Chaotic Modeling and Simulation (CMSIM) 4: 601–607, 2013

The Fractional Laplacian as continuum limit of self-similar lattice models

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Abstract. We show that the fractional Laplacian (FL) $-(-\Delta)^{\frac{\alpha}{2}}$ is the principal characteristic operator of harmonic systems with self-similar interparticle interactions. We demonstrate that the FL can be rigorously defined by Hamilton's variational principle as "*fractional continuum limit*" of a spring model with self-similar, in some cases fractal harmonic interactions which we introduced recently (Michelitsch *et al.*[5]). We generalize that approach to the multi-dimensional physical space of dimensions $n = 1, 2, 3, ...$ In this way we demonstrate the interlink between fractal discrete behavior (discrete self-similar Laplacian) and its fractional continuum field counterpart (FL) and give the latter a physical justification. The dispersion relation of the discrete model is obtained as self-similar Weierstrass-Mandelbrot fractal function which takes in the fractional continuum limit the form of a smooth self-similar power law. The density of states (density of normal modes) takes the form of a characteristic scaling law which depends only on the scaling exponent of the FL and the dimension of the physical space. The approach has a wide range of interdisciplinary applications of self-similar dynamic problems such as anomalous diffusion (Levi flights), self-similar wave propagation, and may also be useful to model self-similar chaotic processes and dynamics in turbulence.
Keywords: Fractional Laplacian, fractional continuum limit, linear chain, Fractals, Weierstrass-Mandelbrot function, self-similarity, scaling laws.

1 Introduction

Despite fractional calculus has a long history, recently a new increasing interest has emerged to employ fractional operators and the so called *fractional Laplacian* (FL) (often also referred to as Riesz fractional derivative) $-(-\Delta)^{\frac{\alpha}{2}}$ where α indicates a fractional in general non-integer exponent. The reason for this

Received: 7 August 2013 / Accepted: 23 September 2013

ISSN 2241-0503

new interest is the conclusion that the fractional approach is a highly powerful mathematical tool to model complex and chaotic phenomena in various disciplines.

The goal of this note is to demonstrate that the FL is the "natural" characteristic linear operator, in a sense most basic operator that can be generated from a physical "self-similar" spring model and its generalizations. Due to its non-local "long tail" and self-similar invariant characteristics of the FL we raise the question what is the interlink of the FL with fractal and chaotic features often chosen in nature.

Recently many models where developed which employ the *FL* in various physical contexts, among them the description of "complex" dynamic phenomena including anomalous diffusion (Lévi flights) [1–3,8,10] and see also the numerous references therein.

This note is organized as follows: As point of departure we introduce a 1D harmonic spring model with harmonic elastic potential energy which describes *self-similar interparticle interactions* which we developed recently [5]. This discrete model leads to fractal dynamic vibrational characteristics such as a dispersion relation of the form of Weierstrass-Mandelbrot fractal functions. Application of Hamilton's variational principle defines a discrete self-similar Laplacian with all good properties of a Laplacian: The self-similar Laplacian is self-adjoint, elliptic, negative (semi-) definite (indicating elastic stability), and translational invariant. We introduce a *fractional continuum limit* which yields in rigorous manner the FL. In this way the FL is physically justified being a continuum description of a self-similar spring model. The approach is generalized to n dimensions of the physical space.

2 Linear chain model with self-similar harmonic interactions

We consider an infinite sequence of points $\{h_p\}$ generated by a non-linear invertible mapping $h \rightarrow N(h)$ with (initial value $h = h_0$)

$$h_p = N(h_{p-1}), \qquad p \in Z_0 \tag{1}$$

where we exclude for convenience periodic orbits and fixed points. All points of the sequence are assumed to fulfil $h_p \neq h_q$ for $p \neq q$ $(-\infty < p < \infty)$. Define a function Φ for a arbitrary generated by the series

$$\Phi(h) = \sum_{s=-\infty}^{\infty} a^{-\delta s} f(h_s) \tag{2}$$

where the sum is performed over the infinite sequence of points h_s of (1). $\Phi(h)$ is defined (convergent) for sufficiently good functions f. Function Φ behaves self-similar under the (in general non-linear) transformation $h \rightarrow N(h)$ of its argument, namely

$$\Phi(N(h)) = a^{\delta} \sum_{s=-\infty}^{\infty} a^{-\delta(s+1)} f(h_{s+1}) = a^{\delta}\Phi(h) \tag{3}$$

For the sake of simplicity but without loss of generality let us consider here a sequence generated by a linear mapping

$$N(h) = ah, \qquad a > 1 \tag{4}$$

Then we introduce the self-similar elastic potential in the form of a self-similar function (3), namely

$$\mathcal{W}(x,h) = \frac{1}{4}\sum_{s=-\infty}^{\infty} a^{-\delta s}\left\{(u(x+ha^s)-u(x))^2 + (u(x-ha^s)-u(x))^2\right\} \tag{5}$$

which is self-similar in the sense of (3) with respect to h. The elastic potential describes a homogeneous mass distribution where each material point x is connected with other material points $x \pm ha^s$ by a self-similar distribution of linear springs of spring constants $\sim a^{-\delta s}$. In general this potential can be defined also for nonlinear sequence h_s of (1). The notion of self-similarity at a point was coined by Peitgen *et al.*[9].

