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# A vision of the Brownian motion models useful in random systems analysis

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**Abstract.** There are many cases when we refer to *chaos* and *chaotic and complex systems* to describe the comportment of some natural phenomena. In this context, we shall discuss, in this paper, some aspects which appear in the study of various systems. Firstly, we shall refer to the *Brownian transition probabilities* in connection with the conditions assumed on the transition probabilities; and then the *standard Brownian motion* is considered in connection with the "passage times" which are the most important Markov times.
**Keywords:** stochastic differential equations, stochastic calculus, Markov processes, Markov property, Brownian motion..

## 1 Introduction

Starting from the observation that many a time we refer to *chaos* and *chaotic and complex systems* to describe the comportment of some natural pheno-mena, it is very useful, from a mathematical point of view, to talk about a passing *from chaotic and complex systems to Brownian motion.* In this way we can refer to the *Brownian motion* which is a more realistic model of such phenomena.

Its fascinating properties and its far-reaching extension of the simplest normal limit theorems to functional limit distributions acted, and continue to act, as a catalyst in random systems analysis. As some authors remarks too, the Brownian motion reflects a perfection that seems closer to a law of nature than to a human invention.

In Physics, the ceaseless and extremely erratic dance of microscopic particles suspended in a liquid or gas, is called *Brownian motion.* It was systematically investigated by Robert Brown (1828, 1829), an English botanist, from movement of grains of pollen in water to a drop of water in oil. He was not the first to mention this phenomenon and had many predecessors but Brown's investigation brought it to the attention of the scientific community.

Brownian motion was frequently explained as due to the fact that particles were alive. It is only in 1905 that kinetic molecular theory led Einstein to

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the first mathematical model of Brownian motion. He began by deriving its possible existence and then only learned that it had been observed.

A completely different origin of mathematical Brownian motion is a game theoretic model for fluctuations of stock prices due to L. Bachélier from 1900. In his doctoral thesis, *Théorie de la spéculation*, Ann. Sci. École Norm. Sup., 17, 1900, 21-86, he hinted that it could apply to physical Brownian motion. Therein, and in his subsequent works, he used the heat equation and, proceeding by analogy with *heat propagation* he found, albeit formally, distributions of various functionals of mathematical Brownian motion. Heat equations and related parabolic type equations were used rigorously by Kolmogorov, Petrovsky, Khintchine.

But Bachélier was unable to obtain a clear picture of the Brownian motion and his ideas were unappreciated at the time. This because a precise definition of the Brownian motion involves a measure on the path space, and it was not until 1908-1909 when É. Borel published his classical me-moir on Bernoulli trials: *Les probabilités dénombrables et leurs applications arithmétique*, Rend. Circ. Math. Palermo 27, 247-271, 1909. But as soon as the ideas of Borel, Lebesgue and Daniell appeared, it was possible to put the Brownian motion on a firm mathematical foundation. And this was achived in 1923 by N. Wiener, in his work: *Differential space*, J. Math. Phys. 2, 131-174, 1923.

Many researchers were fascinated by the great beauty of the theory of Brownian motion and many results have been obtained in the last decades. As for example, among other things, in *Diffusion processes and their sample paths* by K. Itô and H.P. McKean, Jr., in *Theory and applications of stochastic differential equations* by Z. Schuss, or in *Stochastic approximation* by M.T. Wasan as in *Stochastic calculus and its applications to some problems in finance* by J.M. Steele.

In fact, the construction of the Brownian motion as a limit of a rescaled random walk can be generalized to a class of Markov chains. In this context, at the $4^{th}$ CMSIM international Conference, we discussed some aspects relating to the approximation in the study of Markov processes and Brownian motion; also, we referred to the Markov property from a perspective of K. Itô.

Itô's integral and other details and related topics in stochastic calculus and applications in random systems analysis are developed among other by B. Øksendal and A. Sulem, J.M. Steele, P. Malliavin, P. Protter, D.W. Stroock.

