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Extensions of Verhulst Model in Population Dynamics and Extremes

M. Fátima Brilhante[1], M. Ivette Gomes[2], and Dinis Pestana[3]

[1] Universidade dos Açores (DM) and CEAUL, Ponta Delgada, Açores, Portugal
(E-mail: fbrilhante@uac.pt)
[2] CEAUL — Centro de Estatística e Aplicações da Universidade de Lisboa, Lisboa, Portugal
(E-mail: ivette.gomes@fc.ul.pt)
[3] CEAUL — Centro de Estatística e Aplicações da Universidade de Lisboa, Lisboa, Portugal
(E-mail: dinis.pestana@fc.ul.pt)

Abstract. Starting from the Beta(2,2) model, connected to the Verhulst logistic parabola, several extensions are discussed, and connections to extremal models are revealed. Aside from the classical General Extreme Value Model from the independent, identically distributed case, extreme value models in randomly stopped extremes schemes are discussed. In this context, the classical logistic Verhulst model is a max-geo-stable model, i.e. the geometric thinning of the observations curbs down growth to sustainable patterns. The general differential models presented are a unified approach to population dynamics growth, with factors of the form $[-\ln(1-N(t))]^{P-1}$ and the linearization $[N(t)]^{p-1}$ modeling two very different growth patterns, and factors of the form $[-\ln N(t)]^{Q-1}$ and the linearization $[1-N(t)]^{q-1}$ leading to very different ambiental resources control of the growth behavior.
Keywords: Verhulst logistic model, Beta and BeTaBoOp models, population dynamics, extreme value models, geometric thinning, randomly stopped maxima with geometric subordinator.

1 Introduction

Let $N(t)$ denote the size of some population at time t. Verhulst ([22], [23], [24]) imposed some natural regularity conditions on $N(t)$, namely that

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)=\sum_{k=0}^{\infty}A_k[N(t)]^k,$$

with $A_0=0$ since nothing can stem out from an extinct population, $A_1>0$ a 'growing' parameter, $A_2<0$ a retroaction parameter controlling sustainable growth tied to available resources. See also Lotka, [14].

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The second order approximation $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = A_1 N(t) + A_2[N(t)]^2$ can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\left[1 - \frac{N(t)}{K}\right], \tag{1}$$

where $r > 0$ is frequently interpreted as a Malthusian instantaneous growth rate parameter, whenever modeling natural breeding populations, and $K > 0$ as the equilibrium limit size of the population.

The general form of the solution of the differential equation approximation, in (1), is the family of logistic functions

$$N(t) = \frac{K\,N_0}{N_0 + (K - N_0)\,\mathrm{e}^{-rt}},$$

where N_0 is the population size at time $t = 0$. This is the reason why in the context of population dynamics $r\,x\,(1-x)$ is frequently referred to as 'the logistic parabola'.

Due to the seasonal reproduction and time life of many natural populations, the differential equation in (1) is often discretised, first taking r^* such that $N(t+1) - N(t) = r^*\,N(t)\,[1 - N(t)/K]$ and then $\alpha = r^* + 1$, $x(t) = \frac{r^*\,N(t)}{r^*+1}$, to obtain $x(t+1) = \alpha\,x(t)[1 - x(t)]$, and then the associated difference equation

$$x_{n+1} = \alpha\,x_n\,(1 - x_n), \tag{2}$$

where it is convenient to deal with the assumption $x_n \in [0,1],\ n = 1, 2, \ldots$

The equilibrium $x_{n+1} = x_n$ leads to a simple second order algebraic equation with positive root $1 - 1/\alpha$, and to a certain extent it is surprising that anyone would care to investigate its numerical solution using the fixed point method, which indeed brings in many pathologies when a steep curve — i.e., for some values of the iterates $|\alpha\,(1 - 2\,x_n)| > 1$ — is approximated by an horizontal straight line. This numerical investigation, apparently devoid of interest, has however been at the root of many theoretical advances (namely Feigenbaum bifurcations and ultimate chaotic behavior), and *a posteriori* led to many interesting breakthroughs in the understanding of population dynamics.

Observe also that (2) can be rewritten as $x_{n+1} = \frac{\alpha}{6}\,6\,x_n\,[1 - x_n]$, and that $f(x) = 6\,x\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ is the $Beta(2,2)$ probability density function (pdf). Extensions of the Verhulst model using difference equations similar to (2), but where the right hand side is tied to a more general $Beta(p,q)$ pdf,

$$f_{p,q}(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\,\mathrm{I}_{(0,1)}(x), \tag{3}$$

where as usual

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

is Euler's beta function, have been investigated in Aleixo *et al.*, [1], and in Rocha *et al.*, [19].

Herein we consider further extensions of population dynamics first discussed in Pestana *et al.* [15], Brilhante *et al.* [5] and Brilhante *et al.* [3], whose inspiration has been to remark that $1 - x$ is the linear truncation of the series expansion of $-\ln x$, as well as x is the linear truncation of the series expansion of $-\ln(1-x)$.

In Section 2, we describe the $BeTaBoOp(p, q, P, Q)$, $p, q, P, Q > 0$ family of pdfs, with special focus on subfamilies for which one at least of those shape parameters is 1. In Section 3, we discuss generalised Verhulst differential equations and connect them to extreme value theory (EVT). In Section 4, some further points tying population dynamics and statistical extreme value models are discussed, namely the connection of the instantaneous growing factors x^{p-1} and $[-\ln(1-x)]^{P-1}$ to models for minima, and of the retroaction control factors $(1-x)^{q-1}$ and $[-\ln x]^{Q-1}$ to modeling population growth using maxima extreme value models — either in the classical extreme value setting or in the geo-stable setting, where the geometric thinning curbs down growth to sustainable patterns. Section 5 discusses what should be expected from some specially remarkable differential description of growth in terms of products of independent uniform random variables, and products of maxima and minima of two independent uniforms.

