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Exponential dichotomy and bounded solutions of the Schrӧdinger equation

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Abstract. Necessary and sufficient conditions for existence of bounded on the entire real axis solutions of Schrödinger equation are obtained under assumption that the homogeneous equation admits an exponential dichotomy on the semi-axes. Bounded analytical solutions are represented using generalized Green's operator.

Keywords: exponential dichotomy, normally-resolvable operator, pseudoinverse Moore-Penrose operator.

Numerous papers deal with problems of the existence of bounded solutions of linear and nonlinear differential equations in Banach spaces and condition of exponential dichotomy on both semi-axes. We note the well-known paper [1], where such problems were considered in finite-dimensional spaces. Boundary value problems for linear differential equations in Banach spaces which admit exponential dichotomy on both semi-axes with bounded and unbounded operators in linear part was investigated in [2], [3]. The normal solvability of a differential operator was considered in [4]. The present paper dealt with the derivation of necessary and sufficient conditions for the existence of generalized bounded solutions of the Schrӧdinger equation in the Hilbert space.

1 Linear case

1.1 Statement of the Problem

Consider the next differential Shrӧdinger equation

$$\frac{d\varphi(t)}{dt} = -iH(t)\varphi(t) + f(t), t \in J \tag{1}$$

in a Hilbert space $\mathcal{H}$, where, for each $t \in J \subset R$, the unbounded operator $H(t)$ has the form $H(t) = H_0 + V(t)$ (here $H_0 = H_0^*$ is unbounded self-adjoint operator with domain $D = D(H_0) \subset \mathcal{H}$), the mapping $t \to V(t)$ is strongly continuous. Define as in [5] operator-valued function

$$\tilde{V}(t) = e^{itH_0}V(t)e^{-itH_0}.$$

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In this case for $\tilde{V}(t)$ Dyson's [5, p.311] representation is true and its propagator we define as $\tilde{U}(t,s)$. If $U(t,s) = e^{-itH_0}\tilde{U}(t,s)e^{isH_0}$ then $\psi_s(t) = U(t,s)\psi$ is a weak solution of (1) with condition $\psi_s(s) = \psi$ in the sense that for any $\eta \in D(H_0)$ function $(\eta, \psi_s(t))$ is differentiable and

$$\frac{d}{dt}(\eta, \psi_s(t)) = -i(H_0\eta, \psi_s(t)) - i(V(t)\eta, \psi_s(t)), t \in J.$$

The present part dealt with the derivation of necessary and sufficient conditions for the existence of weak (in different senses) bounded solutions of the inhomogeneous equation (1) with $f \in BC(J,H) = \{f : J \to \mathcal{H}$; the function f is continuous and bounded $\}$. Here the boundedness is treated in the sense that $|||f||| = sup_{t\in J}||f(t)|| < \infty$. For simplicity we suppose that D dense in $\mathcal{H}$. The operator $U(t,s)$ is a bounded linear operator for fixed t, s, and since the set D is dense in $\mathcal{H}$, we find that it can be extended to the entire space $\mathcal{H}$ by continuity, which is assumed in forthcoming considerations. The extension of the family of evolution operators to the entire space is denoted in the same way.

1.2 Bounded solutions

Throughout the following, we use the notion of exponential dichotomy in the sense of [6]. It is of special interest to analyze the exponential dichotomy on the half-lines $R_s^- = (-\infty, s]$ and $R_s^+ = [s; \infty)$. [In this case, the projection-valued functions defined on half-lines will be denoted by $P_-(t)$ for all $t \geq s$ and $P_+(t)$ for all $t < s$ with constants M_1, α_1 and M_2, α_2, respectively (α_1, α_2 - entropy or Lyapunov coefficients on the half-lines).] Most of the results obtained below follows directly from [3]. The main result of this section can be stated as follows.

Theorem 1. *Let $\{U(t,s), t \geq s \in R\}$ be the family of strongly continuous evolution operators associated with equation (1). Suppose that the following conditions are satisfied.*

1. The operator $U(t,s)$ admits exponential dichotomy on the half-lines R_0^+ and R_0^- with projection-valued operator-functions $P_+(t)$ and $P_-(t)$, respectively.

