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Control and Synchronization of Fractional–Order Differential Equations of Phase–Locked Loop

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Abstract. Previous published researches on chaos, controlling and synchronization in phased–locked loops focused only on integer–order phase–locked loops. In this paper, we study control and synchronization of phase–locked loop systems based on fractional–order differential equations. Stability analyses of commensurate fractional–order linear system are utilized to control chaotic behaviour exhibited by fractional–order differential equation–based phase–locked loop. Furthermore, chaos synchronization is obtained by employing the nonlinear state observer method. Finally, numerical simulations verify the effectiveness and applicability of our approaches.
Keywords: Fractional–order equation, Phase–locked loop, Chaos control, Chaos synchronization.

1 Introduction

Fractional calculus was introduced in the early 17^{th} century and has been applied to describe various real systems such as transmission lines, electrical noises, dielectric polarization and heat transfer phenomena Arena *et al.*[1]. Recently, there are amount of efforts to discover the chaos of fractional–order systems. Specifically, chaotic features have been proofed in fractional–order Lorenz system, fractional–order Chua's system, fractional–order Duffing's oscillator, fractional–order Genesio–Tesi system and fractional–order Lotka–Volterra system Caponetto *et al.*[2]. Moreover, the synchronization and controlling fractional–order chaotic systems are the topics which received more attention because of their practical applications. Li et al. implemented synchronization of fractional Lorentz system, Chen system and Chua circuit by the aid of controller and driving signals Li and Yan[3]. Matouk[4] proposed the feedback control

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and synchronization of a fractional–order modified Van der Pol–Duffing circuit using fractional Routh–Hurwitz conditions.

Phase–locked loop (PLL) plays a vital role in communication and control systems Gardner[5] where applications of PLL include clock synchronization, carrier recovery, frequency or phase modulation and demodulation, frequency synthesis, and PLL controlled motors. Generally, PLL works in the locked range in which average frequency of the voltage controlled oscillator (VCO) exactly equal to the average frequency of the input signal. However, chaotic behaviours of PLLs have been observed and studied with second–order loop filter Endo and Chua[6] under certain conditions. In order to control undesirable chaos effects in PLLs, Zhao *et al.*[7] represented the state observer to design a non–linear feedback controller for second–order non–autonomous PLL. Experimental synchronization of two PLLs driven by a common chaotic signal derived from a master PLL was also observed Endo and Chua[6] if the detuning of the VCO free-running frequencies was not large. Even through, a fractional–order differential equation–based phase–locked loop is still not considered. There is an expectation that fractional–order differential equation–based phase–locked loop (FOPLL), which processes key features of classical PLL, will have important potential applications in such areas as communications and control. Motivated by this expectation, in this work, we introduce a new model of the FOPLL and propose control and synchronization methods for it.

This paper is organized as follows. In the next Section, we review the fractional calculus and the stability of the fractional–order systems. The model of FOPLL will be given in Section 3. After explaining chaos controlling for FOPLL in Section 4, the synchronization between two FOPLLs is described in Section 5. Finally, Section 6 draws some concluding remarks.

2 Fractional calculus review

The fractional–order differentiator can be denoted by a general fundamental operator ${}_aD_t^\alpha$ as a generalization of the differential and integral operators Caponetto *et al.*[2], which is defined as follows

$$
{}_aD_t^\alpha = \begin{cases} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} & R(\alpha) > 0, \\ 1 & R(\alpha) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-\alpha} & R(\alpha) < 0, \end{cases}
$$

where a is the initial value, in addition, α is the fractional order which can also be complex, and $R(\alpha)$ is the real part of the fractional order. The commonly used definition of fractional derivative is Grunwald-Letnikov definition which is described as the following form

$$
{}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t-jh),
$$

where [.] means the integer part. Other way to study fractional–order function is applied the Laplace transform of fractional derivative as

$$L\left\{{}_0D_t^{\alpha}f\left(t\right)\right\} = s^{\alpha}F\left(s\right).$$

The commensurate fractional–order linear time–invariant system can be presented by the state–space model

