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# Control of a Simple Chaotic Flow having a Line Equilibrium by means of a Single Passive Controller

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**Abstract.** Recently, Jafari and Sprott (2013) have found nine simple chaotic flows with quadratic nonlinearities which include the unusual feature of having a line equilibrium. This study investigates the control of a simple chaotic system having a line equilibrium by means of the passive control method. Lyapunov function is used to realize that the passive controller ensures the global asymptotic stability of the system. In order to validate all the theoretical analyses, numerical simulations are demonstrated. Owing to the single passive controller, the chaotic flow stabilizes towards its line equilibrium in the state space effectively.
**Keywords:** Simple Chaotic Flow, Line Equilibrium, Passive Control, Chaos Control.

## 1 Introduction

Lorenz introduced the first chaotic attractor in 1963 [1]. It is an interesting nonlinear phenomenon, therefore chaos generation has received a great deal of attention from researchers. Rössler proposed a simple three-dimensional chaotic system in 1976 [2]. A double-scroll attractor was shown from Chua's circuit in 1984 [3]. Sprott focused on simpler chaotic systems in 1994 and uncovered 19 distinct chaotic flows which have either five terms and two nonlinearities or six terms and one nonlinearity [4]. In 1999, Chen and Ueta found a novel chaotic attractor called Chen chaotic system [5]. Lü et al. developed a new chaotic system, which represents the transition between the Lorenz and Chen systems in 2002 [6]. Then, Lü et al. proposed a generalized form of the Lorenz, Chen and Lü systems called unified chaotic system in 2002 [7]. In recent years, several new chaotic attractors have been revealed [8–10], and many more will be discovered on account of their potential applications especially in cryptology and secure communication [11, 12]. Recently, Jafari and Sprott have focused on the chaotic systems that have a line equilibrium and found nine simple chaotic flows [13]. They are three-dimensional continuous autonomous chaotic attractors and consist of six terms and two parameters [13].

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In addition to searching for new chaotic systems, chaos control has become an important task. Its goal is to eliminate the chaotic trajectories and stabilize towards an equilibrium point of the system. At first, it was believed that controlling chaos cannot be done because chaotic systems are very sensitive to initial conditions. However, Ott, Grebogi, and Yorke applied the control of a chaotic system successfully in 1990 [14]. Afterwards, the chaos control has also received extensive attention. Many effective control methods such as linear feedback control [15], nonlinear feedback control [16], adaptive control [17], sliding mode control [18], passive control [19-25], impulsive control [26], and backstepping design [27] have been proposed for the control of chaos. Among them, the passive control method has been gaining significance due to using only one state controller which provides considerable benefits in reducing the complexity and cost. In this method, the main idea is to keep the system internally stable by using a controller which renders the closed loop system passive upon the properties of the system. In recent years, the passive control method has been applied for the control of Lorenz [19], Chen [20], unified [21], Rabinovich [22], Rucklidge [23] and some other chaotic systems [24, 25].

According to the literature review, the control of a chaotic system having a line equilibrium has not been investigated. Motivated by the chaos control studies, in this paper, the control of a chaotic flow having a line equilibrium has been implemented with a single state passive controller. The rest of this paper is organized as follows. In Section 2, a chaotic system which has a line equilibrium is described. In Section 3, a single passive controller is designed for the control. In Section 4, numerical simulations are demonstrated to validate the control. Finally, concluding remarks are given in Section 5.