2. The operator $D = P_+(0) - (I - P_-(0))$ is generalized-invertible.

Then the following assertions hold.

1. There exist weak solutions of equation (1) bounded on the entire line if and only if the vector function $f \in BC(R, \mathcal{H})$ satisfies the condition

$$\int_{-\infty}^{+\infty} H(t)f(t)dt = 0, \quad (2)$$

where $H(t) = \mathcal{P}_{N(D^*)}P_-(0)U(0,t)$.

2. Under condition (2), the weak solutions of (1) bounded on the entire line have the form

$$\varphi_0(t,c) = U(t,0)P_+(0)\mathcal{P}_{N(D)}c + (G[f])(t,0)\forall c \in \mathcal{H}, \quad (3)$$

where

$$(G[f])(t,s) = \begin{cases} \int_s^t U(t,\tau)P_+(\tau)f(\tau)d\tau - \int_t^{+\infty} U(t,\tau)(I-P_+(\tau))f(\tau)d\tau + \\ \quad + U(t,s)P_+(s)D^+[\int_s^{\infty} U(s,\tau)(I-P_+(\tau))f(\tau)d\tau + \\ \quad + \int_{-\infty}^{s} U(s,\tau)P_-(\tau)f(\tau)d\tau], \quad t \geq s \\ \int_{-\infty}^t U(t,\tau)P_-(\tau)f(\tau)d\tau - \int_t^{s} U(t,\tau)(I-P_-(\tau))f(\tau)d\tau + \\ \quad + U(t,s)(I-P_-(s))D^+[\int_s^{\infty} U(s,\tau)(I-P_+(\tau))f(\tau)d\tau + \\ \quad + \int_{-\infty}^{s} U(s,\tau)P_-(\tau)f(\tau)d\tau], \quad s \geq t \end{cases}$$

is the generalized Green operator of the problem on the bounded, on the entire line, solutions

$$(G[f])(0+,0) - (G[f])(0-,0) = -\int_{-\infty}^{+\infty} H(t)f(t)dt;$$

$$\mathcal{L}(G[f])(t,0) = f(t), \quad t \in R$$

and

$$(\mathcal{L}x)(t) = \frac{dx}{dt} - iH(t)x(t),$$

D^+ *is the Moore-Penrouse pseudoinverse operator to the operator* D*;* $\mathcal{P}_{N(D)} = I - D^+D$ *and* $\mathcal{P}_{N(D^*)} = I - DD^+$ *are the projections [7] onto the kernel and cokernel of the operator* D*.*

Remark 1. A similar theorem holds for the case in which the family of evolution operators $U(t,s)$ admits exponential dichotomy on the half-lines R_s^+ and R_s^-.

Now we show that condition 2 in theorem 1 can be omitted and in the different senses equation (1) is always resolvable. From the proof of the theorem 1 follows that equation (1) have bounded solutions if and only if the operator equation

$$D\xi = g, \tag{4}$$

$$g = \int_{-\infty}^{0} U(0,\tau)P_-(\tau)f(\tau)d\tau + \int_0^{+\infty} U(0,\tau)(I-P_+(\tau))f(\tau)d\tau$$

is resolvable and its number depends from the dimension of $N(D)$.

Consider next 3 cases.

1) Classical strong generalized solutions.

Consider case when the operator D is normally-resolvable ($R(D) = \overline{R(D)}$ is the set of values of D). Then [7] $g \in R(D)$ if and only if $\mathcal{P}_{N(D^*)}g = 0$ and the set of solutions of (4) can be represented in the form [7] $\xi = D^+g + \mathcal{P}_{N(D)}c, forallc \in \mathcal{H}$.

2) Strong generalized solutions.

Consider the case when $R(D) \neq \overline{R(D)}$. We show that operator D may be extended to $\overline{D}$ in such way that $R(\overline{D})$ is closed.

