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Bifurcation theory of dynamical chaos

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Abstract. It is shown that there exists a universal bifurcation scenario of transition to dynamical and spatio-temporal chaos in all kinds of nonlinear differential equations including dissipative and conservative, nonautonomous and autonomous nonlinear systems of ordinary and partial differential equations and differential equations with delay arguments. This scenario consists of several Andronov-Hopf bifurcations and then of Feigenbaum cascade of period doubling bifurcations of stable cycles or n-dimensional tori, and then of Sharkovsky cascade of subharmonic bifurcations of stable cycles or tori of an arbitrary period up to the cycle (torus) of period three, and then of Magnitskii cascade of homoclinic or heteroclinic bifurcations of stable cycles or tori up to the homoclinic or heteroclinic separatrix loop of singular points or up to the homoclinic or heteroclinic separatrix toroidal topological structure. It is shown that this universal FShM bifurcation scenario of transition to chaos takes place in all classical dissipative and conservative systems of ordinary and partial differential equations including Lorenz, Ressler, Chua, Sprott, Duffing-Holmes, Mathieu, Croquette, Rikitaki, Henon-Heiles, Yang-Mills systems, Brusselyator, Kuramoto-Tsuzuki, Mackey-Glass equations and many others. It is shown that transition to turbulence in Navier-Stokes equations has the same scenario.
Keywords: Dissipative and conservative systems, Transition to chaos, Universal bifurcation FShM scenario, Navier-Stokes equations, Turbulence.

1 Introduction

It is proved and shown in some papers of author (Magnitskii *et al.* [1-6]) and others that there exists an unique universal bifurcation scenario of transition to dynamical and spatio-temporal chaos in all kinds of nonlinear differential equations including dissipative and conservative, nonautonomous and autonomous nonlinear systems of ordinary and partial differential equations and differential equations with delay arguments. This scenario consists of several Andronov-Hopf bifurcations and then of Feigenbaum cascade of period doubling bifurcations of stable cycles or n-dimensional tori, and then of Sharkovsky cascade of subharmonic bifurcations of stable cycles or tori of an arbitrary period in accordance with Sharkovsky's order

$$1 \lhd 2 \lhd 2^2 \lhd 2^3 \lhd ... \lhd 2^k(2l+1) \lhd ... \lhd 2\cdot 7 \lhd 2\cdot 5 \lhd 2\cdot 3 \lhd ... \lhd 7 \lhd 5 \lhd 3,$$

up to the cycle (torus) of period three, and then of Magnitskii cascade of homoclinic (heteroclinic) bifurcations of stable cycles or tori up to the

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homoclinic or heteroclinic separatrix loop of singular points or up to the homoclinic or heteroclinic separatrix toroidal topological structure. All irregular attractors born during realization of such scenario are exclusively singular attractors that are the nonperiodic limited trajectories in finite dimensional or infinitely dimensional phase space any neighborhood of which contains the infinite number of unstable periodic trajectories.

Thus, in any nonlinear system there can be an infinite number of various singular attractors, becoming complicated at change of bifurcation parameter in a direction of the cascade of bifurcations. Presence or absence in system of stable or unstable singular points, presence or absence of saddle-nodes or saddle-focuses, and also homoclinic or heteroclinic separatrix contours and Smale's horseshoes, and also positivity of the calculated senior Lyapunov's exponent are not criteria of occurrence in system of chaotic dynamics. And the birth in the system of three-dimensional and even multi-dimensional stable torus not leads to its destruction with birth of mythical strange attractor as it postulated by Ruelle-Takens theory, but leads to another Andronov-Hopf bifurcation or to cascade of its period-doubling bifurcations along one of its frequencies or several frequencies simultaneously. Chaotic dynamics in Hamiltonian and conservative systems also is consequence of cascades of bifurcations of birth of new tori, instead of consequence of destruction of some already ostensibly existing mythical tori of nonperturbed system as it postulated by Kolmogorov-Arnold-Mozer (KAM) theory

.The purpose of the present paper is once again to show on concrete examples that dynamical chaos in nonlinear systems of ordinary differential equations and spatio-temporal or diffusion chaos in nonlinear systems of equations with partial derivatives and chaos in Hamiltonian and conservative systems are generated by cascades of bifurcations in accordance with the FShM scenario.