## 2 In short about transition probabilities

In some previous papers we have dicussed on Markov processes in a vision of K. Itô and we have emphasized the aspects regarding to the Markov pro-perty. In this context a fundamental concept is that of *transition probabilities* which will be considered, in short, below.

Let $S$ be a *state space* and consider a particle which moves in $S$. Also, suppose that the particle starting at $x$ at the present moment will move into the set $A \subset S$ with probability $p_t(x, A)$ after $t$ units of time, "irrespectively of its past motion", that is to say, this motion is considered to have a *Markovian character*.

The *transition probabilities* of this motion are $\{p_t(x,A)\}_{t,x,A}$ and is considered that the time parameter $t \in T = [0, +\infty)$.

The state space $S$ is assumed to be a *compact Hausdorff space with a countable open base*, so that it is homeomorphic with a compact separable metric space by the Urysohn's metrization theorem. The $\sigma$-field generated by the open space (the topological $\sigma$-field on $S$) is denoted by $K(S)$. Therefore, a *Borel set* is a set in $K(S)$.

It will be assumed that the transition probabilities $\{p_t(x,A)\}_{t\in T, x\in S, A\in K(S)}$ satisfy the following conditions:

**(1)** for $t$ and $A$ fixed,
 a) the transition probabilities are Borel measurable in $x$;
 b) $p_t(x,A)$ is a probability measure in $A$;

**(2)** $p_0(x,A) = \delta_x(A)$ (i.e. the $\delta$-measure concentrated at $x$);

**(3)** $p_t(x,\cdot) \overset{weak}{\longrightarrow} p_t(x_0,\cdot)$ as $x \to x_0$ for any $t$ fixed, that is

$$\lim_{x\to x_0} \int f(y) p_t(x,dy) = \int f(y) p_t(x_0,dy)$$

for all continuous functions $f$ on $S$;

**(4)** $p_t(x,U(x)) \longrightarrow 1$ as $t \searrow 0$, for any neighborhood $U(x)$ of $x$;

**(5)** the Chapman-Kolmogorov equation holds:

$$p_{s+t}(x,A) = \int_S p_t(x,dy) p_s(y,A).$$

It is interesting to observe that, if we define,

$$p_t(x,dy) = \frac{1}{t\sqrt{2\pi}}\, e^{-\frac{(y-x)^2}{2t^2}}\, dy \quad \text{in } R$$
$$p_t(\infty,A) = \delta_\infty A.$$

then, the conditions (1) – (5) above are satisfied for *Brownian transition probabilities*.

Let now consider $C = C(S)$ to be the space of all continuous functions (it is a separable Banach space with the supremum norm). Then, the *transition operators* can be defined in a similar manner.

**Definition 21** *The operators $p_t$, defined by*

$$(p_t f)(x) = \int_S p_t(x,dy) f(y), \quad f \in C$$

*are called "transition operators".*

And the conditions for the transition probabilities can be adapted to the transition operators.

Now the Markov process can be defined as follows

**Definition 22** *A Markov process is a system of stochastic processes*

$$\{X_t(\omega), t \in T, \omega \in (\Omega, K, P_a)\}_{a \in S},$$

*that is for each $a \in S$, $\{X_t\}_{t \in S}$ is a stochastic process defined on the probability space $(\Omega, K, P_a)$.*

The transition probabilities of a Markov process will be denoted by $\{p(t, a, B)\}$. Now let us denote by $\{H_t\}$ the transition semigroup and let $R_\alpha$ be the resolvent operator of $\{H_t\}$.

The next results shows that $p(t, a, B)$, $H_t$ and $R_\alpha$ can be expressed in terms of the process as follows:

**Theorem 21** *Let $f$ be a function in $C(S)$. Then*

1. $p(t, a, B) = P_a(X_t \in B)$.
2. *For* $E_a(\cdot) = \int_\Omega \cdot P_a(d\omega)$ *one has* $H_t f(a) = E_a(f(X_t))$.
3. $R_\alpha f(a) = E_a\left(\int_0^\infty e^{-\alpha t} f(X_t) dt\right)$.

