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Relativistic Magnetohydrodynamic Turbulence in the Early Universe

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Abstract. The dynamics of all physical systems can be understood in terms of their invariant quantities. In this letter, we look at the invariant quantities of Relativistic and Non Relativistic fluids of charged particles in order to understand the spontaneous generation of magnetic seed fields which may have been responsible for magnetogenesis in the early universe. We show how the invariants of Relativistic Magnetohydrodynamic systems naturally lead to the development of seed magnetic fields while the invariants of non-relativistic magnetohydrodynamic systems will suppress such a development.
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A significant volume of work has been performed in order to understand the invariants of Non Relativistic Magnetohydrodynamic (MHD) systems in both the compressible and incompressible regimes [1,5,7–13]. In such systems there could be as many as 3 invariants; energy (E), and the psuedoscalars Cross Helicity (H_C) and Magnetic Helicity (H_M).

Table 1. Invariants for Ideal Non Relativistic MHD.

Case	Mean Field	Angular Velocity	Invariants
I	0	0	E, H_C, H_M
II	$B_0 \neq 0$	0	E, H_C
III	0	$\Omega_0 \neq 0$	E, H_M
IV	$B_0 \neq 0$	$\Omega_0 = \sigma B_0$	E, H_P
V	$B_0 \neq 0$	$\Omega_0 \neq 0$ ($B_0 \times \Omega_0 \neq 0$)	E

The Cross Helicity and Magnetic Helicity can be expressed as:

$$H_C = \frac{1}{2} \int \mathbf{V} \cdot \mathbf{B} \ \mathbf{d^3x} \tag{1}$$
$$H_M = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{B} \ \mathbf{d^3x}. \tag{2}$$

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For the magnetofluid, $\mathbf{V}$ is the velocity of the fluid element, $\mathbf{B}$ is the magnetic field and $\mathbf{A}$ is the vector potential. A conservation law is satisfied when the time derivative of one of the terms above is zero. According to work by Yoshida et al [4,14], in addition to 4-momentum, relativistic systems are expected to conserve a quantity called Relativistic Helicity. It is defined below using the canonical 4-momentum density, $\mathcal{P}^\mu$ and vorticity, $\mathbf{\Omega} = \nabla \times \mathcal{P}$, of the system.

$$\kappa = (\mathcal{P} \cdot \mathbf{\Omega}, \mathcal{P}_\mathbf{0}\mathbf{\Omega} + \mathcal{P} \times (\nabla \mathcal{P}_\mathbf{0} + \partial_\mathbf{0} \mathcal{P})) \tag{3}$$

Here the canonical 4-momentum density $\mathcal{P} = (\mathcal{P}_0, \mathcal{P})$ is a combination of mechanical and electromagnetic momentum densities, $\mathcal{P}_\mu = P_\mu + eA_\mu$. The conservation of Relativistic Helicity is then effectively, $\int \partial_\mu \kappa^\mu \mathbf{d^3x} = \mathbf{0}$. If we ignore the electromagnetic momentum and physical vorticity, we recover a relativistic version of Cross Helicity Density, $\kappa_\mathbf{C}$. If we set the particle's kinetic momentum to zero, we recover a relativistic version of Magnetic Helicity Density, $\kappa_\mathbf{M}$.

$$\kappa_\mathbf{C} = (\mathbf{P} \cdot \mathbf{B}, \mathbf{P_0 B} - \mathbf{P} \times \mathbf{E}) \tag{4}$$

$$\kappa_\mathbf{M} = (\mathbf{A} \cdot \mathbf{B}, \mathbf{A_0 B} - \mathbf{A} \times \mathbf{E}) \tag{5}$$