The total elastic energy of (5) is given by

$$V(h) = \int_{-\infty}^{\infty} \mathcal{W}(x,h)\,\mathrm{d}x \tag{6}$$

A self-similar Laplacian is then defined by Hamilton's principle

$$\Delta_{\delta,h}u(x) = -\frac{\delta V}{\delta u(x)} \tag{7}$$

where $\frac{\delta(..)}{\delta u}$ stands for a functional derivative, and where

$$\Delta_{(\delta,a,h)}u(x) = \sum_{s=-\infty}^{\infty} a^{-\delta s}\left(u(x+ha^s)+u(x-ha^s)-2u(x)\right), \qquad 0<\delta<2 \tag{8}$$

fulfilling self-similarity condition $\Delta_{\delta,ah} = a^{\delta}\Delta_{\delta,h}$. This Laplacian has all required good properties. The dispersion relation (negative eigenvalues) of this Laplacian are obtained in the form of Weierstrass-Mandelbrot functions

$$\omega^2_{(\delta,a)}(kh) = 4\sum_{s=-\infty}^{\infty} a^{-\delta s}\sin^2(\frac{kha^s}{2}), \qquad 0<\delta<2 \tag{9}$$

which are self-similar $\omega^2_{(\delta,a)}(kah) = a^{\delta}\omega^2_{(\delta,a)}(kh)$ within its entire interval of existence $0<\delta<2$. The dispersion relation (9) is within $0<\delta<1$ a nowhere differentiable fractal function of estimated Hausdorff dimension $2-\delta$ [4,5]. In figures 1-3 cases of increasing fractal dimension (decreasing δ) are plotted. Note that for $1 \leq \delta < 2$ (9) is a non-fractal function of Hausdorff dimension $D = 1$ (see figure 1). For increasing fractal dimension D (decreasing exponent δ) fractal dispersion curves have increasingly erratic characteristics. For more details we refer to our paper Michelitsch *et* al.[5].

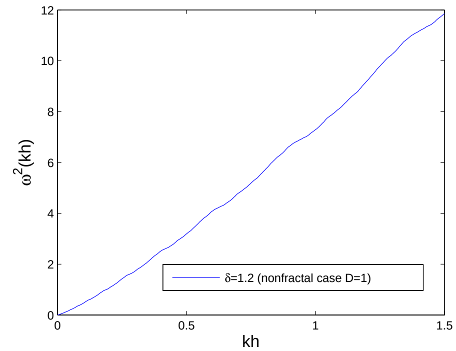

Fig. 1. Dispersion relation (9) for a fractal case.

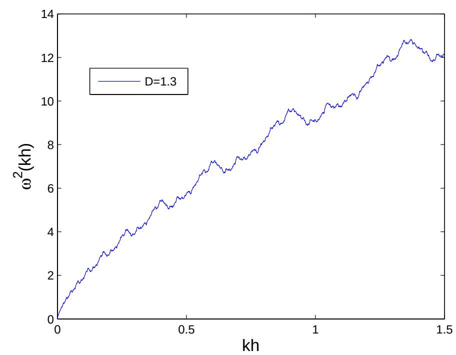

Fig. 2. Dispersion relation (9) for a non-fractal case.

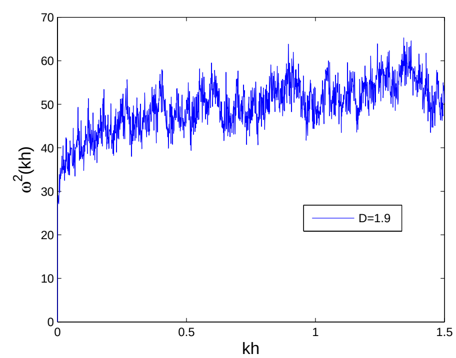

Fig. 3. Dispersion relation (9) for a fractal case.

