Chaotic Modeling and Simulation (CMSIM) 4: 631–635, 2013

Stability and bifurcation in a two species predator-prey model with quintic interactions

I. Kusbeyzi Aybar[1] and I. Hacinliyan[2]

[1] Department of Computer Education and Instructional Technology, Yeditepe University, 26 August Campus, Kayisdagi Street, Istanbul, 34755, Turkey (E-mail: ikusbeyzi@yeditepe.edu.tr)

[2] Department of Mathematics, Istanbul Technical University, Maslak, Istanbul, 34469, Turkey (E-mail: hacinliy@itu.edu.tr)

Abstract. In this work, the generalization of Lotka-Volterra model including the addition of symmetrically coupled quintic polynomial interaction is analyzed. Stability and bifurcation properties of this model are studied. It is also shown that the model has a family of limit cycles bifurcating from the Hopf points by using a numerical method.

Keywords: Predator-Prey Models, Stability, Bifurcation Analysis.

1 Introduction

Predator-prey problem attempts to model the relationship between the populations of two or more species in interaction. The simplest model of predator-prey interactions, called the classical Lotka-Volterra (LV) model, is given by the following system of differential equations [1]:

$$\dot{x} = x(a - by), \qquad \dot{y} = -y(c - dx), \tag{1}$$

where the parameters a, b, c and d characterize the predator-prey environment, dots denote the time derivatives, $x(t)$ and $y(t)$ are the prey and predator populations, respectively. Due to its unrealistic stability characteristics, the LV model serves as a starting point of many generalized models which should predict a single closed orbit, or perhaps finitely many, but not a continuous family of neutrally stable cycles. Among many ways to improve stability in the LV model, a simple approach is to add polynomial interactions. One of the generalizations considered by Nutku has been to suggest a cubic self-interaction term, instead of a quadratic self-interaction [2]. The Nutku generalization introduces additional stability in a simple way; beside a further generalization involving coupling of the form $x^k y$, where k is a positive integer and $k \leq 2$, provides a rich spectrum of equilibrium points leading to Hopf, pitchfork, saddle node

Received: 6 August 2013 / Accepted: 18 September 2013

ISSN 2241-0503

and cusp bifurcations [3]. Moreover, the limit cycles of the Hopf bifurcation point tend to a specific solution of an equation in [3]. Meanwhile, it is shown that the Gause type predator-prey model with holling type III functional response and allee effect on prey, which is another type generalization of the LV model, topologically equivalent to the differential equations, are given by a fifth order polynomial system in [4,5]. On the other hand, Giné and Romanovski have obtained necessary and sufficient integrability co! nditions at the origin for a complex generalization of the LV model where a quintic nonlinearity is introduced [6]. By the help of this motivation, we will examine stability and bifurcation properties of this model with the symmetrically coupled interaction by using approximate techniques near equilibrium points.

2 The Model, Stability and Bifurcation Scenarios

The quintic Lotka-Volterra model with symmetrically coupled interaction is given as,

$$\begin{aligned} \dot{x} &= x(1 - Ax^4 - Bx^3y - Cx^2y^2 - Dxy^3 - Ey^4) \\ \dot{y} &= -y(1 - Ay^4 - Bxy^3 - Cx^2y^2 - Dx^3y - Ex^4), \end{aligned} \tag{2}$$

where parameters A, B, C, D and E are positive. System (2) with $A(-B + 3D) = E(3B-D)$ has an integrating factor of the form $V = (xy)^{(-4B+2D)/(B-D)}$ which allows us to find the algebraic integral

$$(xy)^{\frac{r_1}{r_2}} \left(\frac{r_2}{r_1} + \frac{r_2}{2}xy(x^2+y^2) + \frac{Cr_2}{r_3}x^2y^2 - \frac{Ar_2}{r_1}(x^4+y^4) \right) = \text{constant}, \tag{3}$$

where $r_1 = -3B + D$, $r_2 = B - D$ and $r_3 = B + D$.