## 2 A Simple Chaotic Flow having a Line Equilibrium

Jafari and Sprott [13] were inspired by the structure of the conservative Sprott case A system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + yz, \\ \dot{z} &= 1 - y^2, \end{aligned} \tag{1}$$

and they searched for chaotic flows with a line equilibrium. They considered a general parametric form of Eq. (1) with quadratic nonlinearities of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= a_1 x + a_2 yz, \\ \dot{z} &= a_3 x + a_4 y + a_5 x^2 + a_6 y^2 + a_7 xy + a_8 xz + a_9 yz, \end{aligned} \tag{2}$$

where the system has a line equilibrium in (0, 0, $z$) with no other equilibria. An exhaustive computer search has been done and nine simple cases are found which have only six terms. The ninth chaotic flow is given in the following equation:

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= -ay - xz, \\ \dot{z} &= z - bz^2 + xy, \end{aligned} \tag{3}$$

where $a$ and $b$ are the system parameters [13, 28]. When they are selected as $a$ = 1.62 and $b$ = 0.2, the Lyapunov exponents become 0.0642, 0, and -0.6842 [13]. Thus, system (3) is chaotic for these parameters. Its Kaplan–Yorke dimension is 2.0939 [13].

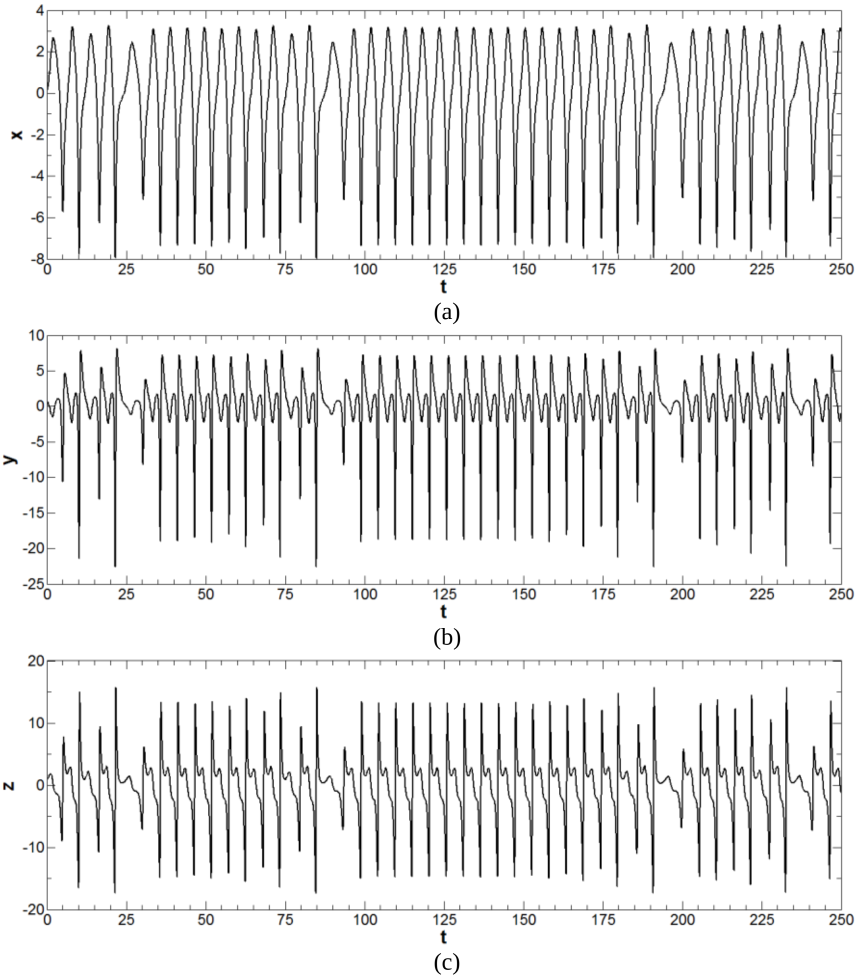



Fig. 1. Time series of the chaotic system having a line equilibrium for (a) $x$ signals, (b) $y$ signals, and (c) $z$ signals.