Since the operator D is bounded the next representations of $\mathcal{H}$ in the direct sum are true

$$H = N(D) \oplus X, \mathcal{H} = \overline{R(D)} \oplus Y,$$

with $X = N(D)^{\perp}$ and $Y = \overline{R(D)}^{\perp}$. Let $E = H/N(D)$ is quotient space of $\mathcal{H}$ and $\mathcal{P}_{\overline{R(D)}}$ is orthoprojector, which projects onto $\overline{R(D)}$. Then operator

$$\mathcal{D} = \mathcal{P}_{\overline{R(D)}} D j^{-1} p : X \to R(D) \subset \overline{R(D)},$$

is linear, continuous and injective (here $p : X \to E$ is continuous bijection and $j : \mathcal{H} \to E$ is a projection. The triple $(\mathcal{H}, E, j)$ is a locally trivial bundle with typical fiber $\mathcal{P}_{N(L)}H$). In this case [8, p.26,29] we can define strong generalized solution of equation

$$\mathcal{D}\xi = g, \xi \in X.$$

We complete the space X with the norm $||\xi||_{\overline{X}} = ||\mathcal{D}\xi||_F$, where $F = \overline{R(D)}$ [8]. Then the extended operator

$$\overline{\mathcal{D}} : \overline{X} \to \overline{R(D)}, X \subset \overline{X}$$

is a homeomorphism of $\overline{X}$ and $\overline{R(D)}$. Operator $\overline{D} := \overline{\mathcal{D}}\mathcal{P}_{\overline{X}} : \overline{\mathcal{H}} \to \mathcal{H}$ is normally-resolvable. By the construction of a strong generalized solution [8], the equation

$$\overline{D}\,\bar{\xi} = g,$$

has a unique generalized solution, which we denote $\overline{D}^{+}g$ which is called the strong generalized solution of (4). Then the set of strong generalized solutions of (4) has the form

$$\xi = \overline{D}^{+}g + \mathcal{P}_{N(D)}c, for\ all\ c \in \mathcal{H}.$$

3) Strong pseudosolutions.

Consider an element $g \notin \overline{R(D)}$. This condition is equivalent $\mathcal{P}_{N(D^*)}g \neq 0$. In this case there are elements from $\overline{\mathcal{H}}$ that minimize norm $||\overline{D}\xi - g||_{\overline{\mathcal{H}}}$ for $\xi \in \overline{\mathcal{H}}$:

$$\xi = \overline{D}^{+}g + \mathcal{P}_{N(D)}c,\ for\ all\ c \in \mathcal{H}.$$

These elements are called strong pseudosolutions by analogy of [7].

Remark 2. It should be noted that in each cases 1) - 3) the form of bounded solutions (4) isn't change.

Remark 3. As follows from 1) and 3) the notion of exponential dichotomy is equivalent of existence of bounded on the entire real axis solutions of (1).

2 Main result (Nonlinear case)

In the Hilbert space $\mathcal{H}$, consider the differential equation

$$\frac{d\varphi(t)}{dt} = -iH(t)\varphi(t) + \varepsilon Z(\varphi, t, \varepsilon) + f(t). \tag{5}$$

We seek a bounded solution $\varphi(t, \varepsilon)$ of equation (5) that becomes one of the solutions of the generating equation (1) for $\varepsilon = 0$.

To find a necessary condition on the operator function $Z(\varphi, t, \varepsilon)$, we impose the joint constraints

$$Z(\cdot,\cdot,\cdot) \in C[||\varphi - \varphi_0|| \le q] \times BC(R, \mathcal{H}) \times C[0, \varepsilon_0],$$

where q is some positive constant.

Let us show that this problem can be solved with the use of the operator equation for generating constants

$$F(c) = \int_{-\infty}^{+\infty} H(t) Z(\varphi_0(t,c), t, 0) = 0. \qquad (6)$$

Theorem 2 *(necessary condition). Let the equation (1) admits exponential dichotomy on the half-lines R_0^+ and R_0^- with projection-valued operator functions $P_+(t)$ and $P_-(t)$, respectively, and let the nonlinear equation (5) have a bounded solution $\varphi(\cdot,\varepsilon)$ that becomes one of the solutions of the generating equation (1) with constant $c = c^0$, $\varphi(t,0) = \varphi_0(t, c^0)$ for $\varepsilon = 0$. Then this constant should satisfy the equation for generating constants (6).*

The proof of this theorem is the same as in [3, Theorem 1].