2 The $X_{p,q,P,Q} \frown BeTaBoOp(p, q, P, Q)$ models, $p, q, P, Q > 0$

Let $\{U_1, U_2, \ldots, U_Q\}$ be independent and identically distributed (iid) standard uniform random variables,

$$V = \prod_{k=1}^{Q} U_k^{\frac{1}{p}}, \quad p > 0,$$

the product of iid $Beta(p, 1)$ random variables. As $-\ln V \frown Gamma(Q, \frac{1}{p})$, the pdf of V is

$$f_V(x) = \frac{p^Q}{\Gamma(Q)}\, x^{p-1}(-\ln x)^{Q-1} \mathrm{I}_{(0,1)}(x).$$

While for the interpretation of V as a product of powers of independent uniform random variables the parameter Q must be an integer, the above expression makes sense for all $Q > 0$. This led Brilhante *et al.* [5] to introduce the so-called $Betinha(p, Q)$ family of random variables $\{X_{p,Q}\}$, $p, Q > 0$, with pdf

$$f_{X_{p,Q}}(x) = \frac{p^Q}{\Gamma(Q)}\, x^{p-1}(-\ln x)^{Q-1}\, \mathrm{I}_{(0,1)}(x),\ p, Q > 0, \tag{4}$$

to derive population growth models that do not comply with the sustainable equilibrium exhibited by the Verhulst logistic growth model. Observe that the $Beta(p, q)$, $p, q > 0$ family, in (3), can be viewed as a truncation approximation of this more flexible $Betinha(p, Q)$, in (4), since $1 - x$ is the linear term of the MacLaurin expansion $-\ln x = \sum_{k=1}^{\infty} (1-x)^k/k$.

On the other hand, if $X_{q,P} \frown Betinha(q,P)$, the pdf of $1 - X_{q,P}$ is

$$f_{1-X_{q,P}}(x) = \frac{q^P}{\Gamma(P)}\,(1-x)^{q-1}(-\ln(1-x))^{P-1}\,\mathrm{I}_{(0,1)}(x),\;\; q,P>0,$$

and the family of such random variables also extends the $Beta(p,q)$ family in the sense that x is the linearization of $-\ln(1-x)$.

Having in mind Hölder's inequality, it follows that

$$x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}(-\ln\, x)^{Q-1} \in \mathcal{L}^1_{(0,1)},\;\; p,q,P,Q>0,$$

and hence

$$f_{p,q,P,Q}(x) = \frac{x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}(-\ln\, x)^{Q-1}\mathrm{I}_{(0,1)}(x)}{\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}(-\ln\, x)^{Q-1}\mathrm{d}x} \tag{5}$$

is a pdf of a random variable $X_{p,q,P,Q}$ for all $p,q,P,Q>0$.
Obviously, $1 - X_{p,q,P,Q} = X_{q,p,Q,P}$.

Brilhante *et al.* [3] used the notation $X_{p,q,P,Q} \frown BeTaBoOp(p,q,P,Q)$ for the random variable with pdf (5) — obviously the $Beta(p,q),\; p,q>0$ family of random variables, in (3), is the subfamily $BeTaBoOp(p,q,1,1)$, and the formerly introduced $Betinha(p,Q),\; p,Q>0$, in (4), is in this more general setting the $BeTaBoOp(p,1,1,Q)$ family. The cases for which some of the shape parameters are 1 and the other parameters are 2 are particularly relevant in population dynamics. In the present paper, we shall discuss in more depth $X_{p,1,1,Q}$ and $X_{1,q,P,1}$, and in particular $X_{2,1,1,2}$ and $X_{1,2,2,1}$.

Some of the 15 subfamilies when one or more of the 4 shape parameters p,q,P,Q are 1 have important applications in modeling. Below we enumerate the most relevant cases, giving interpretations in terms of products of powers of independent $U_k \frown Uniform(0,1)$ random variables, for integer parameters and whenever feasible.

1. $X_{1,1,1,1} = U \frown Uniform(0,1)$,
$$f_{1,1,1,1}(x) = \mathrm{I}_{(0,1)}(x).$$
2. $X_{p,1,1,1} = U^{\frac{1}{p}} \frown Beta(p,1)$,
$$f_{p,1,1,1}(x) = p\,x^{p-1}\mathrm{I}_{(0,1)}(x).$$
3. $X_{1,q,1,1} = 1 - U^{\frac{1}{q}} \frown Beta(1,q)$,
$$f_{1,q,1,1}(x) = q\,(1-x)^{q-1}\mathrm{I}_{(0,1)}(x).$$
4. $X_{1,1,P,1}$, that for $P \in \mathbb{N}$ is 1 minus the product of P iid standard uniform random variables,
$$X_{1,1,P,1} = 1 - \prod_{k=1}^{P} U_k, \quad U_k \frown Uniform(0,1), \text{independent.}$$

More generally, for all $P > 0$,

$$f_{1,1,P,1}(x) = \frac{(-\ln(1-x))^{P-1}}{\Gamma(P)}\,\mathrm{I}_{(0,1)}(x),$$

where $\Gamma(P) = \int_0^\infty x^{P-1}\mathrm{e}^{-x}\mathrm{d}x$ is Euler's gamma function.

5. $X_{1,1,1,Q}$, that for $Q \in \mathbb{N}$ is the product of P iid standard uniform random variables,

$$X_{1,1,1,Q} = \prod_{k=1}^{Q} U_k, \quad U_k \frown Uniform(0,1), \text{independent}.$$

Alternatively, $X_{1,1,1,Q}$ can be described in the following hierarchical construction: Let $Y_1 \overset{\mathrm{d}}{=} X_{1,1,1,1} \frown Uniform(0,1)$, $Y_2 \frown Uniform(0,Y_1)$, $Y_3 \frown Uniform(0,Y_2)$, $\ldots$, $Y_Q \frown Uniform(0,Y_{Q-1})$. Then $Y_Q \overset{\mathrm{d}}{=} X_{1,1,1,Q} \frown BeTaBoOp(1,1,1,Q)$.
More generally, for all $Q > 0$,