The equilibria of the chaotic system (3) can be found by getting $\dot{x} = 0$, $\dot{y} = 0$, and $\dot{z} = 0$ as follows:

$$\begin{aligned} &z = 0, \\ &-ay - xz = 0, \\ &z - bz^2 + xy = 0, \end{aligned} \tag{4}$$

Hence, the chaotic system (3) has a line equilibrium point: ($x$, 0, 0). Under the initial conditions $x(0) = 0$, $y(0) = 1$, and $z(0) = 0.8$, the time series, the 2D phase plots, and the 3D phase plane of chaotic system (3) are demonstrated in Fig. 1, Fig. 2, and Fig. 3, respectively.

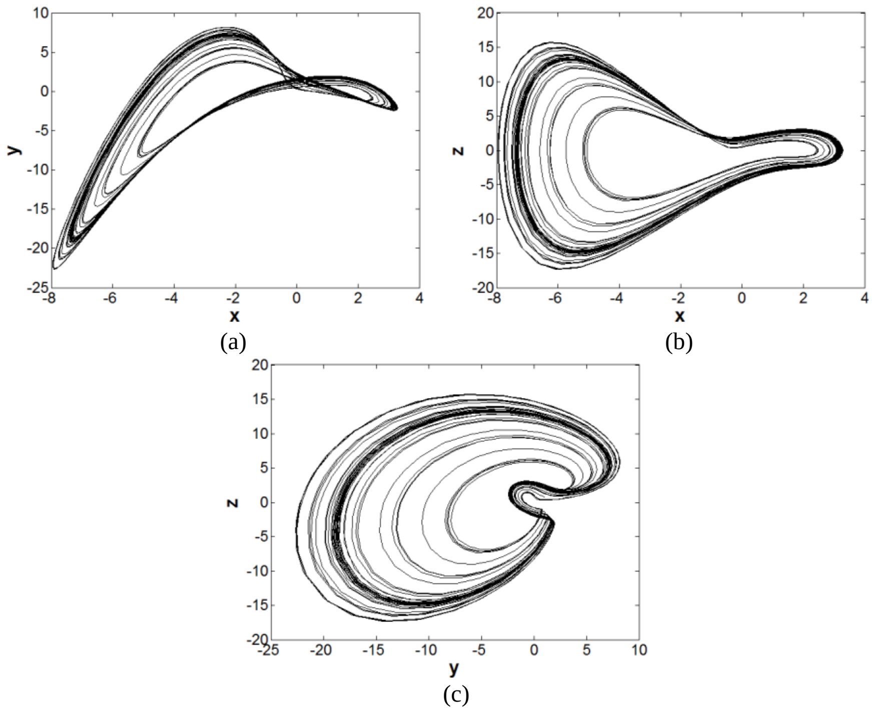



Fig. 2. Phase plots of the chaotic system having a line equilibrium for (a) $x$–$y$ phase plot, (b) $x$–$z$ phase plot, and (c) $y$–$z$ phase plot.

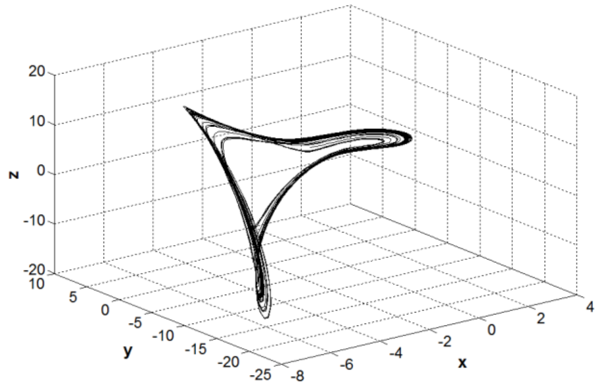



Fig. 3. Phase plane of the chaotic system having a line equilibrium.