2 Chaos in two-dimensional dissipative systems with periodic coefficients.

It is shown (Magnitskii [3]) that nonlinear two-dimensional systems of ordinary differential equations with rotor type singular points have chaotic dynamic in accordance with the FShM theory. The simplest example is the system

$$\dot{u}_1 = 2(\mu - 1 + \cos\omega\, t)u_1 + (2\sin\omega\, t - \omega/2)u_2 - u_2^2,$$
$$\dot{u}_2 = (2\sin\omega\, t + \omega/2)u_1 + 2(\mu - 1 - \cos\omega\, t)u_2,$$

where μ is a bifurcation parameter. This system has all infinitely many stable cycles from FShM cascade up to the homoclinic separatrix loop (Fig.1). More complex systems are: Duffing-Holmes equation

$$\ddot{x} + k\,\dot{x} + \omega^2 x + \mu\, x^3 = f_0 \cos\Omega t,$$

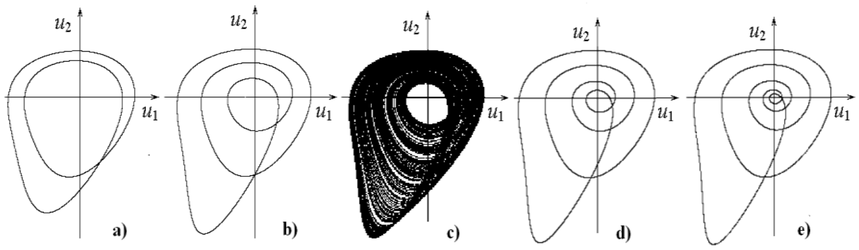

Fig.1.Cycles of periods 2, 3; singular attractor; homoclinic cycles of periods 4,5.

modified dissipative Mathieu equation

$$\ddot{x} + \mu\dot{x} + (\delta + \varepsilon\cos\omega t)x + \alpha x^3 = 0,$$

Croquette dissipative equation

$$\ddot{x} + \mu\dot{x} + \alpha\sin x + \beta\sin(x - \omega t) = 0.$$

All of them have the FShM scenario of transition to chaos and infinitely many singular attractors (see, for example, Fig.2 for Croquette dissipative equation).

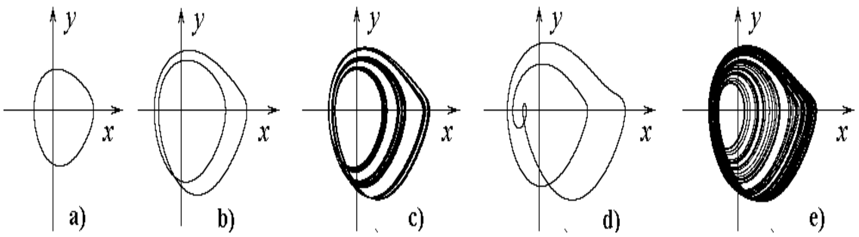

Fig.2.Cycles of periods 1, 2, 3(d); Feigenbaum (c) and more complex (e) singular attractors.

3 Chaos in three-dimensional autonomous dissipative systems

It is proved by author (Magnitskii [3,6]), that nonlinear three-dimensional autonomous dissipative systems have the same scenario of transition to dynamical chaos as two-dimensional systems with rotor type singular points. It takes place for all famous systems such as Lorenz, Ressler, Chua, Sprott, Chen, Magnitskii and many others systems. For example, see Fig.3 for Lorenz system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(r - z) - y, \quad \dot{z} = xy - bz.$$

Homoclinic cascade of bifurcations in Lorenz system which tends to saddle-focus homoclinic separatrix loop and very complex heteroclinic separatrix contour - heteroclinic butterfly are presented in Fig.4.

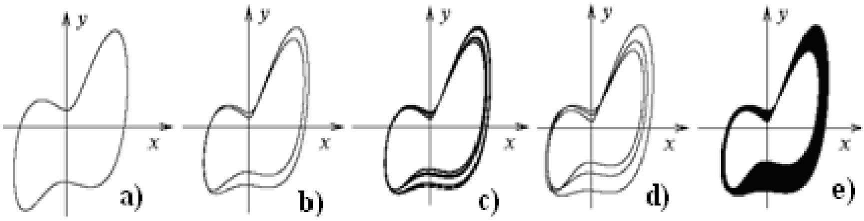

Fig.3.Cycles of periods 1, 2, 3(d); Feigenbaum (c) and more complex (e) singular attractors.