$$f_{1,1,1,Q}(x) = \frac{(-\ln x)^{Q-1}}{\Gamma(Q)}\,\mathrm{I}_{(0,1)}(x).$$

6. $X_{p,q,1,1} \frown Beta(p,q)$, with pdf $f_{p,q,1,1}(x) \equiv f_{p,q}(x)$, already given in (3). Observe that if $p,q \in \mathbb{N}$, we have an interesting interpretation in terms of order statistics of an uniform random sample: $X_{p,q,1,1}$ is then the p-th ascending order statistic from an uniform random sample of size $p+q-1$, usually denoted $U_{q:p+q-1}$.
As already observed, the pdf $f_{2,2,1,1}(x) = 6\,x\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ of $X_{2,2,1,1}$ is proportional to the logistic parabola, a landmark in the development of applications of dynamic systems and chaos to analyze biological phenomena, and namely in population dynamics. Observe also that $X_{2,2,1,1}$ is $U_{2:3}$, the median of an uniform random sample of size 3.
7. $X_{p,1,P,1}$, with pdf

$$f_{p,1,P,1}(x) = C_{p,1,P,1}\, x^{p-1}\,[-\ln(1-x)]^{P-1}\,\mathrm{I}_{(0,1)}(x),$$

where $C_{p,1,P,1} = 1/\int_0^1 x^{p-1}\,[-\ln(1-x)]^{P-1}\mathrm{d}x$.
Observe that for $p \in \mathbb{N}$, $C_{p,1,P,1} = 1/\sum_{k=1}^{p}(-1)^{k+1}\binom{p-1}{k-1}\frac{\Gamma(P)}{k^P}$.
8. $X_{p,1,1,Q}$, with

$$f_{p,1,1,Q}(x) = \frac{p^Q}{\Gamma(Q)}\,x^{p-1}(-\ln\,x)^{Q-1}\,\mathrm{I}_{(0,1)}(x),$$

that for $Q \in \mathbb{N}$ is the product of Q iid $Beta(p,1)$, i.e. standard uniform random variables raised to the power $1/p$, cf. also Arnold *et al.* [2].
As $1-x$ can be viewed as the linear truncation of $-\ln\,x$, the traditional $Beta(p,q)$ family, with pdf given in (3), can be viewed as an approximation, in what concerns the retroactive curbing down factor, of this $X_{p,1,1,Q}$ family. Such a family is thus suited to model more complex growth control patterns. Observe that $(-\ln\,x)^{\nu-1} > (1-x)^{\nu-1}$ for each $\nu > 1$, while the reverse inequality holds for $\nu \in (0,1)$.

9. $X_{1,q,P,1}$, with pdf

$$f_{1,q,P,1}(x) = \frac{q^P}{\Gamma(P)}(1-x)^{q-1}\,[-\ln(1-x)]^{P-1}\,\mathrm{I}_{(0,1)}(x).$$

Similarly to what happens in the previous case, for some fixed value $\nu > 1$, the growing factors $x^{\nu-1} < [-\ln(1-x)]^{\nu-1}$, while for $\nu \in (0,1)$ $x^{\nu-1} > [-\ln(1-x)]^{\nu-1}$. As already underlined, x can be viewed as the linear truncation of $-\ln(1-x)$, and henceforth the traditional $Beta(p,q)$ family can be viewed as an approximation, in what concerns the growing factor, of this $X_{1,q,P,1}$ family, that exhibits more complex growth patterns.

10. $X_{1,q,1,Q}$, with pdf

$$f_{1,q,1,Q}(x) = C_{1,q,1,Q}\,(1-x)^{q-1}\,[-\ln\, x]^{Q-1}\,\mathrm{I}_{(0,1)}(x),$$

where $C_{1,q,1,Q} = 1/\int_0^1 (1-x)^{q-1}\,[-\ln\, x]^{Q-1}\mathrm{d}x$. More generally, the notation $C_{p,q,P,Q} = 1/\int_0^1 x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}\,[-\ln\, x]^{Q-1}\mathrm{d}x$ is used in the sequel.

11. $X_{1,1,P,Q}$, with pdf

$$f_{1,q,1,Q}(x) = C_{1,1,P,Q}\,[-\ln(1-x)]^{P-1}\,[-\ln\, x]^{Q-1}\,\mathrm{I}_{(0,1)}(x).$$

12. $X_{p,q,P,1}$, with pdf

$$f_{p,q,P,1}(x) = C_{p,q,P,1}\,x^{p-1}(1-x)^{q-1}\,[-\ln\,(1-x)]^{P-1}\,\mathrm{I}_{(0,1)}(x).$$

13. $X_{p,q,1,Q}$, with pdf

$$f_{p,q,1,Q}(x) = C_{p,q,1,Q}\,x^{p-1}(1-x)^{q-1}\,[-\ln\, x]^{Q-1}\,\mathrm{I}_{(0,1)}(x).$$

14. $X_{p,1,P,Q}$, with pdf

$$f_{p,1,P,Q}(x) = C_{p,1,P,Q}\,x^{p-1}\,[-\ln\,(1-x)]^{P-1}\,[-\ln\, x]^{Q-1}\,\mathrm{I}_{(0,1)}(x).$$

15. $X_{1,q,P,Q}$, with pdf

$$f_{1,q,P,Q}(x) = C_{1,q,P,Q}\,(1-x)^{q-1}\,[-\ln\,(1-x)]^{P-1}\,[-\ln\, x]^{Q-1}\,\mathrm{I}_{(0,1)}(x).$$

Observe that the denominator of the norming constants

$$C_{p,q,P,Q} = 1/\int_0^1 x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}\,[-\ln\, x]^{Q-1}\mathrm{d}x$$

can be viewed as moments of functions of *BeTaBoOp* random variables with additional shape parameters with value 1. For instance,

$$C_{p,q,1,Q} = \mathbb{E}_{X_{p,q,1,1}}[(-\ln\, X)^{Q-1}] = \mathbb{E}_{X_{1,q,1,1}}[X^{p-1}(-\ln\, X)^{Q-1}].$$

In what concerns the applicability of some of the above models (and namely 11–15), we have to recognize that computations are unfeasible, even if we decide to use only lower moments and approximations instead of more powerful methods using the exact model. However, computational algorithms can, at least partially, resolve this question when dealing with precise practical applications.