## 3 Control with a Passive Controller

The passive control method is applied to system (3) in order to control the chaotic system having a line equilibrium to its equilibrium point. The controlled system is considered as follows:

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= -ay - xz, \\ \dot{z} &= z - bz^2 + xy + u, \end{aligned} \tag{5}$$

where $u$ is the passive controller to be designed. By assuming that the state variable $z$ is the output of the system and supposing that $z_1 = x$, $z_2 = y$, $Y = z$, and $Z = [z_1\ z_2]^T$, the system (5) can be denoted by normal form:

$$\begin{aligned} \dot{z}_1 &= Y, \\ \dot{z}_2 &= -az_2 - z_1 Y, \\ \dot{Y} &= Y - bY^2 + z_1 z_2 + u. \end{aligned} \tag{6}$$

The passive control theory has the following generalized form

$$\begin{aligned} \dot{Z} &= f_0(Z) + p(Z,Y)Y, \\ \dot{Y} &= b(Z,Y) + a(Z,Y)u, \end{aligned} \tag{7}$$

and according to system (6),

$$f_0(Z) = \begin{bmatrix} 0 \\ -az_2 \end{bmatrix}, \tag{8}$$

$$p(Z,Y) = \begin{bmatrix} 1 \\ -z_1 \end{bmatrix}, \tag{9}$$

$$b(Z,Y) = Y - bY^2 + z_1 z_2, \tag{10}$$

$$a(Z,Y) = 1. \tag{11}$$

The storage function is chosen as

$$V(Z,Y) = W(Z) + \frac{1}{2}(Y^2) \tag{12}$$

where

$$W(Z) = \frac{1}{2}(z_1{}^2 + z_2{}^2) \tag{13}$$

is the Lyapunov function of $f_0(Z)$ with $W(0) = 0$. According to the Eq. (8), the derivative of $W(Z)$ is

$$\dot{W}(Z) = \frac{\partial W(Z)}{\partial Z} f_0(Z) = [z_1 \quad z_2]\begin{bmatrix} 0 \\ -az_2 \end{bmatrix} = -az_2{}^2 \le 0. \tag{14}$$

Since $W(Z) \ge 0$ and $\dot{W}(Z) \le 0$, it can be concluded that $W(Z)$ is the Lyapunov function of $f_0(Z)$ and that $f_0(Z)$ is globally asymptotically stable [21].

According to the passivity definition, the controlled system can be equivalent to a passive system and globally asymptotically stabilized at its zero equilibrium by the following controller [19]:

$$u = a(Z,Y)^{-1}\left[-b^T(Z,Y) - \frac{\partial W(Z)}{\partial Z} p(Z,Y) - \alpha Y + v\right]. \tag{15}$$

From the Eq. (15), the passive control function is

$$u = -Y + bY^2 - z_1 z_2 - z_1 + z_1 z_2 - \alpha Y + v, \tag{16}$$

where $\alpha$ is a positive constant and $v$ is an external input signal. By taking back $z_1 = x$, $z_2 = y$, and $Y = z$ conversions, the passive controller $u$ becomes

$$u = -z + bz^2 - xy - x + xy - \alpha z + v. \tag{17}$$

Substituting the Eq. (17) into system (3) yields

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= -ay - xz, \\ \dot{z} &= -x + xy - \alpha z + v. \end{aligned} \tag{18}$$

The equivalent system (18) is a passive system of the chaotic system (3) which has a line equilibrium.

The passive controlled system can stabilize towards its any equilibrium point ( $\bar{x}, \bar{y}, \bar{z}$ ). Let $\dot{x} = 0,\ \dot{y} = 0,\ \dot{z} = 0$ and then system (18) yields

$$\begin{aligned} 0 &= \bar{z}, \\ 0 &= -a\bar{y} - \bar{x}\bar{z}, \\ 0 &= -\bar{x} + \overline{xy} - \alpha\bar{z} + v. \end{aligned} \tag{19}$$

This implies

$$\begin{aligned} \bar{z} &= 0, \\ \bar{y} &= 0, \\ v &= \bar{x}. \end{aligned} \tag{20}$$

The conditions in Eq. (20) maintain the global asymptotical stability of chaotic system (5) towards its $E(x, 0, 0)$ equilibrium point.