To find a sufficient condition for the existence of bounded solutions of (1), we additionally assume that the operator function $Z(\varphi, t, \varepsilon)$ is strongly differentiable in a neighborhood of the generating solution ($Z(\cdot, t, \varepsilon) \in C^1[||\varphi - \varphi_0|| \le q]$).

This problem can be solved with the use of the operator

$$B_0 = \int_{-\infty}^{+\infty} H(t) A_1(t) U(t,0) P_+(0) \mathcal{P}_{N(D)} dt : \mathcal{H} \to \mathcal{H},$$

where $A_1(t) = Z^1(v, t, \varepsilon)|_{v=\varphi_0, \varepsilon=0}$ (the Fréchet derivative).

Theorem 3 *(sufficient condition). Suppose that the equation (1) admits exponential dichotomy on the half-lines R_0^+ and R_0^- with projection-valued functions $P_+(t)$ and $P_-(t)$, respectively. In addition, let the operator B_0 satisfy the following conditions.*

1. The operator B_0 is Moore-Penrose pseudoinvertible.

2. $\mathcal{P}_{N(B_0^*)} \mathcal{P}_{N(D^*)} P_-(0) = 0$.

Then for an arbitrary element of $c = c^0 \in \mathcal{H}$ satisfying the equation for generating constants (6), there is exists bounded solution. This solution can be found with the use of the iterative process

$$\overline{y}_{k+1}(t,\varepsilon) = \varepsilon G[Z(\varphi_0(\tau, c^0 + y_k, \tau, \varepsilon))](t,0),$$

$$c_k = -B_0^+ \int_{-\infty}^{+\infty} H(\tau) \{A_1(\tau) \overline{y}_k(\tau,\varepsilon) + \mathcal{R}(y_k(\tau,\varepsilon), \tau, \varepsilon)\} d\tau,$$

$$\mathcal{R}(y_k(t,\varepsilon)) = Z(\varphi_0(t,c^0) + y_k(t,\varepsilon), t, \varepsilon) - Z(\varphi_0(t,c^0), t, 0) - A_1(t) y_k(t,\varepsilon),$$

$$\mathcal{R}(0,t,0) = 0, \qquad \mathcal{R}_x^{(1)}(0,t,0) = 0,$$

$$y_{k+1}(t,\varepsilon) = U(t,0) P_+(0) \mathcal{P}_{N(D)} c_k + \overline{y}_{k+1}(t,0,\varepsilon),$$

$$\varphi_k(t,\varepsilon) = \varphi_0(t,c^0) + y_k(t,\varepsilon), \quad k = 0,1,2,..., \quad y_0(t,\varepsilon) = 0, \quad \varphi(t,\varepsilon) = \lim_{k\to\infty} \varphi_k(t,\varepsilon).$$

2.1 Relationship between necessary and sufficient conditions

First, we formulate the following assertion.

Corollary. *Let a functional $F(c)$ have the Fréchet derivative $F^{(1)}(c)$ for each element c^0 of the Hilbert space $\mathcal{H}$ satisfying the equation for generating constants (6). If $F^1(c)$ has a bounded inverse, then equation (5) has a unique bounded solution on the entire line for each c^0.*

Remark 4. If assumptions of the corollary are satisfied, then it follows from its proof that the operators B_0 and $F^{(1)}(c^0)$ are equal. Since the operator $F^{(1)}(c)$ is invertible, it follows that assumptions 1 and 2 of Theorem 3 are necessarily satisfied for the operator B_0. In this case, equation (5) has a unique bounded solution for each $c^0 \in \mathcal{H}$. Therefore, the invertibility condition for the operator $F^1(c)$ relates the necessary and sufficient conditions. In the finite-dimensional case, the condition of invertibility of the operator $F^{(1)}(c)$ is equivalent to the condition of simplicity of the root c^0 of the equation for generating amplitudes [7].

In such way we obtain the modification of the well-known method of Lyapunov-Schmidt. It should be emphasized that theorem 2 and 3 give us possible condition of chaotic behavior of (5) [9].

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