$$\begin{cases} {}_0D_t^{\alpha}\mathbf{x}\left(t\right) = \mathbf{A}\mathbf{x}\left(t\right) + \mathbf{B}\mathbf{u}\left(t\right), \\ \mathbf{y}\left(t\right) = \mathbf{C}\mathbf{x}\left(t\right), \end{cases}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^r$ and $\mathbf{y} \in R^p$ are the state, input and output vectors of the system and $\mathbf{A} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times r}$, $\mathbf{C} \in R^{p\times n}$. According to Matignon[8], the system is stable if it satisfies the condition:

$$\left|\arg\left(\mathrm{eig}\left(\mathbf{A}\right)\right)\right| > \alpha\frac{\pi}{2}, \tag{1}$$

where $0 < \alpha < 1$ and $\mathrm{eig}\left(\mathbf{A}\right)$ is the eigenvalues of matrix $\mathbf{A}$.

On the other hand, the commensurate fractional order nonlinear system could be described by

$${}_0D_t^{\alpha}\mathbf{x} = \mathbf{f}\left(\mathbf{x}\right),$$

where $0 < \alpha < 1$ and $\mathbf{x} \in R^n$. The equilibrium points of system are asymptotically stable Tavazoei and Haeri[9] if the following condition is satisfied:

$$\left|\arg\left(\mathrm{eig}\left(\mathbf{J}\right)\right)\right| = \left|\arg\left(\lambda_i\right)\right| > \alpha\frac{\pi}{2}, \tag{2}$$

where λ_i are the eigenvalues of the Jacobian matrix $\mathbf{J}$, which is evaluated at the equilibrium points.

3 Mathematical model of fractional–order differential equation–based phase–locked loop

The model of conventional phase–locked loop is considered firstly. The PLL contains three main components: a phase detector (PD), a loop filter (LF) and a VCO as shown in Fig. 1. PD compares the phase of input signal against the phase of VCO and creates the control voltage which is applied to VCO to change the VCO frequency. As the result, the average phase of the VCO tracks the average phase of input. To analyse the dynamical features of PLL, its model in term of phase is presented as in Fig. 2.

In Fig. 2, θ_i and θ_o denote the input and output phase, respectively; while $\phi = \theta_i - \theta_o$ is the phase error. There, PD is a mixer and LF is a first order filter with the transfer function

$$F_{LPF}\left(s\right) = \frac{1}{1+\tau s},$$

where τ is time constant and s is an operator denoting $\frac{d}{dt}$. The differential equation that characterizes the PLL can be written as

$$\frac{d^2\phi}{dt^2} + \beta\frac{d\phi}{dt} + \sin\phi = \beta\sigma + \beta M\sin\left(\Omega t\right) + M\Omega\cos\left(\Omega t\right),$$

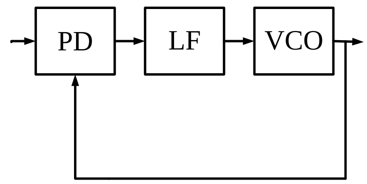

Fig. 1. Block diagram of the typical phase–locked loop.

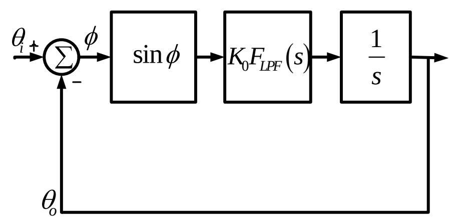

Fig. 2. Phase model of the typical phase–locked loop.

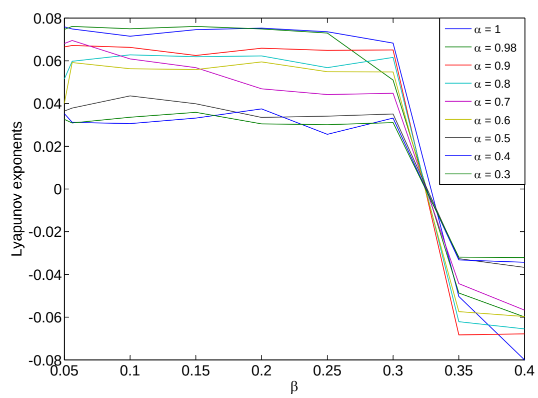

Fig. 3. Largest Lyapunov exponents according to β and of the typical integer–order phase–locked loop and the phase–locked loops based on fractional–order differential equation when $M = 0.8$, $\Omega = 0.7$.