Here the magnetic field is related to the vector potential by the equation, $\mathbf{B} = \nabla \times \mathbf{A}$. The electric field is defined using the MHD conditions, $\mathbf{E} = \mathbf{B} \times \mathbf{V}$. One can see that in the non relativistic limit, the Cross Helicity and Magnetic Helicity should equate to those shown in Equation 1 and 2 assuming that mass density is constant. Previous work by the author [3] has shown that only Relativistic Helicity and Energy are conserved in Relativistic MHD systems, see Figures 1-3. It can be shown that the four divergences of the Relativistic, Cross and Magnetic Helicities for Relativistic MHD reduce to:

$$\partial_\mu \kappa^\mu = 2\Omega \cdot (\nabla \mathcal{P}_0 + \partial_0 \mathcal{P}) = \mathbf{0} \tag{6}$$

$$\partial_\mu {\kappa_\mathbf{C}}^\mu = -(\nabla \cdot V)(\mathbf{P} \cdot \mathbf{B}) \neq \mathbf{0} \tag{7}$$

$$\partial_\mu {\kappa_\mathbf{M}}^\mu = -2(\nabla \cdot V)(\mathbf{A} \cdot \mathbf{B}) \neq \mathbf{0} \tag{8}$$

Because $\partial_0 \mathcal{P} = -\nabla \mathcal{P}_\mathbf{0} = \mathbf{F}$, the four divergence of the Relativistic Helicity is identically zero. However, the four divergences of the cross helicity and magnetic helicity are not identically zero because the divergence of velocity is not necessarily null and the Lorentz Transformations result in components of the magnetic field and vector potentials parallel to the velocity vector. Note that since the magnetic field and velocity perturbations are initially random, there will be components of magnetic field that lie along the velocity vector, relativistic effects will amplify these components.

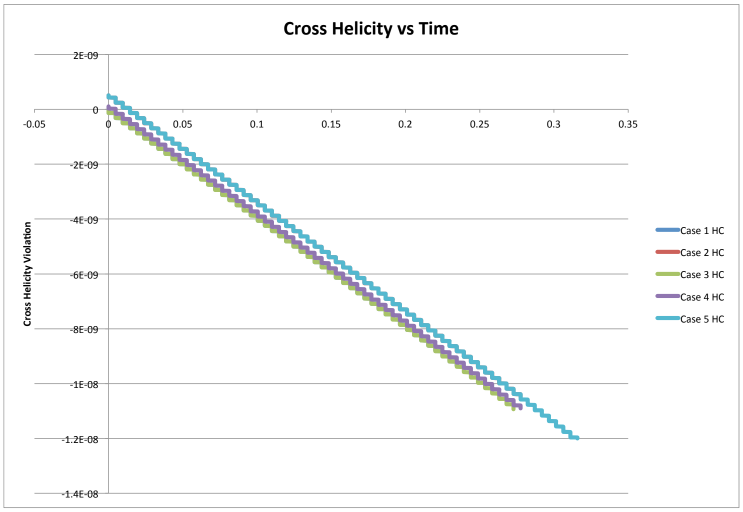
$$\mathbf{B}' = \gamma(\mathbf{B} - \frac{\mathbf{V}(\mathbf{B} \cdot \mathbf{V}) - \mathbf{B}\mathbf{V^2}}{\mathbf{c^2}}) - (\gamma - \mathbf{1})(\mathbf{B} \cdot \hat{\mathbf{V}})\hat{\mathbf{V}} \tag{9}$$

$$\mathbf{A}' = \gamma \mathbf{A} - \frac{\gamma \mathbf{A_0}}{\mathbf{c^2}}\mathbf{V} - (\gamma - \mathbf{1})(\mathbf{A} \cdot \hat{\mathbf{V}})\hat{\mathbf{V}} \tag{10}$$

For the non relativistic versions of Cross and Magnetic Helicity, we evaluate their time derivatives below assuming conservation.

$$\partial_t H_C = \tfrac{1}{2} \int (\tfrac{\partial \mathbf{V}}{\partial t} \cdot \mathbf{B} + \mathbf{V} \cdot \tfrac{\partial \mathbf{B}}{\partial \mathbf{t}})\ \mathbf{d^3x} = \mathbf{0} \rightarrow \tfrac{\partial \mathbf{B}}{\partial \mathbf{t}} = -\tfrac{\hat{\mathbf{V}}}{|\mathbf{V}|}(\mathbf{B} \cdot \tfrac{\partial \mathbf{V}}{\partial \mathbf{t}}) \tag{11}$$

$$\partial_t H_M = \tfrac{1}{2} \int (\tfrac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} + \mathbf{A} \cdot \tfrac{\partial \mathbf{B}}{\partial \mathbf{t}})\ \mathbf{d^3x} = \mathbf{0} \rightarrow \tfrac{\partial \mathbf{B}}{\partial \mathbf{t}} = \tfrac{\hat{\mathbf{A}}}{|\mathbf{A}|}(\mathbf{B} \cdot \nabla \mathbf{A_0}) \tag{12}$$