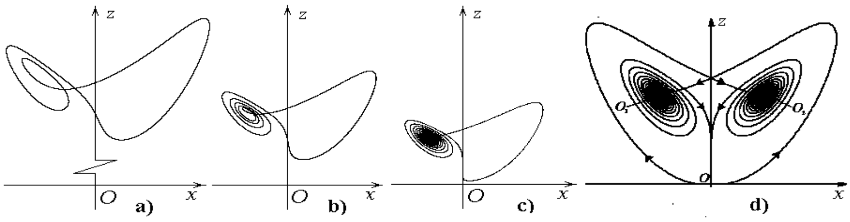

Fig.4. Homoclinic cascade: a),b),c) and heteroclinic butterfly (d)..

4 Chaos in many-dimensional autonomous dissipative systems

In dissipative autonomous systems, which dimension is more than three, one can observe FShM scenario of transition to chaos as through cascade of bifurcations of stable cycles, as through cascade of bifurcations of stable two-dimensional or many-dimensional tori. FShM cascade of bifurcations of stable cycles for four-dimensional Rikitaki system

$$\dot{x} = -\mu x + yz,\quad \dot{y} = -\mu y + xu,\quad \dot{z} = 1 - xy - bz,\quad \dot{u} = 1 - xy - cu$$

is shown in Fig. 5

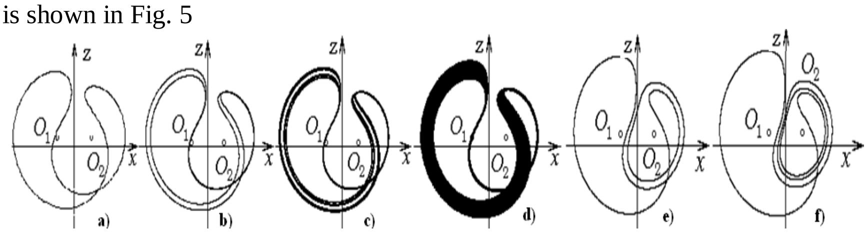

Fig.5. Cycles of periods 1,2; homoclinic cycles e),f); singular attractors c),d).

and FShM cascade of bifurcations of stable two-dimensional tori for five-dimensional complex Lorenz system

$$\dot{X} = -\sigma X + \sigma Y,\quad \dot{Y} = -XZ + rX - aY,\quad \dot{Z} = -bZ + (X^{*}Y + XY^{*})/2,$$

with $X = x_1 + ix_2,\ Y = y_1 + iy_2$ is shown in Fig. 6.

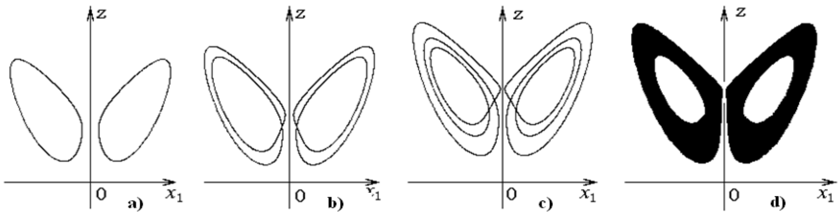

Fig. 6. Poincare sections of tori of periods 1, 2, 3 and singular toroidal attractor.

5 Chaos in infinitely-dimensional system with delay argument

Differential equation with delay argument has infinite dimension. So, it can have transition to chaos as through FShM cascade of bifurcations of stable cycles, as through FShM cascade of bifurcations of tori of any dimension. As example, FShM cascade of bifurcations of stable cycles in Mackey-Glass equation

$$\dot{x}(t) = -ax(t) + \beta_0 \frac{\theta^n x(t-\tau)}{\theta^n + x^n(t-\tau)}.$$

is shown in Fig.7, where delay τ is the bifurcation parameter.