3 Generalised Verhulst differential equations

Looking to the Verhulst equation, in (1), and observing that in it $N(t) \propto f_{2,2,1,1}$, with $f_{p,q,1,1} \equiv f_{p,q}$, given in (3), it seems worth considering similar differential equations with $N(t) \propto f_{p,q,P,Q}$, in (5),

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[N(t)]^{p-1}[1-N(t)]^{q-1}[-\ln(1-N(t))]^{P-1}[-\ln(N(t))]^{Q-1}, \quad (6)$$

$p, q, P, Q > 0$, namely when one at least of the parameters is 1. The situation $p+q+P+Q = 6$, with $p, q, P, Q \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ seems also worth exploring.

The solution is straightforward for very simple cases, such as

- $p = q = P = Q = 1$ — linear growth;
- $p = 2,\ q = P = Q = 1$ — exponential growth;
- $q = 2,\ p = P = Q = 1$ — exponential decay;
- $p = q = 2,\ P = Q = 1$ — logistic growth.

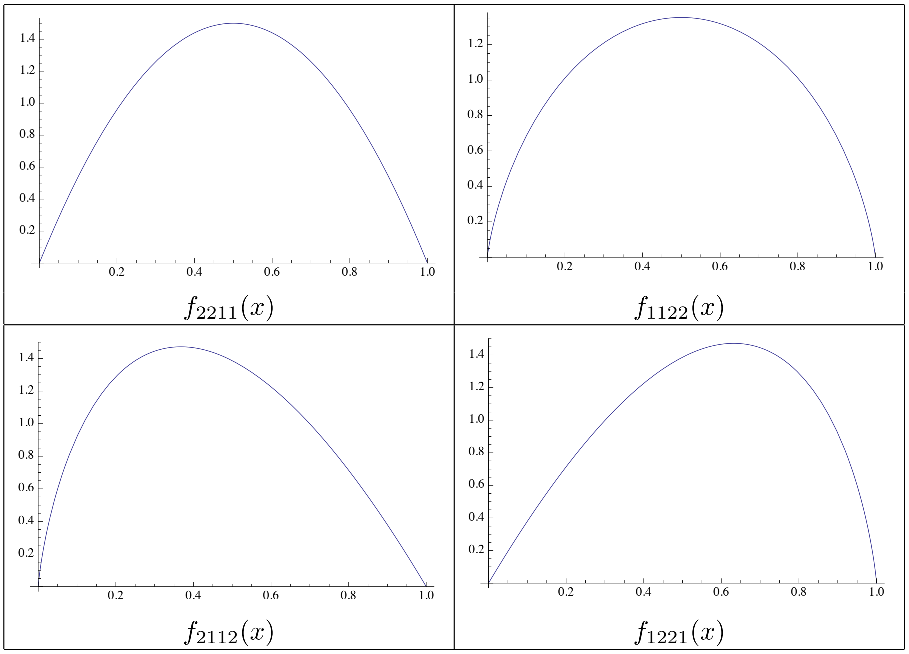
For some combinations of the parameters, *Mathematica*'s procedure DSolve produces explicit (but in general very cumbersome) solutions, for instance:

- $q = P = Q = 1 \implies N(t) = [-(p-2)\,(c+rt)]^{\frac{1}{2-p}}$;
- $p = Q = 2,\ q = P = 1 \implies N(t) = \exp(\mathrm{e}^{-rt+c})$ — Gompertzian (or Gumbel) growth;
- $p = 2,\ q = P = 1,\ Q = 1+\gamma \implies N(t) = \exp\left([-\gamma(rt-c)]^{-\frac{1}{\gamma}}\right)$ — Fréchet growth if $\gamma > 0$, Weibull growth if $\gamma \in (-1, 0)$ (when $\gamma \to 0$, the limiting growth is of Gumbel type).

Looking back at the biological interpretations of (1), it seems reasonable to consider that in (6)

- $[N(t)]^{p-1}$ and $[-\ln(1-N(t))]^{P-1}$ are growing factors;
- $[1-N(t)]^{q-1}$ and $[-\ln(N(t))]^{Q-1}$ are retroaction factors whose role in the model is to take into account bounds imposed by finite environmental resources.

Therefore, some sort of equilibrium is to be expected when $p + P = q + Q$ (although slight deviance from such equilibrium may match some forms of extreme growth or of extinction, as discussed later on). Looking at some plots gives some visual insight on the balance of the expanding and contracting factors:



More rigorous algebraic comparisons can be made. For instance, as shown in [15],

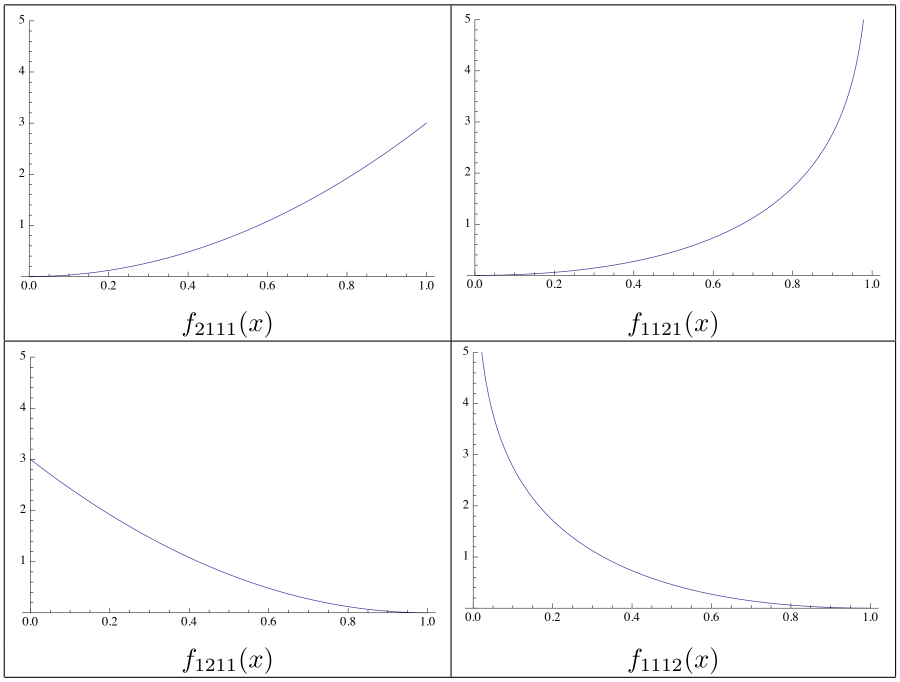
$$f_{2112}(x) = \sum_{k=1}^{\infty} \frac{4}{k(k+1)(k+2)}\, f_{2,k+1,1,1}(x),$$

and as on the other hand

$$f_{2,k+1,1,1}(x) = \sum_{j=0}^{k} \frac{(-1)^j \binom{k}{j}}{(j+2)\, B(2,k+1)}\, f_{j+2,1,1,1},$$

it follows that $X_{2,1,1,2}$ is a pseudo-convex mixture (a term we use to characterise mixtures where negative weights are allowed, provided that the sum of all weights is 1) of power laws, each positive even component forcing population growth, followed by a negative odd component counteracting this growth impetus.