3 The FL as fractional continuum limit of the discrete chain Laplacian and its generalization to n dimensions

Now we define the *fractional continuum limit* as follows [6,7]

$$\Lambda_a(h) = \lim_{a\to 1} \sum_{s=-\infty}^{\infty} a^{-\delta s} f(a^s h) \approx \frac{h^\delta}{\zeta} \int_0^\infty \frac{f(\tau)}{\tau^{\delta+1}} \mathrm{d}\tau \tag{10}$$

where $a = 1+\zeta \to 1$ and $0 < \zeta << 1$. The *fractional continuum limit* of the elastic potential (5) takes then the form

$$\mathcal{W}(x,h) \approx \frac{h^\delta}{4\zeta} \int_0^\infty \frac{(u(x+\tau)-u(x))^2 + (u(x-\tau)-u(x))^2}{\tau^{\delta+1}} \mathrm{d}\tau, \qquad 0<\delta<2 \tag{11}$$

which can be generalized to n dimensions as

$$\mathcal{W}(\mathbf{x},h,\alpha) \approx \frac{h^\alpha}{4\zeta} \int_0^\infty \frac{(u(\mathbf{x}+\mathbf{r})-u(\mathbf{x}))^2 + (u(x-\mathbf{r})-u(\mathbf{x}))^2}{\tau^{\alpha+n}} \mathrm{d}^n\mathbf{r} \tag{12}$$

where $0 < \alpha < 2$. Hamilton's principle yields from (12) the fractional continuum limit of the self-similar Laplacian in n dimensions

$$\Delta_{n,\alpha,h} u(x) =: -\frac{\delta V}{\delta u(x)} = \frac{h^\alpha}{2\zeta} \int_0^\infty \frac{(u(\mathbf{x}+\mathbf{r}) + u(x-\mathbf{r}) - 2u(\mathbf{x}))}{\tau^{\alpha+n}} \mathrm{d}^n\mathbf{r} \tag{13}$$

with $0 < \alpha < 2$. (13) recovers for $n = 1$ also the fractional continuum limit of the self-similar Laplacian (8). The dispersion relation is obtained by $\Delta_{n,\alpha,h} e^{ikx} = -\omega^2_{n,\alpha,h}(kh) e^{ikx}$ and yields [8] a power-law of the form

$$\omega^2_{n,\alpha,h}(kh) = \mathcal{A}_{n,\alpha} k^\alpha, \qquad 0<\alpha<2 \tag{14}$$

with the positive constant [8]

$$\mathcal{A}_{n,\alpha} = \frac{h^\alpha}{\zeta} \frac{\pi^{\frac{n}{2}}}{2^{\alpha-1}\alpha} \frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{\alpha+n}{2})} > 0, \qquad 0<\alpha<2 \tag{15}$$

The positiveness of this constant is a consequence of the elastic stability.

The following observation is crucial: The fractional continuum limit Laplacian (13) coincides (up to a normalization factor) with the FL which is defined, e.g. [2,3,10]

$$\Delta_{n,\alpha,h} = -\mathcal{A}_{n,\alpha} \left(-\Delta\right)^{\frac{\alpha}{2}} \tag{16}$$

where the constant (15) is consistent with the normalization factor given by in the literature e.g. [2,3,8,10] and where (13) recovers with (16) and (15) the standard representation of the FL. Our self-similar chain model represents hence a discrete lattice counterpart which corresponds in the fractional continuum approximation the FL fractional approach.

With (15) it is straight-forward to obtain the density of normal modes ("density of states") $\mathcal{D}(\omega)$ where $\mathcal{D}(\omega)\mathrm{d}\omega$ measures the number of eigenmodes of frequency ω. It is obtained as [8]

$$\mathcal{D}_{\alpha,n}(\omega) = B_{n,\alpha}\omega^{\frac{2n}{\alpha}-1}, \qquad 0 < \alpha < 2 \tag{17}$$

with

$$B_{n,\alpha} = \frac{2^{2-n}}{\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})\alpha\mathcal{A}_{n,\alpha}^{\frac{n}{\alpha}}} \tag{18}$$

We observe that the state density $\mathcal{D}_{\alpha,n}(\omega)$ scales as $\sim \omega^{\frac{2n}{\alpha}-1}$ with a positive exponent where $0 < n-1 < \frac{2n}{\alpha} - 1$ depending only on physical dimension n and α. Because of $0 < \alpha < 2$ the scaling exponent of the self-similar density of states (17) is always greater than the exponent $n-1$ of the standard Laplacian which is asymptotically approached by (17) when α approaches the forbidden value $\alpha \to 2$.

4 Conclusions

We have demonstrated in this brief note that the fractional Laplacian can be rigorously defined as the fractional continuum limit by a self-similar linear spring model and its generalization to $n = 1, 2, 3..$ dimensions. In this way a physical justification for the FL is introduced. The model also reveals the interlink between fractal vibrational Weierstrass-Mandelbrot characteristics and its smooth fractional continuum counterpart. The present approach allows to develop a smooth fractional field theory of phenomena with fractal and erratic - chaotic features [8]. Especially noteworthy is a vast potential of applications which include dynamic processes such as anomalous diffusion (Lévi flights), wave propagation and turbulence problems.

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