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Cascade of Hopf bifurcations in coupled chaotic oscillators

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Abstract. We study exotic patterns appearing in a network of coupled Chen chaotic oscillators. Our network consists of two rings coupled through a 'buffer' cell, with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry group. Numerical simulations of the network reveal steady-states, rotating waves, and quasiperiodic and chaotic states. The different patterns arise through a sequence of Hopf bifurcations. The network architecture explains certain observed features, whereas the properties of the cells' internal dynamics, the Chen chaotic oscillator, may explain others.
Keywords: chaos, quasiperiodic states, symmetry, Hopf bifurcation.

1 Introduction

Stewart, Golubitsky and Pivato [16] and Golubitsky, Stewart and Török [10] have developed a theory concerning coupled dynamical systems, or coupled cell networks. A cell is a system of ordinary differential equations. Issues like synchronization plase-relations synchronized chaos, amongst others [5,8] [14,12] have been particularly focused.

General coupled cell networks may be characterized in two main groups in what concerns symmetry. One group consists of the coupled cells systems that possess some degree of symmetry, the other group gathers the coupled cells systems with no symmetry. The networks with exact symmetry are included in the first group.

The common representation of networks of coupled cells is done by directed graphs. The graphs' nodes correspond to individual cells and the edges to the couplings between them. A 'cell' means a nonlinear dynamical system of ordinary differential equations. In Figure 3, the cells are represented by circles and squares and the couplings between them by arrows. Distinct cells/arrows mean distinct dynamics/couplings.

In this paper we are interested in the dynamical features occurring in a coupled system of two unidirectional rings with $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry. In Section 2, we provide a review of the coupled cells networks formalism. In

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Section 3, we simulate the coupled cells systems associated to the networks of two coupled rings of cells in Fig. 1. In Section 4, we conclude this work and shed some light on future research directions.

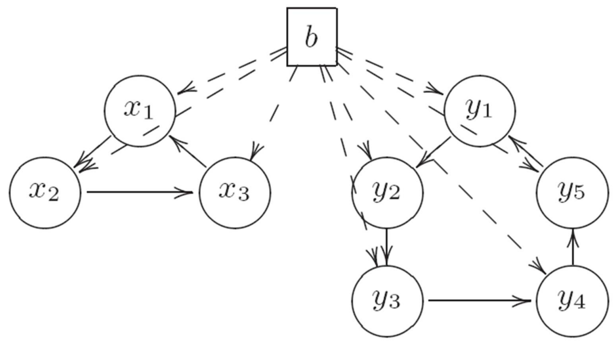

Fig. 1. Network of two coupled unidirectional rings, one with three cells and the other with five, connected through a buffer cell b. The network has $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry group.

2 Coupled cells and symmetry

A coupled cells system consists of a finite set of nodes (or cells) $\mathcal{C}$, and a finite set of edges $\mathcal{E}$. An equivalence relation on cells in $\mathcal{C}$ is defined, where the equivalence class of c is the type of cell c, an input set of cells $\mathcal{I}(c)$, that consists of cells whose edges have cell c as head. Moreover, an equivalence relation on the edges (or arrows), is also defined, where the equivalence class of e is the type of edge e, and it satisfies the condition that 'equivalent edges have equivalent tails and edges'. The last condition means that equivalent edges must have tails and edges of the same equivalence class.

For each cell c an internal phase space P_c is defined. The total phase space of the network is the product $P = \prod_{i=1}^{n} P_c$. The coordinates on P_c are denoted by x_c, the coordinates on P are thus $(x_1, x_2, \ldots, x_n)$. At time t, the system is at state $(x_1(t), x_2(t), \ldots, x_n(t))$.

A vector field f on P that is compatible with the network architecture is said to be *admissible* for that network, and satisfies two conditions: (1) the domain - each component f_i corresponding to cell c_i must be a function of the cells in the $\mathcal{I}(c)$; (2) the pull-back condition - the components f_i and f_j of cells c_i and c_j are identical, up to a suitable permutation of the relevant variables, if the two cells have isomorphic input cells [9].

A symmetry is a transformation of the phase space that sends solutions to solutions. It consists of the group of permutations of the cells (and arrows) that preserves the network structure (including cell-types and arrow-types) and its action on P is by permutation of cell coordinates. The network in Figure 1 is an example of a network with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry.

3 Numerical simulations

The coupled cells system, associated with the network depicted in Fig. 1, is simulated. We use XPPAUT [7] and MATLAB [18] to compute numerically the relevant states. We consider the Chen oscillator as the phase space for each cell of the two rings and an unidimensional phase space for the 'buffer cell'. The total phase space is thus twenty-fifth dimensional. The dynamics of a singular ring cell is given by [6,13]:

$$\begin{aligned} \dot{u} &= a(v-u) \\ \dot{v} &= (c-a)u - uw + cv \\ \dot{w} &= uv - bw \end{aligned} \tag{1}$$

where $a = 35,\ b = 3,\ c$ are real parameters.