where β, σ, Ω, M are normalized natural frequency, normalized frequency detuning, normalized modulation frequency and normalized maximum frequency derivation, respectively. Let $x_1 = \phi$ and $x_2 = \dot{\phi}$, the previous equation has the following form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\beta x_2 - \sin x_1 + \beta\sigma + \beta M \sin(\Omega t) + M\Omega \cos(\Omega t) . \end{cases}$$

The variation of Lyapunov exponents Wolf *et al.*[10] when the parameter β changes in the range $[0.05, 0.4]$ is given in Fig. 3. For the parameter value $\beta = 0.056$, integer–order PLL is chaotic since the Lyapunov exponent is positive. By replacing integer–order derivatives in above equation by fractional–order

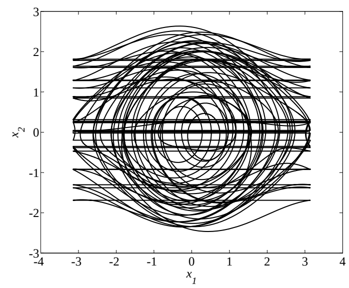

Fig. 4. Chaotic attractor in FOPLL with $\alpha = 0.98$.

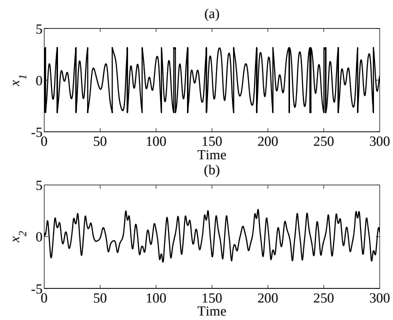

Fig. 5. Chaotic time domain representations of FOPLL with $\alpha = 0.98$: (a) x_1, (b) x_2.

ones, the fractional–order differential equation–based PLL is introduced as

$$\begin{cases} {}_0D_t^{\alpha} x_1 = x_2 \\ {}_0D_t^{\alpha} x_2 = -\beta x_2 - \sin x_1 + \beta\sigma + \beta M \sin(\Omega t) + M\Omega\cos(\Omega t), \end{cases} \tag{3}$$

where α is the derivative order. We found the presence of chaos in fractional–order PLL equation by observing the largest Lyapunov exponent (see Fig. 3). When $\alpha = 0.98$, $\beta = 0.056$, $\sigma = 0.2$, $M = 0.8$, $\Omega = 0.7$ the fractional–order differential equation–based phase–locked loop exhibits chaos behaviour. The phase portrait and time response are illustrated in Figs. 4, 5. It is notable that there are some similarities between two kinds of phase–locked loops in dynamical behaviour (Fig. 3) as well as tracking range (Fig. 6).

4 Chaos control in fractional–order differential equation–based phase–locked loop

Control chaos is a progress to manage the unexpected performances in diverse areas of research such as biology, physiology, fluid mechanics, electronics, chemical engineering, and so on [11]. The study of chaos control in this Section

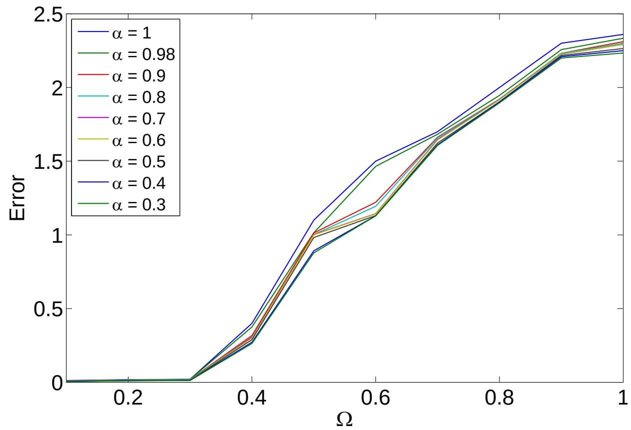

Fig. 6. Tracking errors of the typical integer–order phase–locked loop and the phase–locked loops based on fractional–order differential equation versus the derivative order α.