The cases $p + P \gg q + Q$ and $p + P \ll q + Q$ obviously lead to explosive growth $N(t) \to \infty$ or to ultimate population extinction $N(t) \to 0$, respectively. Once again, visual insight can be gained from some simple plots:



In what follows we shall consider only those cases for which we have some explicit solutions connected to EVT, namely

1. $$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[N(t)][1 - N(t)],$$

   whose normalised solution is the Logistic distribution function, and its extension

   $$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,([N(t)])^{1+\gamma}[1 - N(t)]^{1-\gamma},$$

   whose normalised solutions are the log-logistic or the symmetrised log-logistic distribution functions.

2. $$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[N(t)][-\ln(N(t))],$$

   whose normalised solution is the Gumbel distribution function (for maxima), and its extension

   $$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[N(t)][-\ln(N(t))]^{1+\gamma},$$

   whose normalised solutions are the Fréchet distribution function (for maxima) when $\gamma > 0$, and the max-Weibull distribution function when $\gamma < 0$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[1 - N(t)][-\ln(1 - N(t))],$$
whose normalised solution is the min-Gumbel distribution function, and its extension
$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[1 - N(t)][-\ln(1 - N(t))]^{1+\gamma},$$
whose normalised solutions are the min-Fréchet distribution function when $\gamma > 0$, and the Weibull distribution function (for minima) when $\gamma < 0$.

4 Geo-stable laws for the maxima of iid random variables

Rachev and Resnick [16] developed a theory of stable limits of randomly stopped maxima with geometric subordinator (also called max-geo stability) similar to what had been independently achieved by Rényi [17], Kovalenko [12] and in all generality by Kozubowski [13]. For a panorama cf. also Gnedenko and Korolev [10].

A random variable is max-geo-stable if and only if geometric randomly stopped maxima of independent replicas is of the same Khinchine type. More precisely, if $X_1, X_2, \ldots, X_n, \ldots$ are independent replicas of X, with distribution function F, and $Y \frown \mathit{Geometric}(\theta)$ independent of the X_k's, the distribution function of $\max\{X_1, \ldots, X_Y\}$ is

$$\sum_{k=0}^{\infty} F^k(x)\theta(1-\theta)^{k-1} = \frac{\theta F(x)}{1-(1-\theta)F(x)}. \tag{7}$$

We then say that X is a max-geo-stable random variable (or that F is a max-geo-stable distribution function) if for all $\theta \in (0,1)$ there exist $a_\theta > 0$ and $b_\theta \in \mathbb{R}$ such that
$$F(a_\theta x + b_\theta) = \frac{\theta\, F(x)}{1-(1-\theta)\,F(x)}. \tag{8}$$

Let us define $G(x) = \mathrm{e}^{1-\frac{1}{F(x)}},\ x > \alpha_F$, where α_F denotes the left-endpoint of F, i. e. $\alpha_F = \inf\{x:\ F(x) > 0$. Then (8) is equivalent to

$$G(a_\theta x + b_\theta) = G^{\frac{1}{\theta}}(x), \tag{9}$$

i.e. G is a max-stable distribution. If there is no need of the shift parameter b_θ, we say that X (or F) is strictly max-geo-stable, and we get the max-stability equation $G(a_\theta x) = G^{\frac{1}{\theta}}(x)$, first investigated by Lévy in the context of stability of sums in the iid context (and so for characteristic functions, instead of distribution functions), and adapted by Fréchet, [8], to establish the max-stability of the type $G(x) = \exp\big(-x^{-\frac{1}{\gamma}}\big)\,\mathbb{I}_{[0,\infty)}(x),\ \gamma > 0$.

In fact, Fisher and Tippet, [7], have shown that distribution functions G of the type

$$G(x) \equiv G_\gamma(x) = \exp\left[-(1+\gamma x)^{-\frac{1}{\gamma}}\right] \mathrm{I}_{\{x:\ 1+\gamma x>0\}}(x),\ \gamma \in \mathbb{R}, \tag{10}$$

with the Gumbel limiting form $G_0(x) = \exp^{-e^{-x}}$, $x \in \mathbb{R}$, when $\gamma \to 0$, satisfy the functional equation (9), and Gnedenko, [9], has shown that this general extreme value (GEV) distribution (sometimes presented in three separate branches, for $\gamma > 0$ (Fréchet), $\gamma = 0$ (Gumbel) and $\gamma < 0$ (max-Weibull), while the general expression (10) is known as von Mises-Jenkinson GEV family of distributions). This, together with the characterisation of the domains of attraction of the Fréchet and Weibull types by Gnedenko, [9], and of the Gumbel type by de Haan, [11], form the core of classical EVT.

Hence, the max-geo-stable distribution functions, $F(x) = \frac{1}{1-\ln G_\gamma(x)}$, $x > \alpha_F$, are given by

$$F(x) \equiv F_\gamma(x) = \frac{1}{1 - \ln G_\gamma(x)} = \frac{1}{1 + (1+\gamma x)^{-1/\gamma}}, \quad 1 + \gamma x > 0, \tag{11}$$

The max-geo-stable distribution functions, in (11), can thus be written as one of the following types:

1. $$F(x) = \frac{1}{1 + x^{-1/\gamma}} \, \mathrm{I}_{[0,\infty)}, \;\; \gamma > 0,$$
   a log-logistic distribution (i.e., the distribution of a random variable whose natural logarithm follows the logistic distribution) tied to the classical max-stable Fréchet-γ distribution,
2. $$F(x) = \frac{1}{1 + e^{-x}} \, \mathrm{I}_{\mathbb{R}},$$
   the logistic distribution tied to the classical max-stable Gumbel extreme value distribution,
3. $$F(x) = \frac{1}{1 + (-x)^{-1/\gamma}} \, \mathrm{I}_{(-\infty,0)}, \;\; \gamma < 0,$$
   symmetric to the log-logistic, and tied to the classical max-stable Weibull-γ extreme value distribution,

as first established by Rachev and Resnick [16].