## 4 Numerical Simulations

The third-order Runge-Kutta method with variable time step is used in all numerical simulations of controlling the chaotic system having a line equilibrium. The same parameter values and initial conditions mentioned in Section 2 are considered to ensure the chaotic behaviour of the system. The controller is activated at $t$ = 50 in all simulations. The passive control gain is taken as $\alpha$ = 1. Simulation results for the control of this chaotic system towards (1, 0, 0), (0, 0, 0), and (-1, 0, 0) equilibrium points with a passive controller by setting $v$ = 1, $v$ = 0, and $v$ = -1 are shown in Fig. 4, Fig. 5, and Fig. 6, respectively.

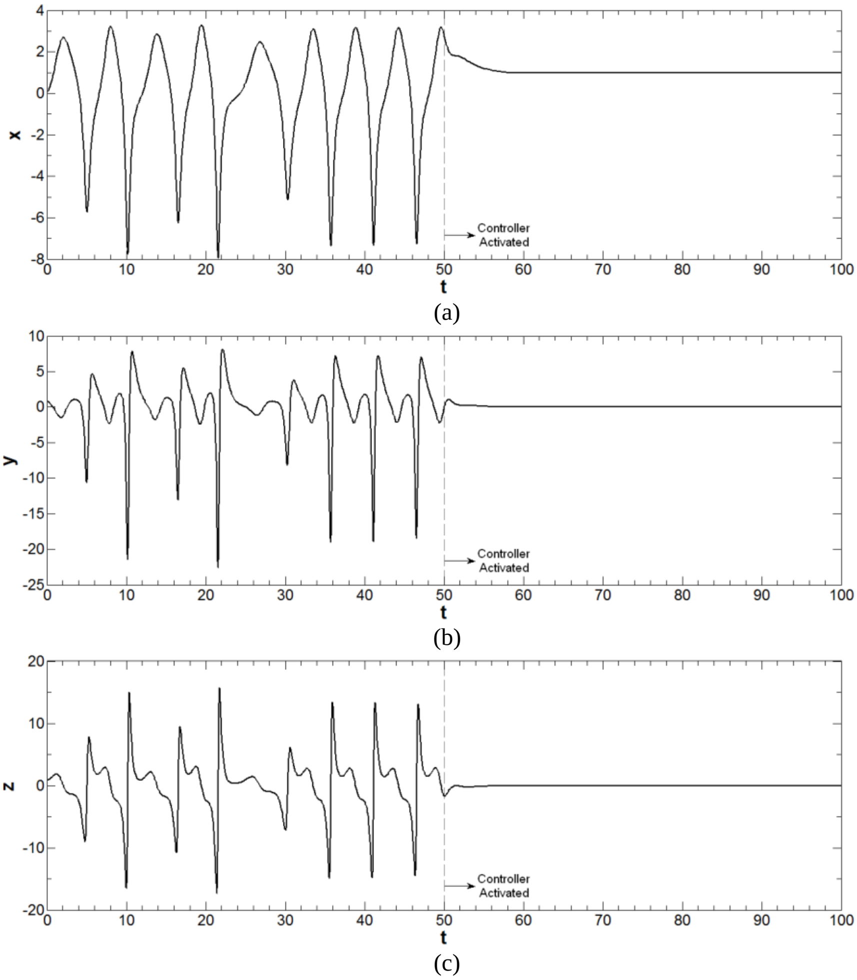
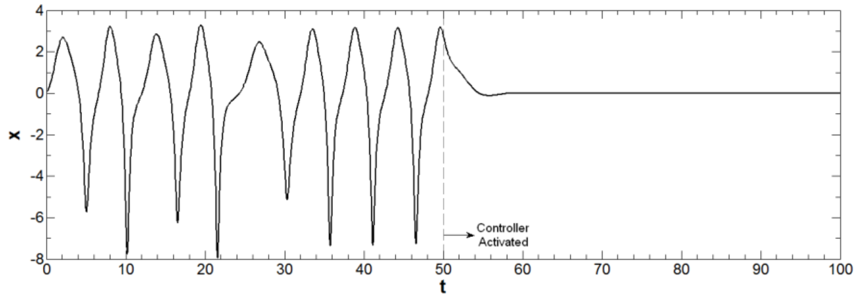
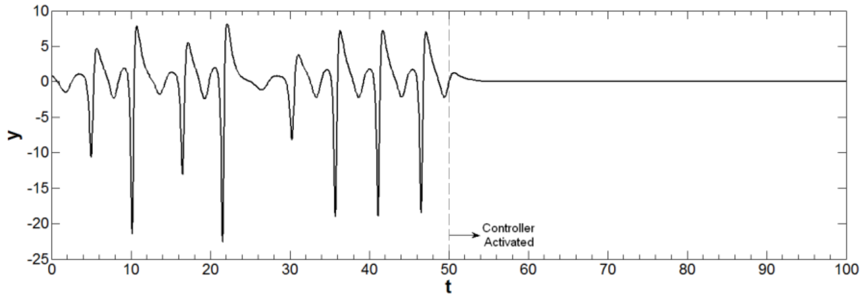


