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Resolving of the 2^N - Lotka - Volterra system of spliting logistical model for the competition problem

Vitalii V. Akimenko, Andrey A.Yefimenko

Taras Shevchenko National University of Kyiv, Ukraine
E-mail: akm15@meta.ua

Abstract: Procedure of spliting of logistic equation of population dynamics allowed to get the hierarchical 2^N Lotka - Volterra system for competitive objects and reduce of this system to the aggregate of independent equations of Bernoulli. This approach allows obtain the analytical solution of Lotka-Volterra system and got asymptotic of solution at $t \to \infty$ for the particular cases - constant coefficients of the system.

Keywords: Logistic function, Method of spliting, 2^N Lotka - Volterra system, Hierarchical competition model.

1. Introduction

The lifecycle of complex biological, economical, demographical objects in the scientific and applied works is described by logistic models, that based on the Cauchy problems for Bernoulli equation [1], [2], [3], [4], [5]. The wide class of evolution models, described the behavior of complex objects and systems, deals with simulation of competition process, interactions between their parts or subsystems, included in them. Lifecycle of complex object with multilevel internal structure of competitive objects is simulated by well known systems of Lotka-Volterra equations [3] - [9]. Formally the model of competition is created as a result of decomposition (one of the basic procedure of system analysis) of logistic model, describing the lifecycle of all system - joint group of objects, on the set of logistic models, where everyone describes the lifecycle of separate competitive object by Lotka-Volterra nonlinear differential equations [10]. In this article, using the approach of the successive hierarchical spliting of logistic problem on two problems of Lotka-Volterra for competitive subsystems and further spliting of every problem on two new, the binary tree – special graph is created with the binary spliting on every node and with the final amount of 2^N equations of Lotka-Volterra for N -level hierarchical system. It is well known that the Lotka-Volterra system in common case doesn't allow to get an analytical solution for the system of three competitive objects and more. The particular case of the binary decomposition of logistic problem considered in the article allows us to get the analytical solution of

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the system of 2^N competitive objects and the asymptotical behavior of the logistical functions for the system of equations with constant coefficients at the $t \to \infty$.

2. The Scheme of Spliting of Logistic Problem

We will suppose, that in the concerning isolated system there is a resource $\alpha_0(t) > 0$, as some non-decreasing function of time $d\alpha_0 / dt \ge 0$ for some complex object, evolving in time and identified through scalar macroparameter $Y(t)$. Lets consider the Cauchy problem for equation of Bernoulli (logistic model) :

$$Y_t = (\alpha_0(t) - Y(t))Y(t), \quad t \in [0,T], \tag{1}$$

$$Y(0) = Y_0 \tag{2}$$

where Y_0 is an initial value of macroparameter, satisfied condition $0 < Y(0) < \alpha_0(0)$. Let the complex object at the first level of hierarchy ($N = 1$) involves two competitive objects $y^{(0)}$ and $y^{(1)}$, so that $Y(t) = \sum_{k \in D_1} y^{(k)}$, where D_N set of binary numbers with N digits. Let put this decomposition in equation (1), initial condition (2) and input additional member from equation of Lotka - Volterra, modelling the process of competitive activity :

$$(y^{(0)} + y^{(1)})_t = (\alpha_0(t) - (y^{(0)} + y^{(1)}))(y^{(0)} + y^{(1)}) \pm \gamma_{0,1}(t) y^{(0)} y^{(1)}$$

$$(y^{(0)}(0) + y^{(1)}(0)) = Y_0 .$$

Break up this system on two, using the principle of symmetry and Lotka-Volterra model for the competitive systems:

$$y_t^{(0)} = (\alpha_0 - Y(t))y^{(0)} + \gamma_{0,1}(t) y^{(0)} y^{(1)} , \tag{3}$$

$$y_t^{(1)} = (\alpha_0 - Y(t))y^{(1)} - \gamma_{1,0}(t) y^{(1)} y^{(0)} , \tag{4}$$