From tail equivalence results obtained by Resnick, [18], and by Cline, [6], it follows that the characterisation of the domains of attraction of max-geo-stable laws are similar to the characterisation of the domains of attraction of the classical maxima extreme value laws.

It is obvious that for the same parent population, the maximum of a geometrically thinned sequence is necessarily stochastically smaller than the maximum of the full sequence, and hence max-geo-stable laws are stochastically smaller than the corresponding classical extreme value laws, as can be seen in Fig. 1.

From the fact that $\min\{X_1, \ldots, X_n\} = -\max\{-X_1, \ldots, -X_n\}$, similar results follow in what concerns min-geo-stability. In what regards stochastic ordering, min-geo-stable laws are stochastically greater than the corresponding classical minimum extreme value laws.

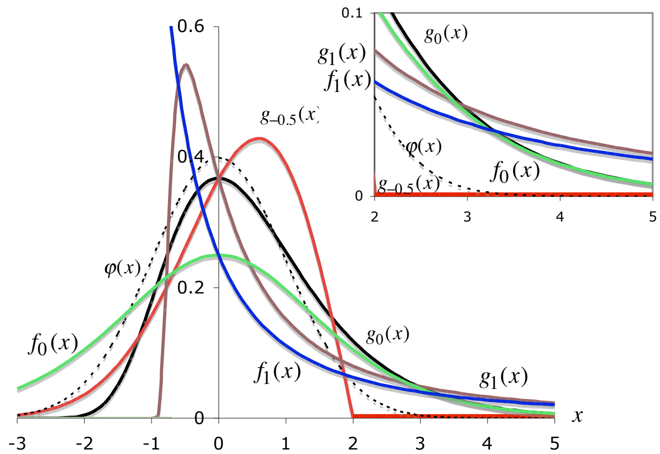


Fig. 1. Pdf's $g_\gamma(x) = dG_\gamma(x)/dx$, for $\gamma = -0.5$, $\gamma = 0$ and $\gamma = 1$, together with the normal pdf, $\varphi(x)$, and the max-geo-stable pdf's $f_0(x)$ and $f_1(x)$.

5 Population Dynamics, $\boldsymbol{BeTaBoOp(p,q,P,Q)}$ and extreme value models

Brilhante *et al.*, [3], used differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,[-\ln[N(t)]]^{1+\gamma}\,, \tag{12}$$

obtaining as solutions the three extreme value models for maxima, max-Weibull when $\gamma < 0$, Gumbel when $\gamma = 0$ and Fréchet when $\gamma > 0$. The result for $\gamma = 0$ has also been presented in Tsoularis [20] and in Waliszewski and Konarski [25], where as usual in population growth context the Gumbel distribution is called Gompertz function. Brilhante *et al.*, [3], have also shown that the associated difference equations

$$x_{n+1} = \alpha\,x_n\,[-\ln\,x_n]^{1+\gamma}$$

exhibit bifurcation and ultimate chaos, when numerical root finding using the fixed point method, when $\alpha = \alpha(\gamma)$ increases beyond values maintaining the absolute value of the derivative limited by 1.

On the other hand, if instead of the right hand side $N(t)\,[-\ln[N(t)]]^{1+\gamma}$ associated to the $BeTaBoOp(2,1,1,2+\gamma)$ we use as right hand side $[-\ln[1-N(t)]]^{1+\gamma}\;[1-N(t)]$, associated to the $BeTaBoOp(1,2+\gamma,2,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\;[-\ln[1-N(t)]]^{1+\gamma}\;[1-N(t)]$$

the solutions obtained are the corresponding extreme value models for minima (and bifurcation and chaos appear when solving the associated difference equations using the fixed point method). In view of the duality of extreme order statistics for maxima and for minima, in the sequel we shall restrict our observation to the case (12) and the associated $BeTaBoOp(2,1,1,2+\gamma)$ model.

As

$$-\ln N(t)=\sum_{k=1}^{\infty}\frac{[1-N(t)]^k}{k}>1-N(t),$$

for the same value of the malthusian instantaneous growth parameter r we have $r\,N(t)\,[1-N(t)]<r\,N(t)\,(-\ln[N(t)])$, and hence while Verhulst differential equation (1) models sustainable growth in view of the available resources, extreme value differential equations (12) model extreme, arguably destructive unsustained growth — for instance cell growth in tumours.

The connection to EVT suggests further observations:

Assume that U_1, U_2, U_3, U_4 are iid standard uniform random variables.