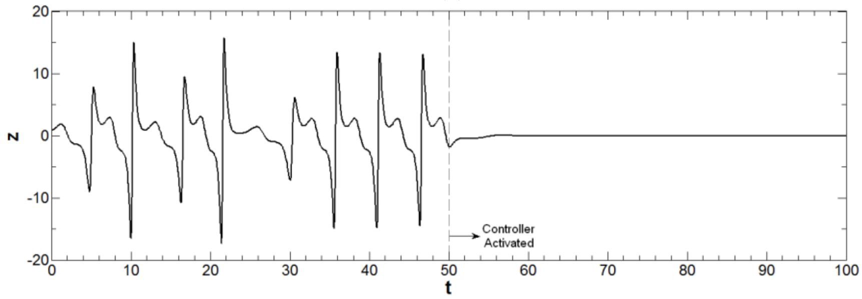

Fig. 4. Time responses of controlled chaotic system having a line equilibrium to (1, 0, 0) equilibrium point when the passive controller is activated at $t$ = 50 for (a) $x$ signals, (b) $y$ signals, and (c) $z$ signals.




(a)

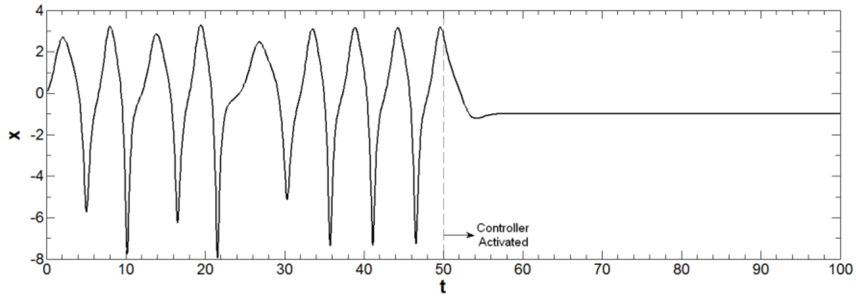



(b)

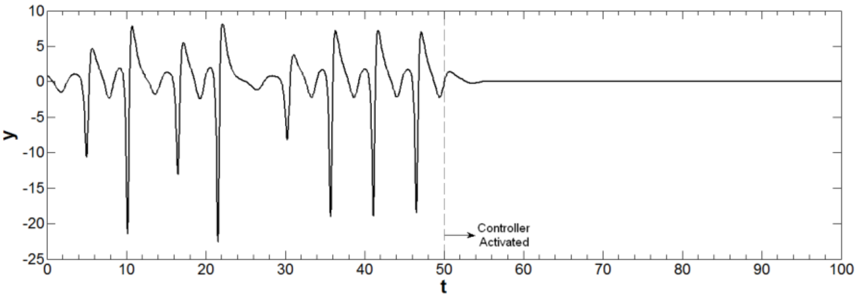



(c)