$$y^{(0)}(0) = y_0^{(0)}, \quad y^{(1)}(0) = y_0^{(1)} . \tag{5}$$

where $\gamma_{0,1}(t) = \gamma_{1,0}(t)$ is a coefficient of competition between objects $y^{(0)}$ and $y^{(1)}$ ($\gamma_{k,s}(t) \in (-1;1)$, $s, k \in D_1$), $y_0^{(k)} > 0$ - initial values of objects macro-parameters, thus $0 < Y(0) = \sum_{k \in D_1} y_0^{(k)} < \alpha_0(0)$. Let each of the objects at the first level of hierarchy contains two competitive objects at the second level ($N = 2$) of hierarchy $y^{(0)} = y^{(00)} + y^{(01)}$, $y^{(1)} = y^{(10)} + y^{(11)}$, so that $Y(t) = \sum_{k \in D_2} y^{(k)}$. Put this decompositions in a problem (3), (4), (5) and, using the algorithm of spliting with additional members, describing the competition process, we get the system for competitive objects at second level of hierarchy :

$$y_t^{(00)} = (\alpha_0(t) - Y(t))y^{(00)} + \gamma_{0,1}(t)(y^{(10)} + y^{(11)})y^{(00)} +$$

$$+ \gamma_{00,01}(t) y^{(00)} y^{(01)}, \qquad (6)$$

$$y_t^{(01)} = (\alpha_0(t) - Y(t))y^{(01)} + \gamma_{0,1}(t)(y^{(10)} + y^{(11)})y^{(01)} -$$

$$- \gamma_{00,01}(t) y^{(00)} y^{(01)}, \qquad (7)$$

$$y_t^{(10)} = (\alpha_0(t) - Y(t))y^{(10)} - \gamma_{0,1}(t)(y^{(00)} + y^{(01)})y^{(11)} +$$

$$+ \gamma_{10,11}(t) y^{(10)} y^{(11)}, \qquad (8)$$

$$y_t^{(11)} = (\alpha_0(t) - Y(t))y^{(11)} - \gamma_{0,1}(t)(y^{(00)} + y^{(01)})y^{(11)} -$$

$$- \gamma_{11,10}(t) y^{(10)} y^{(11)}, \qquad (9)$$

$$y^{(00)}(0) = y_0^{(00)},\ y^{(01)}(0) = y_0^{(01)},\ y^{(10)}(0) = y_0^{(10)},$$

$$y^{(11)}(0) = y_0^{(11)}. \qquad (10)$$

where $\gamma_{00,01}(t)$, $\gamma_{10,11}(t)$ - competition coefficients ($\gamma_{s,k}(t) \in (-1;1)$, $s,k \in D_2$), $y_0^{(k)} > 0$ - initial values of objects macroparameters, thus $0 < Y(0) = \sum_{k \in D_2} y_0^{(k)} < \alpha_0(0)$. Repeating procedure of spliting for N levels, we

get the hierarchical structure of interactions of 2^N objects of the system (1), (2) in a binary graph, shown on Fig.1.

3. Analytical Solution of the System (1) - (10)

Equation (1) over through replacement $Z = Y^{-1}$ is reduced to the linear equation $Z_t = 1 - \alpha_0(t)Z$. Then solution of problem (1), (2) has a view

$$Y(t) = \frac{Y_0 \Lambda(t)}{1 + Y_0 \Pi(t)}, \ \Lambda(t) = \exp\left(\int_0^t \alpha_0(\tau) d\tau \right), \ \Pi(t) = \int_0^t \Lambda(x) dx . \quad (11)$$

From (11) follows that $Y = 0$ is the special point of equation (1) and in the vicinity of this point solution (11) remains limited during any limited interval of time $t \in [0, T]$ at an initial value $Y_0 > 0$. If $\alpha_0 = const > 0$ in (11), then solution (1), (2) has a view:

$$Y(t) = \frac{Y_0 \alpha_0}{Y_0 (1 - \exp(-\alpha_0 t)) + \alpha_0 \exp(-\alpha_0 t)} \quad (12)$$

From (12) follows, that $\lim_{t \to \infty} Y(t) = \alpha_0$ (achievement of satiation by a logistic curve - complete consumption of resource by objects for the infinite interval of time). System of equations (3), (4), (5), with (11), has a view :

$$y_t^{(0)} = \left(\alpha_0(t) - (1 - \gamma_{0,1}) \frac{Y_0 \Lambda(t)}{1 + Y_0 \Pi(t)} \right) y^{(0)} - \gamma_{0,1} \, y^{(0)2} , \quad (13)$$

$$y_t^{(1)} = \left(\alpha_0(t) - (1 + \gamma_{0,1}) \frac{Y_0 \Lambda(t)}{1 + Y_0 \Pi(t)} \right) y^{(1)} + \gamma_{0,1} \, y^{(1)2} , \quad (14)$$

$$y^{(1)}(0) = y_0^{(1)}, \ \ y^{(2)}(0) = y_0^{(2)} . \quad (15)$$

By analogy with equation (1), solution of problem (13), (14), (15) can be obtained in an analytical view:

$$y^{(0)}(t) = \frac{y_0^{(0)} \Lambda^{(0)}(t)}{1 + y_0^{(0)} \Pi^{(0)}(t)} , \quad (16)$$