System (2) has 13 trivial equilibrium points, which are (0,0), $(A^{-1/4}, 0)$, $(-A^{-1/4}, 0)$, $(iA^{-1/4}, 0)$, $(-iA^{-1/4}, 0)$, $(0, A^{-1/4})$, $(0, -A^{-1/4})$, $(0, iA^{-1/4})$, $(0, -iA^{-1/4})$, $(T_1^{-1/4}, T_1^{-1/4})$, $(-T_1^{-1/4}, -T_1^{-1/4})$, $(iT_1^{-1/4}, iT_1^{-1/4})$ and $(-iT_1^{-1/4}, -iT_1^{-1/4})$ with $T_1 = A+B+C+D+E$; and nontrivial ones depending on the values of the coefficients, which are summarized below.

(i) If $T_2 = A - B + C - D + E > 0$ then $(T_2^{-1/4}, -T_2^{-1/4})$, $(-T_2^{-1/4}, T_2^{-1/4})$, $(iT_2^{-1/4}, -iT_2^{-1/4})$ and $(-iT_2^{-1/4}, iT_2^{-1/4})$ are also equilibrium points.

(ii) If $T_2 = A - B + C - D + E < 0$ then there are four complex equilibrium points: $(\sqrt{2}\,(1+i)\,(-T_2)^{-1/4}/2, -\sqrt{2}\,(1+i)\,(-T_2)^{-1/4}/2)$, $(\sqrt{2}\,(-1+i)\,(-T_2)^{-1/4}/2, \sqrt{2}\,(1-i)\,(-T_2)^{-1/4}/2)$ and their complex conjugates.

(iii) If $A = E$ and $B = D$ then there are infinitely many equilibrium points.

(iv) If $A \neq E$, $B = D$ and $T_3 = A - C + E > 0$ then $(T_3^{-1/4}, iT_3^{-1/4})$, $(-T_3^{-1/4}, -iT_3^{-1/4})$, $(iT_3^{-1/4}, -T_3^{-1/4})$, $(-iT_3^{-1/4}, T_3^{-1/4})$ and their complex conjugates are also equilibrium points.

(v) If $A \neq E$, $B = D$ and $T_3 = A - C + E < 0$ then there are eight complex equilibrium points: $(\sqrt{2}\,(1+i)\,(-T_3)^{-1/4}/2, \sqrt{2}\,(-1+i)\,(-T_3)^{-1/4}/2)$,

$(\sqrt{2}\,(1+i)\,(-T_3)^{-1/4}/2, \sqrt{2}\,(1-i)\,(-T_3)^{-1/4}/2)$, $(\sqrt{2}\,(-1+i)\,(-T_3)^{-1/4}/2, \sqrt{2}\,(-1-i)\,(-T_3)^{-1/4}/2)$, $(\sqrt{2}\,(-1+i)\,(-T_3)^{-1/4}/2, \sqrt{2}\,(1+i)\,(-T_3)^{-1/4}/2)$ and their complex conjugates.

(vi) If $A \neq E$, $B \neq D$ and $|B-D| > 2\,|A-E|$ then there are 4 real and 4 complex, or 2 real and 6 complex equilibrium points. One can find these points by solving the system of the equations $x = (-\alpha \pm \sqrt{\alpha^2-1})y$, $2\alpha = (B-D)/(A-E)$, and $Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4 = 1$.

(vii) If $A \neq E$, $B \neq D$ and $|B-D| < 2\,|A-E|$ then one can find equilibrium points by solving the system of the equations $x = (-\alpha \pm i\sqrt{1-\alpha^2})y$ and $Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4 = 1$.