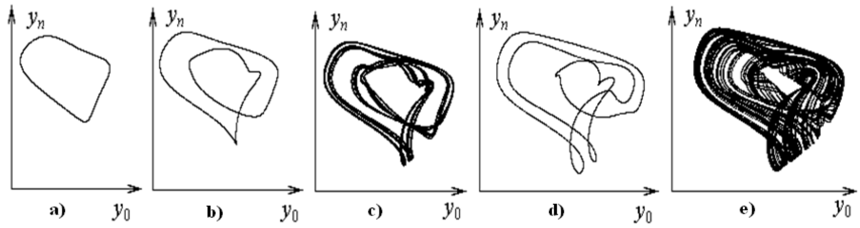

Fig.7.Cycles of periods 1, 2, 3(d); Feigenbaum (c) and more complex (e) singular attractors.

6 Chaos in conservative systems

In papers (Magnitskii [1,3]) it was proposed to consider any conservative (in particular, Hamiltonian) system as a limit of its extended dissipative system when dissipation parameter tends to zero. It turned out that in this case, the extended dissipative system undergoes a cascade of bifurcations of usually two-dimensional tori in accordance with the FShM-theory. Moreover, an infinite number of newly born tori remains in the extended system for all positive values of dissipation parameter, and they generate chaotic dynamics in the conservative system at dissipation parameter is equal to zero.
As a simple example let us consider the conservative Croquette equation

$$\ddot{x} + \alpha \sin x + \beta \sin(x - \omega t) = 0.$$

The extended dissipative system for Croquette equation is the four-dimensional dissipative system

$$\dot{x} = y, \quad \dot{y} = -\mu y - (\alpha + r)\sin x + z\cos x, \quad \dot{z} = \omega r, \quad \dot{r} = -\omega z$$

with the condition $H = z^2 + r^2 = \varepsilon^2, \quad z_0 = z(0) = 0.$ The first cycles of the period-doubling cascade of bifurcations in the parameter $\varepsilon > 0$, found in this method in conservative system, are shown in Fig. 8.

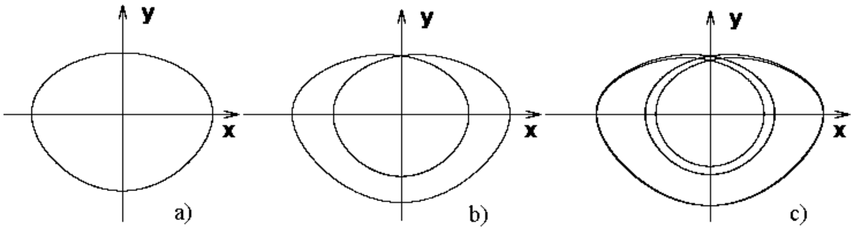

Fig.8. Cycles of periods 1, 2 and 4 in conservative Croquette equation.

FShM-cascades of bifurcations generate infinitely folded heteroclinic separatrix manifolds having in Poincare section a kind of heteroclinic separatrix zigzag. These manifolds are tense on unstable singular cycles of FSM-cascade of dissipative system and they pass at zero dissipation in even more complex separatrix manifolds of conservative (Hamiltonian) system, movement of trajectories on which looks like as chaotic dynamics. Thus there is a stretching of an accordion of infinitely folded heteroclinic separatrix zigzag on some area of phase space of the conservative system. In the remained part of phase space elliptic cycles from the right part of subharmonic and homoclinic cascades can simultaneously coexist with tori around of them. In Poincare section it looks like a family of the hyperbolic singular points connected by separatrix contours. This picture at any shift in initial conditions passes into a family of so-called islands (points in Poincare section forming closed curves around of points of elliptic cycle). Islands in conservative Croquette equation and development and complication of heteroclinic separatrix zigzag in extended dissipative Croquette system are presented in Fig.9.

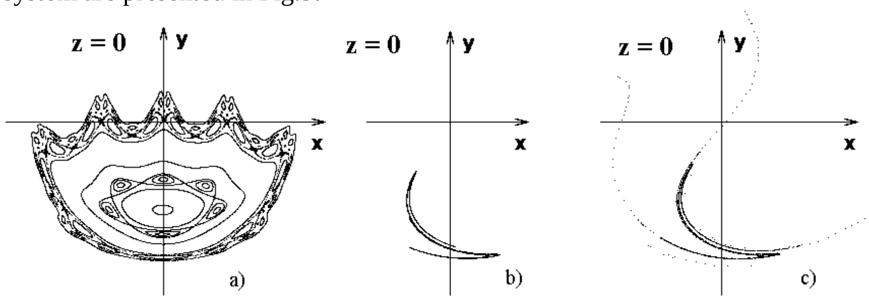

Fig.9. Islands in conservative Croquette equation (a) and development of heteroclinic separatrix zigzag in dissipative Croquette system when $\mu \to 0.$