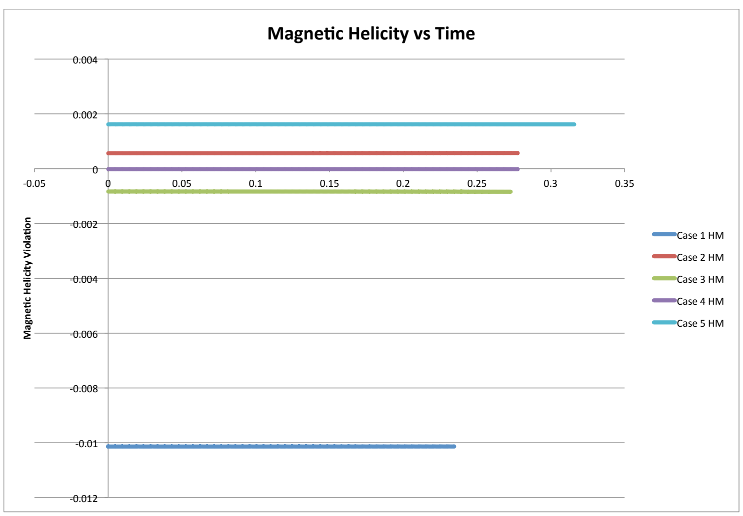


Fig. 1. *The four-divergences of Cross Helicity and Magnetic Helicity Densities do not approach zero. This appears true regardless of whether or not there is a mean magnetic field and/or mean angular velocity. This is evidence that these are not conserved quantities in Relativistic MHD Turbulence. Case 1 - no mean magnetic field or angular momentum, Case 2 - nonzero mean magnetic field, Case 3 - nonzero angular momentum, Case 4 - mean magnetic field and angular momentum aligned, Case 5 - mean magnetic field and angular momentum perpendicular.*

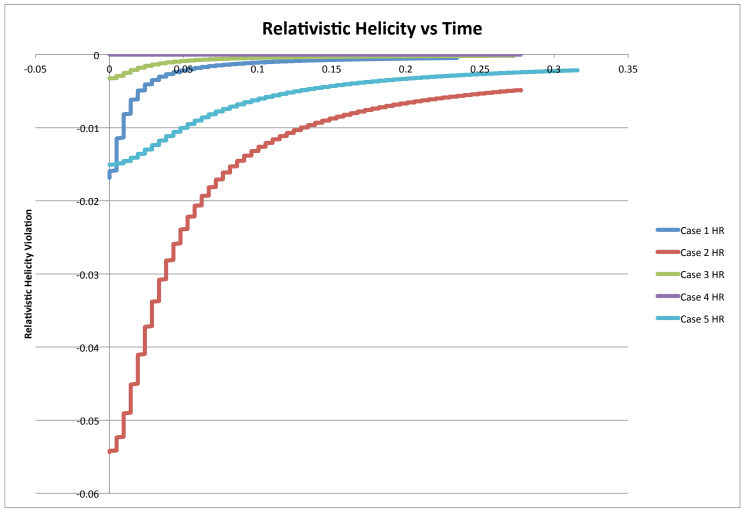


Fig. 2. *The four-divergence of Relativistic Helicity Density approaches zero regardless of whether or not there is a mean magnetic field and/or mean angular velocity. This is evidence that this is a conserved quantity in Relativistic MHD Turbulence. Case 1 - no mean magnetic field or angular momentum, Case 2 - nonzero mean magnetic field, Case 3 - nonzero angular momentum, Case 4 - mean magnetic field and angular momentum aligned, Case 5 - mean magnetic field and angular momentum perpendicular.*

Equation 11 and 12 imply that $\frac{\partial \mathbf{B}}{\partial t}$ is constrained by the relationship between the magnetic field, acceleration and gradient of potential. The magnetic field is on average perpendicular to any acceleration of the fluid. The Lorentz Force and therefore acceleration should also be zero because of the MHD condition. In addition, Equation 12 implies that a changing magnetic field can only result from a magnetic field that is aligned with a potential gradient. This does not occur on average unless there is a mean magnetic field. This could explain why Magnetic Helicity is only conserved when a mean magnetic field does not exist in the system. Because of this the non relativistic MHD equations will tend to suppress any net change in the magnetic field of the system unless a mean magnetic field exists initially. If the initial magnetic field is zero, these equations clearly imply that the net field should remain zero. Relativistic MHD systems don't seem to suffer from the same problem. While the absence of an initial magnetic field will cause the Cross and Magnetic Helicities to be zero, there is no reason why they should remain as such since they are independent of $\frac{\partial \mathbf{B}}{\partial t}$.