*Proof.* One can observe that *1.* and *2.* follow immediately.

To prove *3.*, we will use the following equality:

$$R_\alpha f(a) = \int_0^\infty e^{-\alpha t} H_t f(a) dt = \int_0^\infty e^{-\alpha t} E_a(f(H_t)) dt.$$

Since $f(X_t(\omega))$ is right continuous in $t$ for $\omega$ fixed, and measurable in $\omega$ for $t$ fixed, it is therefore measurable in the pair $(t, \omega)$. Thus, we can use Fubini's theorem and therefore we obtain

$$R_\alpha f(a) = E_a\left(\int_0^\infty e^{-\alpha t} f(X_t) dt\right),$$

which proves *3.*

## 3 Elements of stochastic differential equations

To describe the motion of a particle driven by a *white noise* type of force (due to the collision with the smaller molecules of the fluid) the Langevin equation

$$\frac{d\nu(t)}{dt} = -\beta\nu(t) + \mathbf{f}(t) \tag{1}$$

is used, where $\mathbf{f}(t)$ is the white noise term. Its solution is the following

$$y(t) = y_0 e^{-\beta t} + e^{-\beta t} \int_0^t e^{-\beta t} \mathbf{f}(s) ds. \tag{2}$$

If we denote by $\mathbf{w}(t)$ the Brownian motion, then it is given by

$$\mathbf{w}(t) = \frac{1}{q} \int_0^t \mathbf{f}(s) ds, \tag{3}$$

so that $\mathbf{f}(s) = \dfrac{qd\mathbf{w}(s)}{ds}$. But $\mathbf{w}(t)$ is nowhere differentiable, such that $\mathbf{f}(s)$ is not a function. Therefore, the solution (2), of Langevin's equation, is not a well-defined function. This difficulty can be overcome, in the simple case, as follows. Integrating (2) by parts, and using (3), it results

$$y(t) = y_0 e^{-\beta t} + q\mathbf{w}(t) - \beta q \int_0^t e^{-\beta(t-s)}\mathbf{w}(s)ds. \tag{4}$$

But all functions in (4) are well defined and continuous, such that the solution (3) can be interpreted by giving it the meaning of (4). Now, such a procedure can be generalized in the following way. Let us given two functions $f(t)$ and $g(t)$ that are considered to be defined for $a \leq t \leq b$. For any partition $P : a \leq t_0 < t_1 < \cdots < t_n$, we denote

$$S_P = \sum_{i=1}^{n} f(\xi_i)[g(t_i) - g(t_{i-1})],$$

where $t_{i-1} \leq \xi_i \leq t_i$. If a limit exists

$$\lim_{|P|\to 0} S_P = I$$

where $|P| = \max_{1\leq i\leq n}(t_i - t_{i-1})$, then it is said that $I$ is the *Stieltjes integral* of $f(t)$ with respect to $g(t)$. It is denoted

$$I = \int_a^b f(t)dg(t).$$

Now the stochastic differential equation

$$\begin{aligned} &dx(t = a(x(t), t)dt + b(x(t), t)dw(t) \\ &x(0) = x_0 \end{aligned} \tag{5}$$

is defined by the Itô integral equation

$$x(t) = x_0 + \int_0^t a(x(s), s)ds + \int_0^t b(x(s), s)dw(s). \tag{6}$$

The simplest example of a stochastic differential equation is the following equation

$$\begin{aligned} &dx(t) = a(t)dt + b(t)dw(t) \\ &x(0) = x_0 \end{aligned} \tag{7}$$

which has the solution

$$x(t) = x_0 + \int_0^t a(s)ds + \int_0^t b(s)dw(s).$$

The *transition probability density* of $x(t)$ is a function $p(x,s;y,t)$ satisfying the condition