More complex system is the Hamiltonian system with two degrees of freedom

$$\dot{x} = y, \quad \dot{y} = -(\delta + z)\,x - x^3, \quad \dot{z} = r, \quad \dot{r} = -z - x^2/2$$

and with Hamiltonian

$$H(x, y, z, r) = (\delta\, x^2 + y^2 + z^2 + r^2)/2 + zx^2/2 + x^4/4 = \varepsilon.$$

The extended dissipative system is the four-dimensional dissipative system

$$\dot{x} = y,\;\; \dot{y} = -(\delta + z)\,x - x^3 - \mu\, y,\;\; \dot{z} = r,\;\; \dot{r} = -z - x^2/2 + (\varepsilon - H(x, y, z, r))r.$$

Development and complication of heteroclinic separatrix zigzag in extended dissipative system as a result of FShM cascades of bifurcations when bifurcation parameter $\mu \to 0$ is presented in Fig.10.

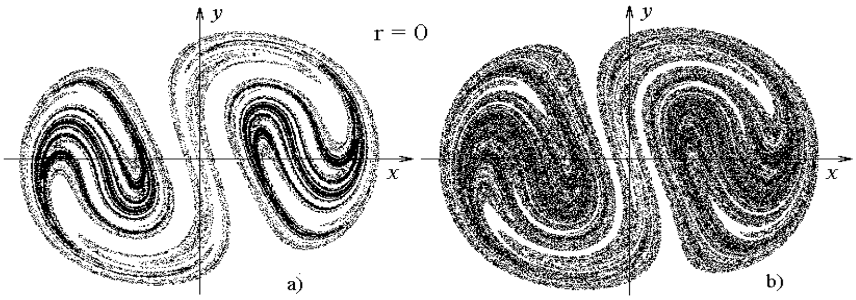

Fig.10. Development of heteroclinic separatrix zigzag in extended dissipative system when $\mu \to 0$.

Else more complex system is the Hamiltonian system with three degrees of freedom

$$\dot{x} = y,\;\; \dot{y} = -(\delta + z)x - x^3,\;\; \dot{z} = r,\;\; \dot{r} = -z - x^2/2 - u^2/2,\;\; \dot{u} = v,\;\; \dot{v} = -(\gamma + z)u - u^3$$

and with Hamiltonian

$$H(x, y, z, r, u, v) = (\delta\, x^2 + y^2 + z^2 + \gamma\, u^2 + v^2)/2 + z(x^2 + u^2)/2 + (x^4 + u^4)/4 = \varepsilon.$$

The extended dissipative system is the six-dimensional dissipative system

$$\dot{x} = y,\;\; \dot{y} = -(\delta + z)x - x^3 - \mu\, y,\;\; \dot{z} = r,\; \dot{r} = -z - x^2/2 - u^2/2 + (\varepsilon - H)\,r,\;\; \dot{u} = v,\;\; \dot{v} = -(\gamma + z)u - u^3 - \mu\, v.$$

Development and complication of heteroclinic separatrix zigzag in extended dissipative system as a result of FShM cascades of bifurcations when bifurcation parameter $\mu \to 0$ is presented in Fig.11.

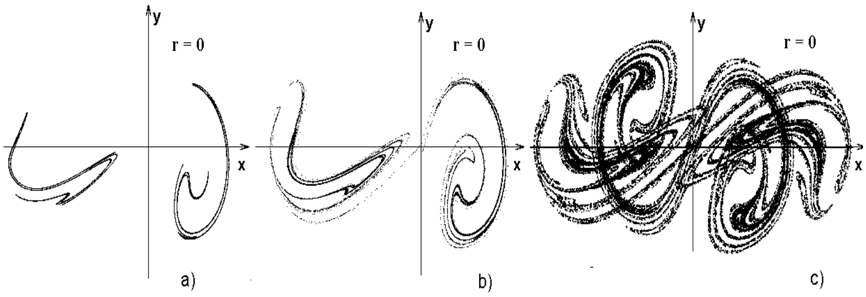

Fig.11. Development of heteroclinic separatrix zigzag in extended dissipative system when $\mu \to 0$.

