lindeberg, Gnedenko is discussed. [See for example [5], [6], [12], [15], [16]].

Thus, a sequence of sums with ever larger numbers of summands is considered and it is assumed that the solutions of the problems which are of interest, are given by limiting distribution functions for a sequence of distribution functions of the sums.

In this context let us consider that a sequence of mutually independent random variables is given

$$X_1, X_2, X_3, \cdots, X_n, \cdots.$$

Also it is supposed that they have finite expectations and variances. To shorten the calculus the following notations are introduced

$$m_k = EX_k, \quad \text{and} \quad D_n^2 = \sum_{k=1}^{n} \sigma^2 X_k = \sigma^2 \sum_{k=1}^{n} X_k.$$

The main question is teh following: *what conditions must be imposed on the variables X_k so that the distribution functions of the sums*

$$\frac{1}{D_n} \sum_{k=1}^{n} (X_k - m_k) \tag{20}$$

converge to the normal distribution law ?

An answer to this question is given by a theorem due tu A.M. Lyapunov. We do not insist on the details, this being a known problem. Nevertheless, more details can be found in some works as [13], [6], or more recently [23].

But we remind that it is sufficient to be satisfied the following known condition due to Lindeberg

Lemma 2. (***Lindeberg condition***)**.** *For any $\tau > 0$*

$$\lim_{n\to\infty} \frac{1}{D_n^2} \sum_{k=1}^{n} \int\limits_{|x-m_k|>\tau D_n} (x - m_k)^2 \, dF_k(x) = 0 \tag{21}$$

where $F_k(x)$ denotes the distribution function of the random variable X_k.

Regarding to the meaning of this condition we may conclude that *the Lindeberg condition is a peculiar kind of demand for the uniform smallness of the terms*

$$\frac{1}{D_n} (X_k - m_k)$$

in the sum (20).

The following result is now obtained

Lemma 3. *If the independent random variable $X_1, X_2, \cdots, X_n, \cdots$ are identically distributed and have a finite variance different from zero, then*

$$P\left[\frac{1}{D_n} \sum_{k=1}^{n} (X_k - EX_k) < x\right] \to \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz \tag{22}$$

uniformly in x as n tends to infinity.

For this it is sufficient to verify that the Lindeberg condition is satisfied under the given assumptions. To this case one has

$$D_n = \sigma\sqrt{n}$$

where σ^2 denotes the variance of a separate summand. Let $EX_k = m$ be and one finds the following equation

$$\sum_{k=1}^{n} \frac{1}{D_n^2} \int\limits_{|x-m|>\tau D_n} (x-m)^2\, dF_k(x) =$$

$$= \frac{1}{n\sigma^2}\, n \int\limits_{|x-m|>\tau D_n} (x-m)^2\, dF_1(x) =$$

$$= \frac{1}{\sigma^2} \int\limits_{|x-m|>\tau D_n} (x-m)^2\, dF_1(x)$$

But the variance was supposed to be finite and positive so that it results that the integral on the right-hand side of this equation tends to zero as n tends to infinity. ■

Now the main result is given below

Theorem 6. (*Lyapunov's theorem*). *If for a sequence of mutually independent random variables $X_1, X_2, X_3, \cdots, X_3, \cdots$ it is possible to choose a number $\delta > 0$ such that*

$$\frac{1}{D_n^{2+\delta}} \sum_{k=1}^{n} E|X_k - m_k|^{2+\delta} \to 0 \tag{23}$$

as $n \to \infty$, then

$$P\left[\frac{1}{D_n} \sum_{k=1}^{n} (X_k - m_k) < x\right] \to \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{z^2}{2}}\, dz \tag{24}$$

uniformly in x as $n \to \infty$.