$$P(x(t)\in A\,|\,x(s)=x)=\int_A p(x,s;y,t)dy$$

for $t>s$ where $A$ is any set in $R$. It is supposed that $a(t)$ and $b(t)$ are deterministic functions. The stochastic integral

$$\chi(t)=\int_0^t b(s)dw(s)$$

is a limit of linear combinations of independent normal variables

$$\sum_i b(t_i)[w(t_{i+1})-w(t_i)].$$

Thus, the integral is also a normal variable. But, then

$$\chi(t)=x(t)-x_0-\int_0^t a(s)ds$$

is a normal variable, and therefore

$$p(x,s;y,t)=\frac{1}{\sqrt{2\pi\sigma}}\,e^{-\frac{(y-m)^2}{2\sigma}}$$

where

$$m=E(x(t)\,|\,x(s)=x).$$

Now

$$E(x(t)\,|\,x(s)=x)=x+\int_s^t a(u)du$$

is the expectation of the stochastic integral vanishes.

The variance is given by the relation

$$\sigma=Var\,x(t)=E\left[\int_s^t b(u)dw(u)\right]^2=\int_s^t b^2(u)du.$$

Thus, $p(x,s;y,t)$ is given by the following equation

$$p(x,s;y,t)=\left[2\pi\int_s^t b^2(u)du\right]^{-\frac{1}{2}}\cdot e^{-\frac{\left(y-x-\int_s^t a(u)du\right)^2}{2\int_s^t b^2(u)du}}.$$

For proofs and other aspects see [3], [8], [13], [23], [20].

## 4 From chaotic motion to Brownian motion

In our days the Brownian motion is of ever increasing importance not only in Probability theory but also in classical analysis and its applications.

Frequently, Brownian motion was explained as due to the fact that particles were alive. Today we know that this motion is due to the bombardament of the particles by the molecules of the medium. In a liquid, under normal conditions, the order of magnitude of the number of these impacts is of 1020 per second.

Let us imagine a chaotic motion of a particle of colloidal size immersed in a fluid. Such a chaotic motion of a particle is called, usually, *Brownian motion* and the particle which performs such a motion is referred to as a *Brownian particle*. Such a chaotic perpetual motion of a Brownian particle is the result of the collisions of particle with the molecules of the fluid in which there is.

But this particle is much bigger and also heavier than the molecules of the fluid which it collide, and then each collision has a negligible effect, while the superposition of many small interactions will produce an observable effect.

On the other hand, for a Brownian particle such molecular collisions appear in a very rapid succession, their number being enormous. For a so high frequency, evidently, the small changes in the particle's path, caused by each single impact, are too fine to be observable. For this reason the exact path of the particle can be described only by statistical methods.

Thus, the influence of the fluid on the motion of a Brownian particle can be described by the combination of two forces in the following way:

**1.** The considered particle is much larger than the particle of the fluid so that the cumulated effect of the interaction between the Brownian particle and the fluid may be taken as having a hydrodynamical character. Thus, the first of the forces acting on the Brownian particle may be considered to be the forces of *dynamical friction*. It is known that the frictional force exerted by the fluid on a small sphere immersed in it is determined from the Stockes's law: *the drag force per unit mass acting on a spherical particle of radius a is given by* $-\beta\mathbf{v}$, *with* $\beta = \dfrac{6\pi a\eta}{m}$, where $m$ is the mass of the particle, $\eta$ is the coefficient of dynamical viscosity of the fluid, and $\mathbf{v}$ is the velocity of particle.

**2.** The other force acting on the Brownian particle is caused by the individual collisions with the particles of the fluid in which there is. This force produces instantaneous changes in the acceleration of the particle. Furthermore, this force is *random both in direction and in magnitude*, and one can say that it is a *fluctuating force*. It will be denoted by $\mathbf{f(t)}$. For $\mathbf{f(t)}$ the following assumptions are made:

*a*) The function $\mathbf{f(t)}$ is statistically independent of $\mathbf{v}(t)$.
*b*) $\mathbf{f(t)}$ has variations much more frequent than the variations in $\mathbf{v}(t)$.
*c*) $\mathbf{f(t)}$ has the average equal to zero.