7 Spatio-temporal chaos in nonlinear partial differential equations.

We shall consider three wide classes of nonlinear partial differential equations describing many physical, chemical, biological, ecological and economic processes. They are: reaction-diffusion system of equations

$$u_t = D_1 u_{xx} + f(u,v,\mu), \quad v_t = D_2 v_{xx} + g(u,v,\mu),$$

FitzHugh-Nagumo system of equations

$$u_t = Du_{xx} + f(u,v,\mu), \quad v_t = g(u,v,\mu)$$

and Kuramoto-Tsuzuki (Time-Dependant Ginzburg-Landau) equation

$$W_t = W + (1+ic_1)W_{xx} - (1+ic_2)W|W|^2, \qquad W = u + iv.$$

As an example of the reaction-diffusion system of equations let's consider the system of the brusselator equations offered for the first time by the Brussels school of I. Prigoging as a model of some self-catalyzed chemical reaction with diffusion

$$u_t = D_1 u_{xx} + A - (\mu+1)u + u^2 v, \;\; v_t = D_2 v_{xx} + \mu u - u^2 v$$

Brusselator is infinitely-dimensional system and so it can have transition to chaos as through FShM cascade of bifurcations of stable cycles, as through FShM cascade of bifurcations of tori of any dimension. FShM cascade of bifurcations of stable cycles in brusselator system when parameter μ increases is presented in Fig.12 and FShM cascade of bifurcations of stable two-dimensional tori is presented in Fig. 13.

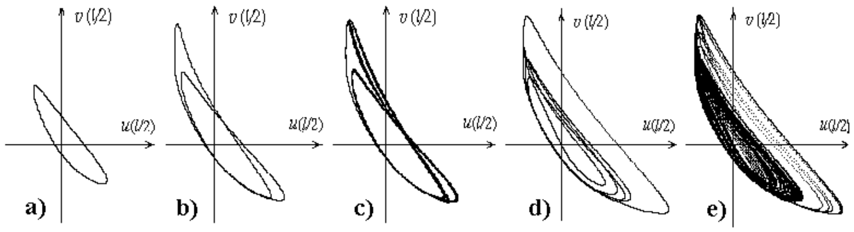

Fig.12.Cycles of periods 1, 2, 5(d); Feigenbaum (c) and more complex (e) singular attractors.

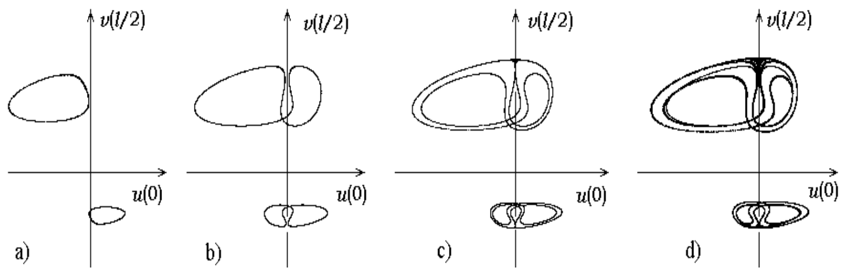

Fig.13. Poincare sections of 2D tori of periods 1,2,4 and Feigenbaum attractor.

FitzHugh-Nagumo system of equations can be reduced by replacement $\xi = x - ct$ to three-dimensional system of ordinary differential equations

$$\dot{u} = y, \;\; \dot{y} = -(c\,y + f(u,v,\mu))/D, \quad \dot{v} = -g(u,v,\mu)/c,$$

Then running waves and running impulses of FitzHugh-Nagumo system are described by limit cycles and homoclinic separatrix loops of ODE system. FShM cascade of bifurcations in three-dimensional ODE system, as the values of bifurcation parameter c decrease, is shown in Fig.14 and running waves in the FitzHugh-Nagumo system corresponding with the homoclinic cycles d) and e) are shown in Fig.15.