It is easy to see that it will suffice to verify that the Lyapunov condition (23) implies that the Lindeberg condition holds. But this fact follows from the following inequalities

$$\frac{1}{D_n^2} \sum_{k=1}^{n} \int\limits_{|x-m_k|>\tau D_n} (x-m_k)^2\, dF_k(x) \le$$

$$\le \frac{1}{D_n^2 (\tau D_n)^\delta} \sum_{k=1}^{n} \int\limits_{|x-m_k|>\tau D_n} |x-m_k|^{2+\delta}\, dF_k(x) \le$$

$$\le \frac{1}{\tau^\delta} \frac{\sum_{k=1}^{n} \int |x-m_k|^{2+\delta}\, dF_k(x)}{D_n^{2+\delta}}$$

and the theorem follows.■

References

1. Bharucha-Reid, A.T., **Elements Of The Theory Of Markov Processes And Their Applications**, Dover Publications, Inc., Mineola, New York, 1997.
2. Feller, W., *On the Normal Approximation to the Binomial Distribution*, Ann. Math. Stat., Vol. XVI, 1945.
3. Feller, W., **An Introduction to Probability Theory and its Applications, vol.I**, John Wiley & Sons, Inc., New York, London, 1971.
4. Feller, W., **An Introduction to Probability Theory and its Applications, vol.II**, John Wiley & Sons, Inc, New York, London, 1971.
5. Gnedenko, B.V., *The local limit theorem for densities*, Doklady Akad. Nauk S.S.S.R., Vol 95, No. 1, 1954.
6. Gnedenko, B.V., **The Theory of Probability**, Mir Publisher, Moskow, 1976.
7. Gordin, M.I., *The central limit theorem for stationary processes*, Soviet Math. Doklady, 10, 1969, pp. 1174-1176.
8. Iosifescu, M, Mihoc, Gh., Theodorescu, R., **Teoria probabilităţilor şi statistică matematică**, Ed. Tehnică, Bucureşti, 1966.
9. Iosifescu, M., Theodorescu, R., **Random Processes and Learning**, Springer-Verlag New York Inc., 1969.
10. Itô K., McKean H.P. Jr., **Diffusion Processes and Sample Paths**, Springer-Verlag, Berlin Heidelberg, 1996.
11. Itô K., **Stochastic Processes**, Eds. Ole E. Barndorff-Nielsen, Ken-iti Sato, Springer, 2004.
12. Lindeberg, J. W., *Eine neue Herleitung des Exponentialgesetz in der Wahrscheinlichkeitsrechnung*, Math. Zeitschr., Vol. 15, 1922.
13. Loève, M., **Probability theory, vol I**, Springer-Verlag, New York, Heidelberg Berlin, 1977.
14. Loève, M., **Probability Theory, vol II**, Springer-Verlag, New York, Heidelberg Berlin, 1978.
15. Lyapunov, A.,M., *Sur une proposition de la théorie des probabilités*, Bull. Acad. Sc. Pèter., 13, 1900.
16. Lyapunov, A.,M., *Nouvelle forme du théoreme sur la limite des pro-babilités*, ibid., 1901.
17. Orman, G.V.. **Lectures on Stochastic Approximation Methods and Related Topics**, Preprint. "Gerhard Mercator" University, Duisburg, Germany, 2001.
18. Orman, G.V.. *On Stochastic Approximation Methods in Reliability*, Invited lecture & Preprint, ICRSA-2003, University of South Carolina, Columbia, 2003.
19. Orman, G.V., **Handbook of limit theorems and stochastic approximation**, Transilvania University Press, Brasov, 2003.
20. Orman, G.V., *On Markov Processes: A Survey Of The Transition Probabilities And Markov Property.* In: **Chaotic Systems: Theory and Applications**, Editors: Christos H. Skiadas and Ioannis Dimotikalis, World Scientific (Publisher), 2010.
21. Orman, G.V., **Aspects of Convergence and Approximation in Random Systems Analysis**, LAP LAMBERT Academic Publishing, 2012.
22. Orman, G.V., **Basic Probability Theory, Convergence, Stochastic Processes and App;ications**, Transilvania University Press, Brasov, 2016.
23. Petrov, V.V., **Limit Theorems of Probability Theory; Sequences of Independent Random Variables**, Clarendon Press, Oxford, 1995.
24. Rosenblatt, M., **Random Processes**, Springer-Verlag, New York Heidelberg Berlin, 1974.