The unidimensional dynamics of the 'buffer cell' is given by [8,3]:

$$f(u) = \mu u - \frac{1}{10}u^2 - u^3 \tag{2}$$

where $\mu = -1.0$ is a real parameter.

The coupled cells system of equations associated to the network in Fig. 1 is given by:

$$\begin{aligned} \dot{x_j} &= g(x_j) + k\,(x_j - x_{j+1}) + d\,b \quad j = 1,\ldots,3 \\ \dot{b} &= f(b) \\ \dot{y_j} &= g(y_j) + k\,(y_j - y_{j+1}) + d\,b \quad j = 1,\ldots,5 \end{aligned} \tag{3}$$

where $g(u)$ represents the dynamics of each Chen oscillator, $k = -5.0,\ d = 0.2$, and the indexing assumes $x_4 \equiv x_1$ and $y_6 \equiv y_1$. We assume that the coupling between all cells is linear and is done only in the first variable of each Chen oscillator.

We vary parameter $c \in [15, 22]$, going from lower to higher values, and start from a steady state of the whole system.

In Figure 2, we plot (top) the time series solution of the coupled cell system (3) and (bottom) we represent the phase plane of oscillator y_1 of the 5-ring. The solution is a rotating wave state in the 5-ring, obtained by a Hopf bifurcation (HB1), from the trivial equilibrium branch. Cells in the 3-ring are at equilibrium. These solutions can be explained using the Equivariant Hopf Theorem for coupled cell systems in the symmetric case [11]. The bifurcation has occurred in the 5-ring.

We increase c again, and another Hopf bifurcation occurs (HB2). In Figure 3, we plot (top) the time series solution of the coupled cell system (3), (center) we show the phase planes for the oscillator x_1 of the 3-ring (center, left) and of the oscillator y_1 of the 5-ring (center, right). The solution is a rotating $\mathbf{Z}_3$ wave in the 3-ring and a rotating $\mathbf{Z}_5$ wave in the other ring. The full solution is quasiperiodic (Fig. 3, bottom).

Figure 4 shows the time series further away from the tertiary Hopf bifurcation (HB3) in the coupled cell system (3). Unlike the previous cases (Figures 2-3), the amplitude of the solution is higher and the wave form is qualitatively different, displaying typical relaxation oscillatory features. Relaxation oscillations are

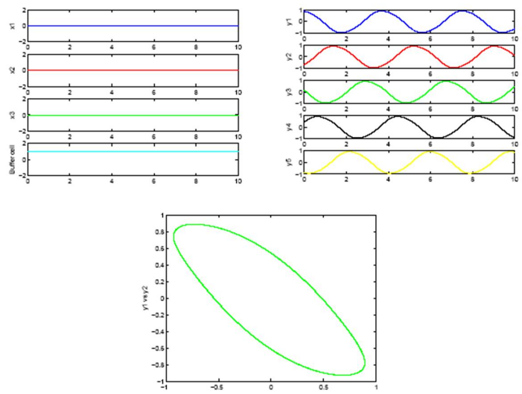

Fig. 2. Simulation of the coupled system (3) with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry. Time series from the nine cells after the first Hopf bifurcation point (HB1). (Top, left) Cells in the 3-ring are at equilibrium and cells in the 5-ring display a $\mathbf{Z}_5$ rotating wave (top, right). (Bottom) Phase plane of oscillator y_1 of the 5-ring.

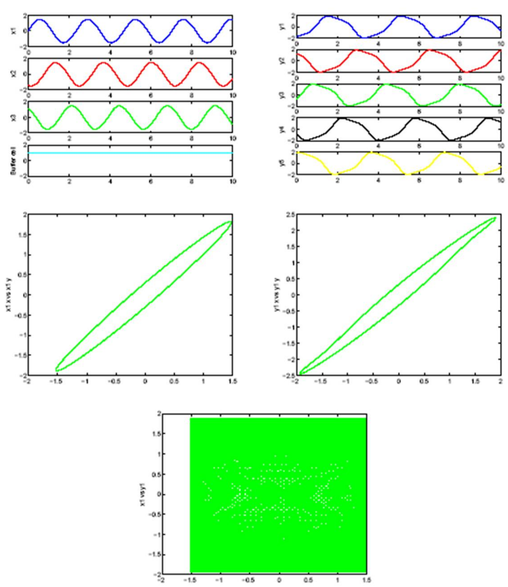

Fig. 3. Simulation of the coupled system (3) with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry, after the second Hopf bifurcation point (HB2). The cells in the 3-ring exhibit a rotating $\mathbf{Z}_3$ wave (top, left), and the cells in the other ring show a rotating $\mathbf{Z}_5$ wave (top, right). Phase planes of the oscillator x_1 (center, left) and of the oscillator y_1 (center, right). Cell x_1 vs cell y_1 (bottom). For more information see text.

solutions characterized by long periods of quasi-static behaviour interspersed with short periods of rapid transition. These solutions are studied in the context of the canard phenomenon [17] in fast-slow systems.