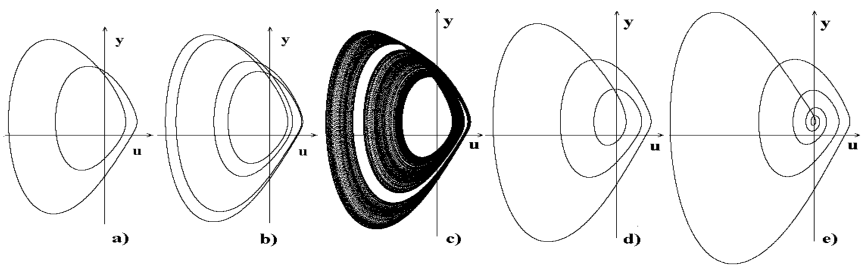

Fig.14.Cycles of periods 2, 4, 3(d); singular attractor (c), homoclinic cycle of period 4(e).

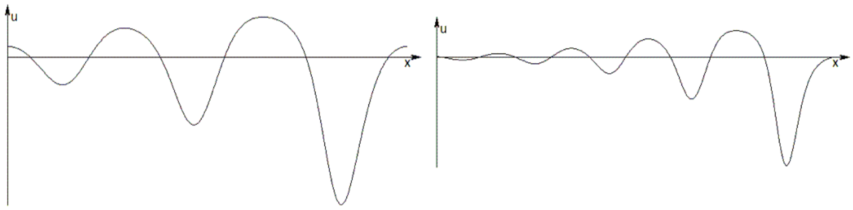

Fig.15. Running waves in the FitzHugh-Nagumo system corresponding with the homoclinic cycles d) and e) in Fig.14.

Kuramoto-Tsuzuki (Time-Dependant Ginzburg-Landau) equation has very many different ways of transition to spatio-temporal chaos through FShM cascades of bifurcations of two-dimensional tori for different values of bifurcation parameters c_1 and c_2. Two such ways are presented in Fig.16 and Fig.17

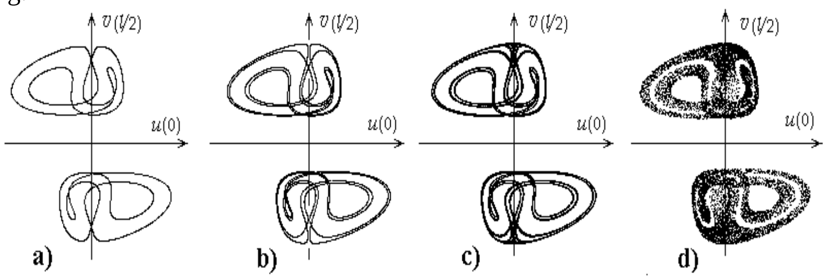

Fig.16. Poincare sections of 2D tori of periods 4, 8; Feigenbaum and more complex singular attractors.

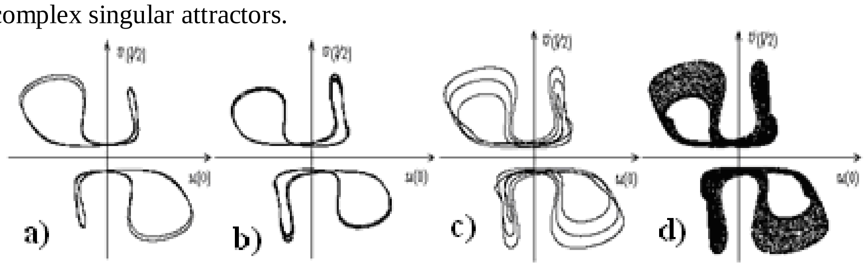

Fig.17. Poincare sections of 2D tori of periods 2(a), 3(c); Feigenbaum (b) and more complex singular attractor (d).

8 Spatio-temporal chaos (turbulence) in Navier-Stokes equations

The problem of turbulence - the disordered chaotic motion of a nonlinear continuous medium describing by Navier-Stokes equations, named by Clay Mathematics Institute as one of seven millennium mathematical problems, and it is also in the list of 18 most significant mathematical problems of XXI century formulated by S. Smale. Several models trying to explain the mechanisms of turbulence generation in nonlinear continues media were suggested at different time. Among such models the most known are Landau-Hopf and Ruelle-Takens models, explaining generation of turbulence by the infinite cascade of Andronov-Hopf bifurcations and, accordingly, by destruction of three-dimensional torus with generation of strange attractor. However, these models have not been justified by experiments with hydrodynamic turbulence. In recent years, the author and his pupils have proved (Magnitskii *et al.* [2,4-5]) that the universal bifurcation FShM mechanism for the transition to spatio-temporal chaos in nonlinear systems of partial differential equations through subharmonic cascades of bifurcations of stable cycles or two-dimensional and many-dimensional tori also takes place in problems of laminar-turbulent transitions for Navier-Stokes equations