In these conditions, the Newton's equations of motion are given by the following stochastic differential equation

$$\frac{dbfv(t)}{dt} = -\beta\mathbf{v}(t) + \mathbf{f}(t) \tag{8}$$

which is called the *Langevin's equation*.

From the Langevin's equation, the statistical properties of the function $\mathbf{f}(\mathbf{t})$ can be obtained if its solution will be in correspondence with known physical laws. One can observe that the solution of (8) determines the *transition probability density* (in brief *the transition density*) $\rho(\mathbf{v}, t\mathbf{v}_0)$ of the random process $\mathbf{v}(t)$, which verifies the equation

$$P(\mathbf{v}(t) \in A) \,|\, \mathbf{v}(0) = \mathbf{v}_0) = \int_A \rho(\mathbf{v}, t, \mathbf{v}_0) d\mathbf{v}. \tag{9}$$

We do not insist on these aspects, our purpose has been to introduce the concept of transition density.

Now following K. Itô ([7], [5]) we shall refer shortly to the *k-dimensional Brownian motion* and emphasize some of its results.

But, firstly, we shall remind some aspects ragarding to the *3-dimensional Brownian motion* discussed at the $6^{th}$ CMSIM international Conference.

It is not difficult to observe that a definition of a *Markov process* as in Definition 22 not correspond to many processes that are of a real interest. For this reason it is useful to obtain an extension of this notion (such an extended notion has been proposed by K. Itô).

Let $E$ be a separable Banach space with real coefficients and norm $||\cdot||$ and let also $L(E, E)$ be the space of all bounded linear operators $E \longrightarrow E$. It can be observed that $L(E, E)$ is a linear space.

**Definition 41** *The collection of stochastic processes*

$$X = \{X_t(\omega) \equiv \omega(t) \in S,\, t \in T,\, \omega \in (\Omega, K, P_a)\}_{a \in S}$$

*is called a "Markov process" if the following conditions are satisfied:*

**1)** *the "state space" $S$ is a complete separable metric space and $K(S)$ is a topological $\sigma$-algebra on $S$;*
**2)** *the "time internal" $T = [0, \infty)$;*
**3)** *the "space of paths" $\Omega$ is the space of all right continuous functions $T \longrightarrow S$ and $K$ is the $\sigma$-algebra $K[X_t : t \in T]$ on $\Omega$;*
**4)** *the probability law of the path starting at a, $P_a(H)$, is a probability measure on $(\Omega, K)$ for every $a \in S$ which satisfy the following conditions:*
*4a)* $P_a(H)$ *is $K(S)$-measurable in a for every $H \in K$;*
*4b)* $P_a(X_0 = a) = 1$;
*4c)* $P_a(X_{t_1} \in E_1, \cdots, X_{t_n} \in E_n) =$
$$\int \cdots \int_{a_i \in E_i} P_a(X_{t_1} \in da_1) P_{a_1}(X_{t_2 - t_1} \in da_2) \ldots$$
$$\ldots P_{a_{n-1}}(X_{t_n - t_{n-1}} \in da_n) \qquad \textit{for} \quad 0 < t_1 < t_2 < \ldots < t_n.$$

According to Definition 41, $X$ will be referred as a *Markov process in the generalized sense.*

Now let $X$ be a Markov process in a generalized sense and let us denote by $\mathbf{B}$(S) the space of all bounded real $K(S)$-measurable functions. Also let us consider a function $f \in \mathbf{B}(S)$.

It is supposed that

$$E_a\left(\int\limits_0^\infty |f(X_t)|dt\right) \tag{10}$$

is bounded in $a$. Therefore

$$Uf(a) = E_a\left(\int\limits_0^\infty f(X_t)dt\right) \tag{11}$$

is well-defined and is a bounded $K(S)$-measurable function of $a \in S$.