provides the designer one tool to develop applications involving FOPLL without undesired chaos. Specifically, FOPLL (3) could be rewritten in the matrix form as follows

$${}_0D_t^{\alpha}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f}\left(\mathbf{x}\right) + \mathbf{u},$$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{f}\left(\mathbf{x}\right) = \begin{bmatrix} 0 \\ \sin x_1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 0 \\ \beta\sigma + \beta M \sin\left(\Omega t\right) + M\Omega\cos\left(\Omega t\right) \end{bmatrix}$. To control chaos, one addition control term $\mathbf{u}_c$ is applied in FOPLL. Hence, the controlled FOPLL system can be obtained as

$${}_0D_t^{\alpha}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f}\left(\mathbf{x}\right) + \mathbf{u} + \mathbf{u}_c.$$

There, by combining the feedback control method Schöll and Schuster[11] and condition (2), $\mathbf{u}_c$ is selected as

$$\mathbf{u}_c = \begin{bmatrix} 0 \\ k\left(x_1 + x_2\right) - \beta\sigma - \beta M \sin\left(\Omega t\right) - M\Omega\cos\left(\Omega t\right) \end{bmatrix}.$$

Therefore, the controlled FOPLL system has the reduced form

$${}_0D_t^{\alpha}\mathbf{x} = \mathbf{g}\left(\mathbf{x}\right), \tag{4}$$

in which

$$\mathbf{g}\left(\mathbf{x}\right) = \begin{bmatrix} x_2 \\ -\sin x_1 + kx_1 + \left(k-\beta\right)x_2 \end{bmatrix}.$$

The Jacobian matrix of (4) is given by

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ k - \cos x_1 & k - \beta \end{bmatrix}. \tag{5}$$

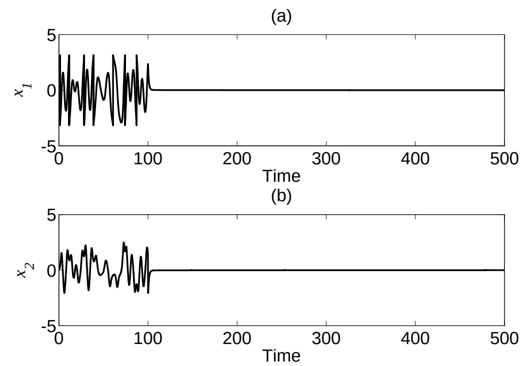

Fig. 7. Times series plots of the controlled FOPLL (a) $x_1(t)$, (b) $x_2(t)$. Controller is turned on at $t = 100$.

Replacing equilibrium points of system (4) $\mathbf{x}^* = \begin{bmatrix} x_1^* \; x_2^* \end{bmatrix} = \begin{bmatrix} 0 \; 0 \end{bmatrix}$ into the Jacobian matrix, we have

$$\mathbf{J}\,|_{\mathbf{x}=\mathbf{x}^*} = \begin{bmatrix} 0 & 1 \\ k-1 & k-\beta \end{bmatrix}. \tag{6}$$

The eigenvalues λ_i of $\mathbf{J}\,|_{\mathbf{x}=\mathbf{x}^*}$ are the solutions to the equation

$$\det\left(\mathbf{J}\,|_{\mathbf{x}=\mathbf{x}^*} - \lambda\mathbf{I}\right) = 0. \tag{7}$$

By choosing the parameter k such that the condition (2) satisfies, equilibrium points of FOPLL (4) are asymptotically stable. To illustrate the effectiveness of the proposed controlling approach, numerical simulation is implemented with the chosen parameter $k = -16$, which makes (7) has two separated negative real solutions. Simulation results of chaos control are displayed in Fig. 7. Obviously, after applying the controlling process, FOPLL works in locked region where phase and frequency errors equal zeros.