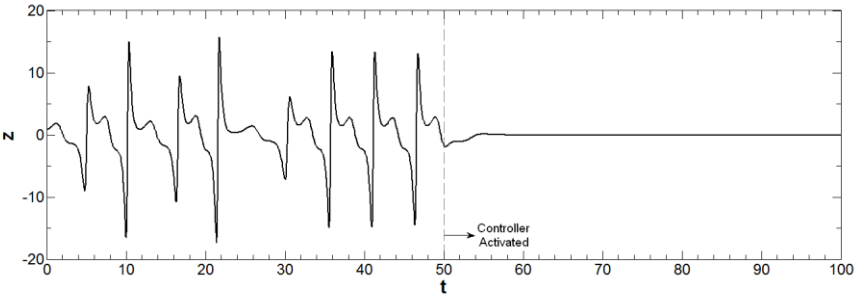
Fig. 5. Time responses of controlled chaotic system having a line equilibrium to (0, 0, 0) equilibrium point when the passive controller is activated at $t$ = 50 for (a) $x$ signals, (b) $y$ signals, and (c) $z$ signals.




(a)




(b)




(c)

Fig. 6. Time responses of controlled chaotic system having a line equilibrium to (-1, 0, 0) equilibrium point when the passive controller is activated at $t$ = 50 for (a) $x$ signals, (b) $y$ signals, and (c) $z$ signals.

As seen in Figs. 4–6, the outputs of chaotic system converge to the (1, 0, 0), (0, 0, 0), and (-1, 0, 0) equilibrium points after the passive controller is activated. Therefore, the simulation results validate all the theoretical analyses. As seen in Fig. 4, when the passive controller is activated at $t$ = 50, the control is provided at $t \geq 58$. Also, the control is observed after 8 time period in Fig. 5 and Fig. 6. Hence, the simulation results confirm the effectiveness of proposed passive control method.

## 5 Conclusions

In this paper, the control of a chaotic system having a line equilibrium is applied with a single state passive controller. The conditions of the asymptotic stability of the steady states of the controlled system are ensured with a Lyapunov function. Numerical simulations show that this three-dimensional continues time chaotic system can be controlled to its line equilibrium point in an appropriate amount of time with a passive controller. Hence, computer simulations have validated the effectiveness of passive control method in the control of the chaotic system having a line equilibrium.

## References

1. E.N. Lorenz. Deterministic nonperiodic flow. Journal of Atmospheric Sciences, 20(2), 130-141, 1963.
2. O.E. Rössler. An equation for continuous chaos. Physics Letters A, 57(5), 397-398 1976.
3. T. Matsumoto. A chaotic attractor from Chua's circuit. IEEE Transactions on Circuits and Systems, CAS-31(12), 1055-1058, 1984.
4. J.C. Sprott. Some simple chaotic flows. Physical Review E, 50(2), 647-650, 1994.
5. G.R. Chen and T. Ueta. Yet another chaotic attractor. International Journal of Bifurcation and Chaos, 9(7), 1465-1466, 1999.
6. J.H. Lü, G.R. Chen and S.C. Zhang. The compound structure of a new chaotic attractor. Chaos, Solitons & Fractals, 14(5), 669-672, 2002.
7. J.H. Lü, G.R. Chen, D.Z. Cheng and S. Celikovsky. Bridge the gap between the Lorenz system and the Chen system. International Journal of Bifurcation and Chaos, 12(12), 2917–2926, 2002.
8. X.Y. Zhou. A novel chaotic system and its circuit simulation. Acta Physica Sinica, 61(3), Article Number: 030504, 2012.
9. B.J. Zhang and H.X. Li. A new four-dimensional autonomous hyperchaotic system and the synchronization of different chaotic systems by using fast terminal sliding mode control. Mathematical Problems in Engineering, 2013, Article Number: 179428, 2013.
10. V.T. Pham, C. Volos, S. Jafari, Z.C. Wei and X. Wang. Constructing a novel no-equilibrium chaotic system. International Journal of Bifurcation and Chaos, 24(5), Article Number: 1450073, 2014.
11. X.J. Wu, C.X. Bai and H.B. Kan. A new color image cryptosystem via hyperchaos synchronization. Communications in Nonlinear Science and Numerical Simulation, 19(6), 1884-1897, 2014.
12. C.C. Cheng, Y.S. Lin and S.W. Wu. Design of adaptive sliding mode tracking controllers for chaotic synchronization and application to secure communications. Journal of the Franklin Institute - Engineering and Applied Mathematics, 349(8), 2626-2649, 2012.
13. S. Jafari and J.C. Sprott. Simple chaotic flows with a line equilibrium. Chaos, Solitons & Fractals, 57, 79-84, 2013.
14. E. Ott, C. Grebogi and J.A. Yorke. Controlling chaos. Physical Review Letters, 64(11), 1196-1199, 1990.