The $Uf$ is called *the potential* of $f$ with respect to $X$. Having in view that $Uf = \lim_{\alpha\downarrow 0} R_\alpha f$, it is reasonable to write $R_0$ instead of $U$. Based on this fact, $R_\alpha f$ will be called *the potential of order* $\alpha$ of $f$.

*Remark 1.* It is useful to retain that $R_\alpha f \in \mathbf{B}(S)$ for $\alpha > 0$; and generally $f \in \mathbf{B}(S)$ while $R_0 f(= Uf) \in \mathbf{B}(S)$ under the condition (10).

Now the name *potential* is justified by the following theorem on *the 3-dimensional Brownian motion*

**Theorem 41** *Let $X$ be the 3-dimensional Brownian motion. If $f \in \mathbf{B}(S)$ has compact support, then $f$ satisfies (10) and*

$$Uf(a) = \frac{1}{2\pi}\int\limits_{R^3}\frac{f(b)db}{|b-a|} = \frac{1}{2\pi} \times Newtonian\ potential\ of\ t. \tag{12}$$

Let us denote by $D$ a bounded domain in $R^n, n \geq 1$.

**Definition 42** *A function $g$ is called "harmonic" in $D$ if $g$ is $C^\infty$ in $D$ and if $\Delta g = 0$ (where $C^\infty$ is the class of functions differentiable infinitely many times.).*

Now let $f$ be a continuous function defined on the boundary $\partial D$ and let us denote by $X$ a k-dimensional Brownian motion defined as follows

**Definition 43** *The k-dimensional Brownian motion is defined on $S = R^k$ by the equality*

$$p_t(a, db) = (2\pi t)^{-\frac{k}{2}} e^{-\frac{|b-a|^2}{2t}} db = N_t(b-a)db,$$

*where $|b-a|$ is the norm of $b-a$ in $R^k$.*

Given a k-dimensional Brownian motion $X$, if there exists a solution $g$ for the Dirichlet problem $(D, f)$, then

$$g(a) = E_a(f(X_\lambda)), \tag{13}$$

The Diriclet problem $D, f$ is to find a continuous function $g = g_{D,f}$ on the closure $\overline{D} \equiv D \cup \partial D$ such that $g$ is harmonic in $D$ and $g = f \circ g\, \partial D$.

where $\lambda \equiv \lambda_D$ = exit time from $D$ (that is to say $\lambda_D = inf\{t > 0 : X_t \notin D\}$, the hitting time of $D^C$).

In this context an interesting result is given in the following theorem

**Theorem 42** *If D is a bounded domain and g is a solution of the Dirichlet problem (D, f), then*

$$g(a) = E_a(f(X_\lambda))$$

*where $a \in D$ and $\lambda = \lambda_D$.*

On the other hand, the Dirichlet problem $(D, f)$ has a solution if $\partial D$ is smooth as it is prooved in the following theorem

**Theorem 43** *If $\partial D$ is smooth, then*

$$g(a) = E_a(f(X_\lambda)),$$

*where $\lambda = \lambda_D$ = exit time from $D$, is the solution of the Dirichlet problem (D, f).*

**Note 41** *The expression "$\partial D$ is smooth" means that $\partial D$ has a unic tangent plane at each point $x$ of $\partial D$ and the outward unit normal of the tangent plane at $x$ moves continously with $x$.*

*Remark 2.* Many other details regarding to the topics just discussed, proofs and some related problems can be found in [7], [6], [1], [5], [25], [15], [23], [14], [20], [18].

**Conclusion 41** *The Brownian motion can be represented as a random sum of integrals of orthogonal functions. Such a representation satisfies the theoretician's need to prove the existence of a process with the four defining properties of Brownian motion, but it also serves more concrete demands, one of the most important being the "chaotic and complex systems analysis".*

*Especially, the series representation can be used to derive almost all of the most important analytical properties of Brownian motion.*

*It can also give a powerful numerical method for generating the Brownian motion paths that are required in computer simulation.*

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