In a turbulent MHD system, a magnetic field could be generated by the electric field that results from variations in the concentrations of electric charges in the plasma field. Biermann showed that this is related to small variations in the density and temperature of the fluid [2,6].

$$\mathbf{E} = \frac{\nabla \mathbf{p}}{\mathbf{nq}} = -\frac{\nabla(\mathbf{n}\frac{\mathbf{T}}{\gamma})}{\mathbf{nq}}. \tag{13}$$

Here n is the number density of charges q. Such density variations can be linked to temperature fluctuations, T, that are expected in a turbulent fluid. These electric fields may then lead to the development of magnetic seed fields. The seed fields can then be amplified by the dynamics of the MHD system. In the case of non relativistic MHD, the potential amplification may by limited by the constraints of Cross and Magnetic Helicity conservation described above, resulting in a zero mean magnetic field. For relativistic systems, these constraints are nonexistent, allowing the magnetic fields to grown unconstrained. Because of this, we see that Relativistic MHD systems have a more natural tendency to develop seed magnetic fields while Non Relativistic MHD systems tend to suppress the development of seed magnetic fields.

References

1. S. Banerjee, “Compressible turbulence in space and astrophysical plasmas: Analytical approach and in-situ data analysis for the solar wind” *Diss. Universit Paris Sud-Paris* **XI** , 2014.
2. L. Bermann, Nature **A5**, 65 (1950)
3. D. Garrison and P. Nguyen, ”Invariants in Relativistic MHD Turbulence” J. of Modern Phys. **7**, 281-289 (2016)
4. Y. Kawazura, Z. Yoshida, and Y. Fukumoto, “Relabeling symmetry in relativistic fluids and plasmas” **47 (46)**, 465501. doi:10.1088/1751-8113/47/46/465501 (2014)
5. E. Marsch, and A. Mangeney, “Ideal MHD equations in terms of compressive Elssser variables” *J. Geophys. Res.* **92(A7)**, 73637367 (1987), **doi:10.1029/JA092iA07p07363**
6. E. Miller, *Magnetogenesis through Relativistic Velocity Shear*, Ph.D. thesis, Dartmouth College (2016)
7. J. V. Shebalin and D. Montgomery, “Turbulent magnetohydrodynamic density fluctuations” J. of Plasma Phys. **39**, 339-367 (1988)
8. J. V. Shebalin, “The Statistical Mechanics of Ideal Homogeneous Turbulence” NASA TP-2002-210783 (2002)
9. J. V. Shebalin, “Theory and simulation of real and ideal magnetohydrodynamic turbulence” Discrete & Continuous Dynamical Systems B **5**, 153-175 (2005)
10. J. V. Shebalin, “Broken symmetries and magnetic dynamos” Physics of Plasmas **14**, 102301 (2007)
11. J. V. Shebalin, “Broken Ergodicity in Magnetohydrodynamic Turbulence” Geophys. Astrophys. Fluid Dyn., Online DOI:10.1080/03091929.2011.589385 (2011)
12. J. V. Shebalin, “Broken ergodicity, magnetic helicity and the MHD dynamo” Geophys. Astrophys. Fluid Dyn., Online DOI:10.1080/03091929.2012.689299 (2012)
13. N. Yokoi, “Cross Helicity and related dynamo” Geophys. Astrophys. Fluid Dyn., Online DOI:10.1080/03091929.2012.754022 (2013)

14. Z. Yoshida, Y. Kawazura and T. Yokoyama, “Relativistic helicity and link in Minkowski space-time” J. of Math. Phys. **55 (4)**, 043101. doi:10.1063/1.4872236 (2014)