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + R^{-1}\Delta\vec{u} + \vec{f}, \qquad \nabla\vec{u} = 0,$$

where Reynolds number R is the bifurcation parameter. FShM cascade of bifurcations of two-dimensional tori in Navier-Stokes equations is shown in Fig.18 and Fig.19.

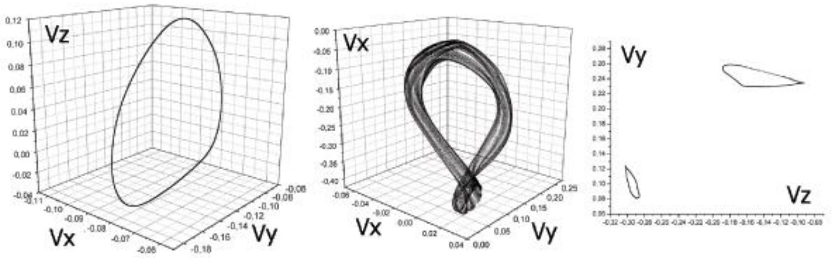

Fig.18. Cycle, 2D-torus and its Poincare section in Navier-Stokes equations

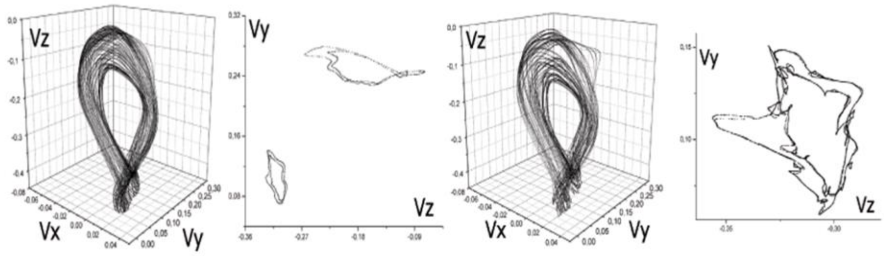

Fig.19. 2D-torus of period 2 and its Poincare section; 2D-torus of period 4 and its Poincare section in Navier-Stokes equations

Transition to chaos in Navier-Stokes equations can be realized also through FShM cascade of bifurcations of three-dimension or many-dimension tori. Stable three-dimension torus and its Poincare sections are shown in Fig.20.

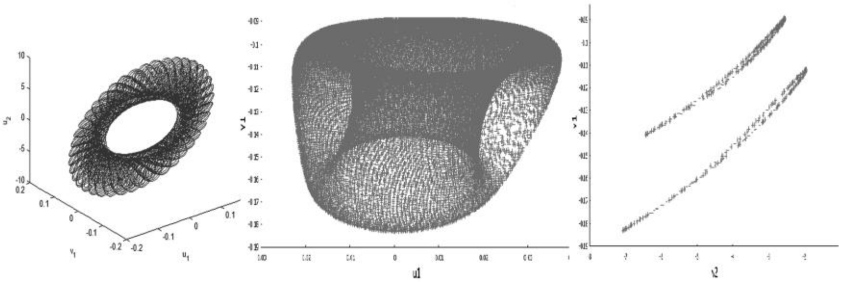

Fig.20. Projection of stable 3D torus and its first and second Poincare sections

Conclusions

There exists universal bifurcation scenario of transition to dynamical or spatio-temporal chaos in all nonlinear dissipative and conservative, autonomous and nonautonomous, ordinary and partial systems of differential equations. This scenario can consist of: FShM scenario with subharmonic and homoclinic (heteroclinic) bifurcations of stable limit cycles; or cascade of Andronov-Hopf bifurcations with forming of n-dimensional torus and then FShM scenario with subharmonic and homoclinic (heteroclinic) bifurcations of stable n-dimensional tori. All nonregular attractors are the only singular attractors which are limits of period-doubling cascades of bifurcations of regular attractors (cycles, tori).

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