1. The pdf of $\min(U_1,U_2)$ is $f_{\min(U_1,U_2)}(x)=2\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ and the pdf of $\max(U_1,U_2)$ is $f_{\max(U_1,U_2)}(x)=2\,x\,\mathrm{I}_{(0,1)}(x)$. Hence the $Beta(2,2)\equiv BeTaBoOp(2,2,1,1)$ tied to the Verhulst model (1) is proportional to the product of the pdf of the maximum and the pdf of the minimum of independent standard uniforms.
2. The pdf of the product U_3U_4 is $f_{U_3U_4}(x)=-\ln x\,\mathrm{I}_{(0,1)}(x)$ — and more generally, the pdf of n independent standard uniform random variables is a $BeTaBoOp(1,1,1,n)$ — and hence the pdf of the $BeTaBoOp(2,1,1,n)$ tied to (12) is proportional to the product of $f_{\max(U_1,U_2)}$ by $f_{U_3U_4}$. Interpreting $f_{\max(U_1,U_2)}\,f_{U_3U_4}$ and $f_{\max(U_1,U_2)}\,f_{\min(U_1,U_2)}$ as 'likelihoods', this shows that the model (12) favors more extreme population growth than the model (1).
   More explicitly, the pdfs $f_{1,1,1,2}f_{U_3U_4}(x)=-\ln\,x\,\mathrm{I}_{(0,1)}(x)$ and $f_{1,2,1,1}f_{\min(U_1,U_2)}(x)=2\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ intersect each other at $x\approx 0.203188$, and scrutiny of the graph shows that the probability that $U_3\,U_4$ takes on very small values below that value is much higher than the probability of $\min(U_1,U_2)<0.203188$, and therefore the controlling retroaction tends to be smaller, allowing for unsustainable growth.
   For more on product of functions of powers of products of independent standard uniform random variables, cf. Brilhante *et al.*, [4], and Arnold *et al.*, [2].
3. The max-geo-stable laws are the logistic, the log-logistic and the symmetrised log-logistic (corresponding to the Gumbel, Fréchet and max-Weibull when there is no geometric thinning, and with a similar characterisation of domains of attraction). Hence, the classical Verhulst population growth model, in (1), can also be looked at as an extreme value model, but in a context where there exists a natural thinning that maintains sustainable growth.
   As shown in [3], non-stable extreme values (arising when the hypothesis of identically distributed random variables is dropped out) may arise when the retroaction factor is delayed.

More involved population dynamics growth differential equation models do have explicit solution for special combinations of the shape parameters. For instance, the solution of

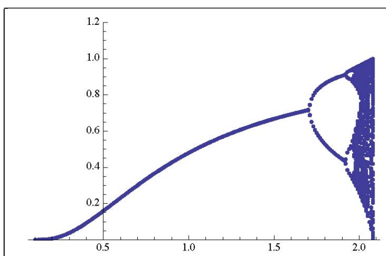
$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[N(t)]^{2-\gamma}\left[1-\frac{N(t)}{K}\right]^{\gamma}, \quad \gamma < 2, \tag{13}$$

is

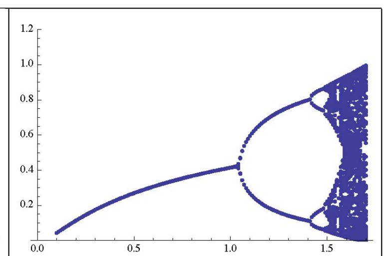
$$N(t) = \frac{K}{1+\left\{(\gamma-1)\,r\,K^{1-\gamma}t + \left(\frac{K}{N_0}-1\right)^{1-\gamma}\right\}^{\frac{1}{1+\gamma}}}$$

as shown by Turner *et al.*, [21], cf. also Tsoularis, [20].

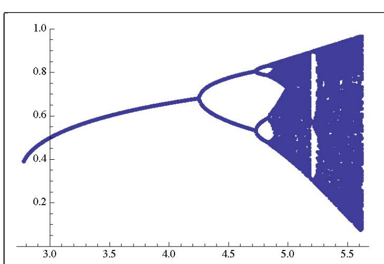
As in the case of the Verhulst parabola, the difference equations corresponding to the differential equations with BeTaBoOp kernel describing other population growth equilibria do exhibit bifurcation and ultimate chaos when the corresponding unimodal curve slope brings in instability to the fixed point algorithm, indicating that the reproduction rate and ensuing growth rate is too high (and therefore resources and sustainability are endangered, and growth rate of competing species rises concomitantly). In the figure below we show the bifurcation graphs for some combinations of the parameters:



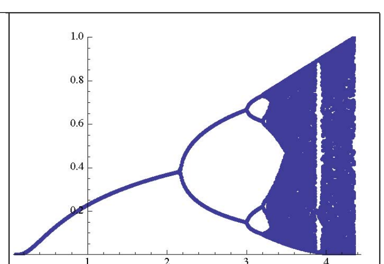
Betaboop$(1, 1, 2, 2)$



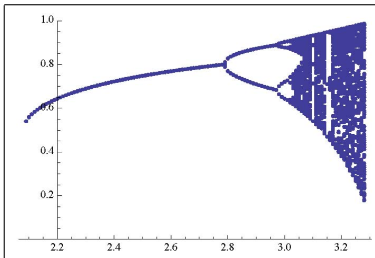
Betaboop$(1, 1, 2, 3)$



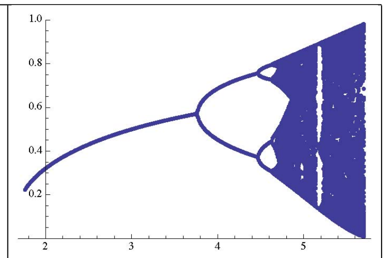
BetaBoop$(1, 2, 3, 2)$



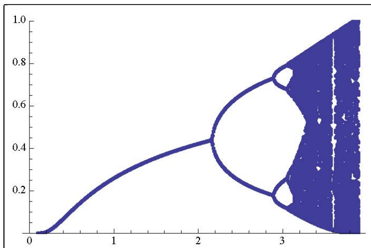
BetaBoop$(1, 3, 2, 2)$



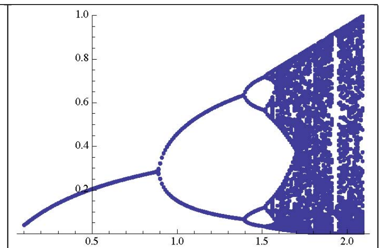
Betaboop(2, 1, 2, 2)



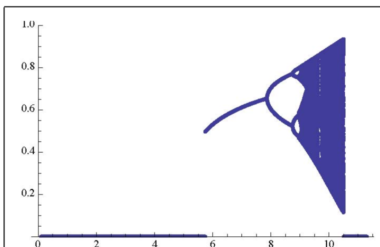
Betaboop(2, 1, 2, 3)



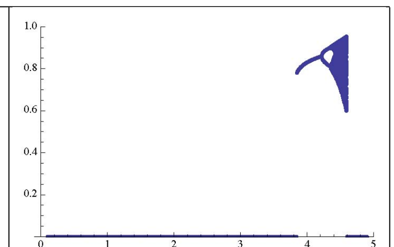
Betaboop(2, 2, 1, 2)



Betaboop(2, 2, 1, 3)