On the other hand, system (2) is Lyapunov unstable for the chosen values of the parameters, which can be very easily demonstrated using the Lyapunov function $V = (E-A)(x^2+y^2) + 2Bxy$ which is positive definite if and only if $E > A$ and $E - A > B$. Therefore, we obtain

$$\dot{V} = 2(x^2-y^2)\,[\beta A(x^4+y^4) + ((A+E)^2 + B(D-B) + \beta C)x^2y^2 - \beta], \quad (4)$$

where $\beta = A - E < 0$. Although the second factor has negative definite dominant term, the first factor changes sign as $|x| = |y|$. Hence there is a regime where the system is Lyapunov unstable so that we can limit our discussion to local stability. At this stage, we focus on trivial equilibrium points to examine stability. Nontrivial ones will be taken into account for a spacial case.

Linearized eigenvalues about the first real trivial equilibrium point $(0,0)$ are $\{\pm 1\}$; thus the origin is a saddle point. Eigenvalues for the points $(A^{-1/4}, 0)$ and $(-A^{-1/4}, 0)$ are $\{-4,\ -1+E/A\}$, so these points are saddle when $A < E$, and stable nodes when $A > E$. Eigenvalues associated with points $(0, A^{-1/4})$ and $(0, -A^{-1/4})$ are $\{4,\ 1-E/A\}$. If $A < E$, these equilibrium points are saddle, otherwise they are unstable nodes. On the other hand the eigenvalues for both of equilibrium points $(T_1^{-1/4}, T_1^{-1/4})$ and $(-T_1^{-1/4}, -T_1^{-1/4})$ are $\{\pm i\sqrt{8[2(E-A)+(D-B)]/T_1}\}$, a pair of purely imaginary eigenvalues, if $2(E-A)+(D-B) > 0$ and $\{\pm\sqrt{8[2(A-E)+(B-D)]/T_1}\}$ if $2(E-A) + (D-B) < 0$. Thus the first purely imaginary values satisfy the resonance conditions and the system can be expanded into a resonant normal form, which gives Hopf bifurcation under the condition $2(E-A)+(D-B) > 0$. For the other condition, these points are also saddle.

Let $A = 1$ and $B = C = D = E = 2$. In this special case, the real equilibrium points of the system are (0,0), (1,0), (-1,0), (0,1), (0,-1), $A_1(1/\sqrt{3}, 1/\sqrt{3})$, $A_2(-1/\sqrt{3}, -1/\sqrt{3})$, $A_3(1,-1)$, $A_4(-1,1)$; and there are 16 complex equilibrium points. Trivial equilibrium point at the origin is a saddle point with the eigenvalues $\{\pm 1\}$. (1,0) and (-1,0) are also saddle points with the eigenvalues $\{-4,\ 1\}$. Similarly (0,1) and (0,-1) are saddle points with the eigenvalues $\{4,\ -1\}$. On the other hand, the points A_1 and A_2 with the eigenvalues $\{\pm i4/3\}$; and also the points A_3 and A_4 with the eigenvalues $\{\pm i4\}$ are also Hopf points. The third order normal form about the point A_1 is

$$\dot{u} = 4iu(1-14uv)/3, \qquad \dot{v} = -4iv(1-14uv)/3, \quad (5)$$

where u and v refer to the variables in the near identity transformation. This normal form indicates Hopf bifurcation. From the linearized eigenvalues of system (5), it is clear that the normal form will be $\dot{u} = i\alpha u f(uv)$, $\dot{v} = -i\alpha v f(uv)$ which admits the solution uv =constant. Hence the inclusion of higher order terms in the normal form will only change the purely imaginary eigenvalues, since the only change will be the constant value of $f(uv)$ to the normal form approximation. This implies that the character of the local bifurcation will not change by including further terms. Normal form analysis for the other equilibrium points is omitted for brevity.