5 Synchronization of chaos in fractional–order differential equation–based phase–locked loops

Synchronization in chaotic systems has received amount of attention because of practical application such as secure chaotic communication. As reporting in Section 3, FOPLL can be chaotic; hence, the synchronization in two FOPLL systems can derive the secure communication systems. In this Section, the synchronous scheme between two FOPLLs, named master and slave system Boccaletti[12] is demonstrated. This synchronous scheme is based on the nonlinear state observer. Similar to the previous Section, the master is defined as

$${}_0D_t^{\alpha}\mathbf{x}_m = \mathbf{A}\mathbf{x}_m + \mathbf{B}\mathbf{f}\left(\mathbf{x}_m\right) + \mathbf{u},$$

where $\mathbf{x}_m = \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix}$, $\mathbf{f}\left(\mathbf{x}_m\right) = \begin{bmatrix} 0 \\ \sin x_{1m} \end{bmatrix}$. While the slave system is built as follows

$${}_0D_t^{\alpha}\mathbf{x}_s = \mathbf{A}\mathbf{x}_s + \mathbf{B}\mathbf{f}\left(\mathbf{x}_m\right) + \mathbf{u} + \mathbf{K}\mathbf{e},$$

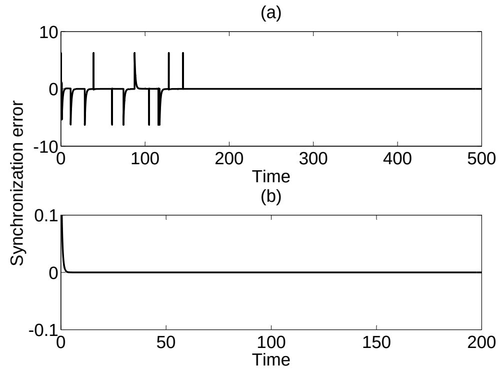

Fig. 8. Synchronization behaviour of two FOPLLs (a) $x_{1m} - x_{1s}$, (b) $x_{2m} - x_{2s}$.

where $\mathbf{x}_s = \begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix}$, $\mathbf{K} \in R^{2\times 2}$ is the feedback gain matrix and $\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_{1m} - x_{1s} \\ x_{2m} - x_{2s} \end{bmatrix}$ is the synchronization error. The dynamical synchronization error of system could be written as

$$ {}_0D_t^\alpha \mathbf{e} = {}_0D_t^\alpha \mathbf{x}_m - {}_0D_t^\alpha \mathbf{x}_s = (\mathbf{A} - \mathbf{K})\,\mathbf{e}. \tag{8} $$

The synchronization occurs when $\mathbf{K}$ is chosen appropriately such that

$$ \lim_{t\to\infty} \|\mathbf{x}_m - \mathbf{x}_s\| = \lim_{t\to\infty} \|\mathbf{e}\| = 0. $$

It is clear to see that (8) is the fractional–order linear time–invariant system; hence, the condition (1) could be applied to find matrix $\mathbf{K}$. There, $\mathbf{K}$ is achieved as

$$ \mathbf{K} = \begin{bmatrix} 1 & 1 \\ 0 & 2-\beta \end{bmatrix}, $$

which makes $|\arg(\mathrm{eig}(\mathbf{A} - \mathbf{K}))| > \alpha\frac{\pi}{2}$ by the eigenvalues $\mathrm{eig}(\mathbf{A} - \mathbf{K}) = \begin{bmatrix} -1 & -2 \end{bmatrix}$. Simulation results in Fig. 8 show the feasibility of the proposed synchronization scheme.

6 Conclusion

We have introduced a novel model of fractional–order differential equations of phase–locked loops. Some principal characteristics of FOPLL, such as phase tracking and chaos, are investigated. The control of chaotic behaviour in FOPLL is implemented in order to guarantee the precision on phase tracking in practice. Moreover, synchronization scheme based on nonlinear state observer is performed. We have also obtained the simulation results which fixed with theoretical analyses. In our future researches on this subject, the discovery of novel features and promising applications of the FOPLL will be estimated.

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