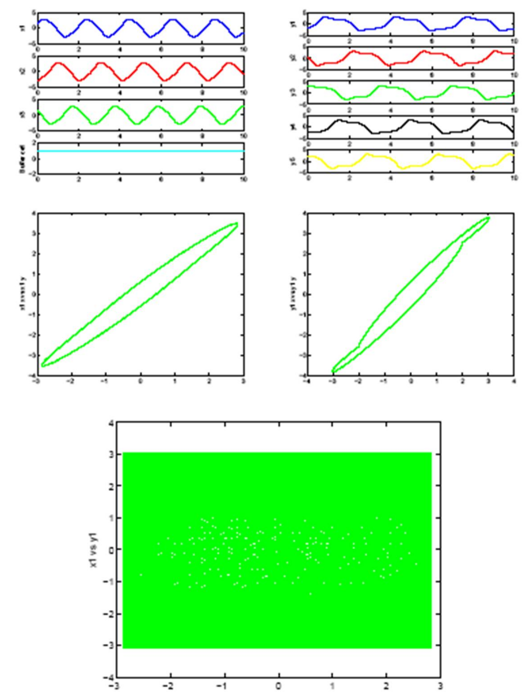

Fig. 4. Simulation of the coupled system (3) with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry, after the third Hopf bifurcation point (HB3). The cells in the 3-ring exhibit a $\mathbf{Z}_3$ rotating wave (top, left), whereas cells in the other ring depict a relaxation oscillation (top, right). Phase planes of the oscillator x_1 (center, left) and of the oscillator y_1 (center, right). Cell x_1 vs cell y_1 (bottom). For more information see text.

Further away of this third Hopf bifurcation point, ‘unusual’ and complex behaviors start to appear. In Fig. 5, the cells in the 3-ring appear to show a quasiperiodic motion and the cells in the 5-ring seem to depict a chaotic state. The full solution is quasiperiodic or chaotic (see Figure 5, bottom).

Thus, from the numerical results, we conclude that there is a richness of dynamic features produced by the network of two coupled rings with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry. The dynamical behavior is much more complex than the one found in [3,4,2,15], for the same network of two coupled rings with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry, but with simpler internal dynamics for each cell. This suggests that the network structure is important for these patterns to be observed but it seems not to be able to explain them fully.

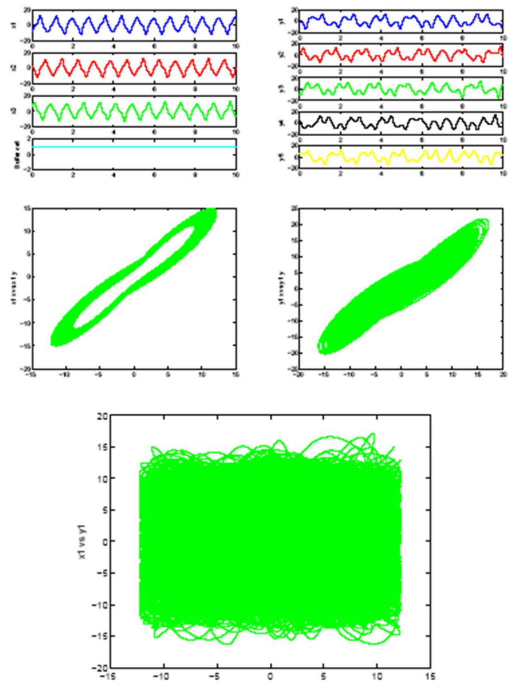

Fig. 5. Simulation of the coupled system (3) with $\mathbf{Z}_3 \times \mathbf{Z}_5$ symmetry, further away of the third Hopf bifurcation point (HB3). The cells in the 3-ring exhibit a quasiperiodic motion (top, left), whereas the cells in the other ring show a chaotic state (top, right). Phase planes of the oscillator x_1 (center, left) and of the oscillator y_1 (center, right). Cell x_1 vs cell y_1 (bottom). For more information, see text.

4 Conclusions

In this paper we study the dynamical behavior of a network consisting of two rings of chaotic Chen oscillators, that admit $\mathbf{Z}_3 \times \mathbf{Z}_5$ exact symmetry group. We find interesting patterns, some of them explained by local bifurcation theorems and some probably by the properties of the cells' internal dynamics, in this case, the Chen chaotic attractor. More work is needed to explain thoroughly these features.

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