15. G. Gambino, M.C. Lombardo and M. Sammartino. Global linear feedback control for the generalized Lorenz system. Chaos, Solitons & Fractals, 29(4), 829-837, 2006.
16. W.N. Zhou, L. Pan, Z. Li and W.A. Halang. Non-linear feedback control of a novel chaotic system. International Journal of Control, Automation and Systems, 7(6), 939-944, 2009.
17. Y.M. Zeng and S.N. Singh. Adaptive control of chaos in Lorenz system. Dynamics and Control, 7(2), 143-154, 1997.
18. M.J. Jang, C.L. Chen and C.K. Chen. Sliding mode control of chaos in the cubic Chua's circuit system. International Journal of Bifurcation and Chaos, 12(6), 1437–1449, 2002.
19. W. Yu. Passive equivalence of chaos in Lorenz system. IEEE Transaction on Circuits and Systems-I: Fundamental Theory and Applications, 46(7), 876-878, 1999.
20. D. Qi, G. Zhao and Y. Song. Passive control of Chen chaotic system. In: Proceedings of the 5th World Congress on Intelligent Control and Automation, Hangzhou, China, 1284–1286, 15–19 June 2004.
21. X. Chen and C. Liu. Passive control on a unified chaotic system. Nonlinear Analysis: Real World Applications, 11, 683–687, 2010.
22. S. Emiroglu and Y. Uyaroglu. Control of Rabinovich chaotic system based on passive control. Scientific Research and Essays, 5(21), 3298–3305, 2010.
23. U.E. Kocamaz and Y. Uyaroglu. Controlling Rucklidge chaotic system with a single controller using linear feedback and passive control methods. Nonlinear Dynamics, 75(1-2), 63–72, 2014.
24. G.M. Mahmoud, E.E. Mahmoud and A.A. Arafa. Passive control of n-dimensional chaotic complex nonlinear systems. Journal of Vibration and Control, 19(7), 1061–1071, 2013.
25. D.Q. Wei and Y.H. Qin. Controlling chaos in single-machine-infinite bus power system by adaptive passive method. Fourth International Workshop on Chaos-Fractals Theories and Applications, IWCFTA, Hangzhou, China, 295–297, 2011.
26. X. He, C.D. Li, X.M. Pan and M. Peng. Impulsive control and Hopf bifurcation of a three-dimensional chaotic system. Journal of Vibration and Control, 20(9), 1361-1368, 2014.
27. J.H. Lu and S.C. Zhang. Controlling Chen's chaotic attractor using backstepping design based on parameters identification. Physics Letters A, 286(2-3), 148-152, 2001.
28. U.E. Kocamaz and Y. Uyaroglu. Comments on "Simple chaotic flows with a line equilibrium" [Chaos, solitons & fractals 57 (2013) 79-84]. Chaos, Solitons & Fractals, 77, 340, 2015.