Betaboop(2, 3, 2, 1)



Betaboop(3, 2, 2, 1)

References

1. Aleixo, S., Rocha, J.L., and Pestana, D., Probabilistic Methods in Dynamical Analysis: Population Growths Associated to Models Beta (p,q) with Allee Effect, in Peixoto, M. M; Pinto, A.A.; Rand, D.A.J., editors, *Dynamics, Games and Science, in Honour of Maurício Peixoto and David Rand*, vol II, Ch. 5, pages 79–95, New York, 2011, Springer Verlag.
2. Arnold, B.C., Coelho, C.A., and Marques, F.J., The distribution of the product of powers of independent uniform random variables – A simple but useful tool to address and better understand the structure of some distributions, *J. Multivariate Analysis*, 2012, doi:10.1016/j.jmva.2011.04.006.
3. Brilhante, M.F., Gomes, M.I., and Pestana, D., BetaBoop Brings in Chaos. *Chaotic Modeling and Simulation*, 1: 39–50, 2011.

4. Brilhante, M.F., Mendonça, S., Pestana, D., and Sequeira, F., Using Products and Powers of Products to Test Uniformity, In Luzar-Stiffler, V., Jarec, I. and Bekic, Z. (eds.), *Proceedings of the ITI 2010, 32nd International Conference on Information Technology Interfaces*, IEEE CFP10498-PRT, pages 509–514, Zagreb, 2010.
5. Brilhante, M.F., Pestana, D., and Rocha, M.L., Betices, *Bol. Soc. Port. Matemática*, 177–182, 2011.
6. Cline, D., Convolution tails, product tails and domains of attraction, *Probab. Theor.* **72**, 529–557, 1986.
7. Fisher, R.A. and Tippett, L.H.C., Limiting form of the frequency distribution of the largest or smallest member of a sample, *Proc. Cambridge Philos. Soc.* **24**, 180–190, 1928.
8. Fréchet, M., Sur la loi de probabilité de l'écart maximum, *Ann. Soc. Polonaise Math.*, **6**, 93–116, 1927.
9. Gnedenko, B.V., Sur la distribution limite du terme maximum d'une série aléatoire, *Ann. Math.* **44**, 423–453, 1943.
10. Gnedenko, B.V., and Korolev, V.Yu., *Random Summation: Limit Theorems and Applications*, Bocca Raton, 1996. CRC-Press.
11. Haan, L. de, *On Regular Variation and its Application to the Weak Convergence of Sample Extremes*, Math. Centre Tracts **32**, Amsterdam, 1970.
12. Kovalenko, I.N., On a class of limit distributions for rarefied flows of homogeneous events, *Lit. Mat. Sbornik* **5**, 569–573, 1965. (Transation: On the class of limit distributions for thinning streams of homogeneous events, *Selected Transl. Math. Statist. and Prob.* **9**, 75– 81, 1971, Providence, Rhode Island.)
13. Kozubowski, T.J., Representation and properties of geometric stable laws, *Approximation, Probability, and Related Fields*, New York, 321–337, 1994, Plenum.
14. Lotka, A.J., *Elements of Physical Biology*, Baltimore, 1925, Williams and Wilkins Co (reprinted under the title *Elements of Mathematical Biology*, New York, 1956, Dover; original edition downloadable at `http://ia600307.us.archive.org/35/items/elementsofphysic017171mbp/elementsofphysic017171mbp.pdf`).
15. Pestana, D., Aleixo, S., and Rocha, J.L., Regular variation, paretian distributions, and the interplay of light and heavy tails in the fractality of asymptotic models. In C. H. Skiadas, I. Dimotikalis and C. Skiadas, editors, *Chaos Theory: Modeling, Simulation and Applications*, pages 309–316, Singapore 2011. World Scientific.
16. Rachev, S.T., and Resnick, S., Max-geometric infinite divisibility and stability, *Communications in Statistics — Stochastic Models*, 7:191-218, 1991.
17. Rényi, A., A characterization of the Poisson process, *MTA Mat. Kut. Int. Kzl.* **1**, 519–527, 1956. (English translation in *Selected Papers of Alfred Rényi,* **1***, 1948–1956*, P. Turán, editors, 622–279 Akadémiai Kiadó, Budapest, with a note by D. Szász on posterior developments untill 1976).
18. Resnick, S., Products of distribution functions attracted to extreme value laws, *J. Appl. Prob.* **8**, 781–793, 1972.
19. Rocha, J.L., Aleixo, S.M. and Pestana, D.D., $Beta(p, q)$-Cantor Sets: Determinism and Randomness. In Skiadas, C. H., Dimotikalis, Y. and Skiadas, C., editors, *Chaos Theory: Modeling, Simulation and Applications*, pages 333–340, Singapore, 2011, World Scientific.
20. Tsoularis, A., Analysis of logistic growth models. *Res. Lett. Inf. Math. Sci.*, vol. 2:23–46, 2001.
21. Turner, M.E., Bradley, E., Kirk, K., and Pruitt, K., A theory of growth. *Mathematical Biosciences*, vol. 29:367–373, 1976.
22. Verhulst, P.-F., Notice sur la loi que la population poursuit dans son accroissement. *Corresp. Math. Physics* 10:113–121, 1838. (easily available in

http://books.google.pt/books?id=8GsEAAAAYAAJ&printsec=frontcover&hl=pt-PT&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false
23. Verhulst, P.-F., La loi de l'accroissement de la population, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles*, 18:1–42, 1845. (Partially reproduced in David, H. A., and Edwards, A. W. F. *Annotated Readings in the History of Statistics*, New York, 2001, Springer Verlag; available at http://gdz.sub.uni-goettingen.de/dms/load/img/.)
24. Verhulst, P.-F., Deuxième mémoire sur la loi d'accroissement de la population. *Mémoires de l'Académie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique* 20:1–32, 1847. (http://gdz.sub.uni-goettingen.de/dms/load/img/)
25. Waliszewski, J., and Konarski, J., A Mystery of the Gompertz Function, in G.A. Losa, D. Merlini, T. F .Nonnenmacher and E.R. Weibel, editors, *Fractals in Biology and Medicine*, Basel, 277–286, 2005, Birkhuser.

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