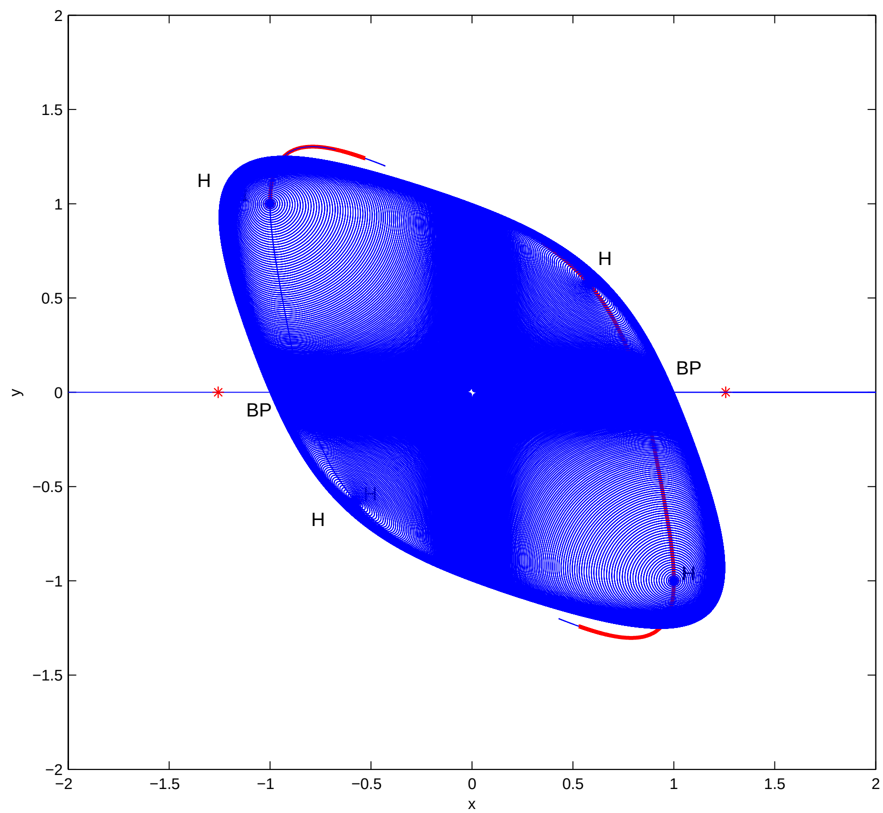

Fig. 1. Family of limit cycles of the system (2) when A is varied

The bifurcation analysis when A is varied is given in Figure 1. In this special case, two supercritical Hopf bifurcation points, A_1 and A_2, and two subcritical Hopf bifurcation points, A_3 and A_4, are observed. All of the limit cycles lie between the coordinate axes and the curve in one of quadrants. They also form a double throw-and-catch mechanism around a pitchfork bifurcation point in the middle.

3 Conclusion

In this work, a special case of the quintic generalization of the LV model has been studied. The model is globally Lyapunov unstable, however local stability indicates several instances of Hopf bifurcation to a family of bounded orbits. It is also numerically observed that there is a discontinuous family of stable cycles in the same way as in the cubic nonlinear intersection.

References

1.A. Lotka. *Elements of Physical Biology*, 1st edition, Williams and Wilkins Company, Baltimore, 1925.
2.Y. Nutku. Hamiltonian structure of the Lotka-Volterra equation. *Phys. Lett. A*, 145: 27-28, 1990.
3.I. Kusbeyzi, O. O. Aybar, and A. Hacinliyan. Stability and bifurcation in two species predotor-prey models. *Nonlinear Anal. RWA*, 12: 377-387, 2011.
4.F. Dumortier, J. Llibre and J. C. Artés. *Qualitative theory of planar differential systems*, Spring, Berlin, 2006.
5.E. González-Olivares and A. Rojas-Palma. Multiple limit cycles in a Gause Type Predator-Prey Model with Holling Type III Functional Response and Allee Effect on Prey. *Bull. Math. Biol*, 73: 1378-1397, 2011.
6.J. Giné and V. G. Romanovski. Integrability conditions for Lotka-Volterra planar complex quintic systems. *Nonlinear Anal. RWA*, 11: 21002105, 2010.