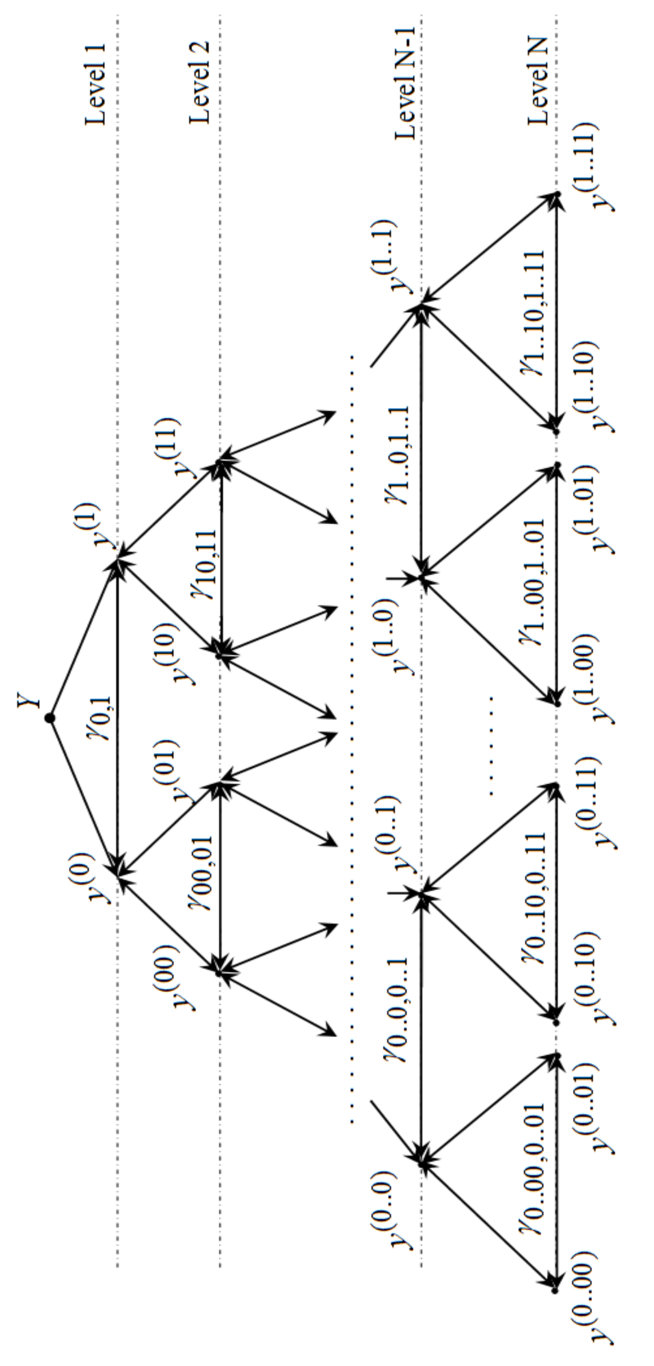

Fig. 1. A binary graph - decomposition of initial object **Y** on 2^N competitive objects $\left\{y^{(00..00)},...,y^{(11..11)}\right\}$.

$$\Lambda^{(0)}(t)=\exp\left(\int_0^t\left(\alpha_0(\tau)-(1-\gamma_{0,1}(\tau))\frac{Y_0\Lambda(\tau)}{1+Y_0\Pi(\tau)}\right)d\tau\right),$$

$$\Pi^{(0)}(t)=\int_0^t\Lambda^{(0)}(x)\gamma_{0,1}(x)dx,$$

$$y^{(1)}(t)=\frac{y_0^{(1)}\Lambda^{(1)}(t)}{1-y_0^{(1)}\Pi^{(1)}(t)}, \tag{17}$$

$$\Lambda^{(1)}(t)=\exp\left(\int_0^t\left(\alpha_0(\tau)-(1+\gamma_{0,1}(\tau))\frac{Y_0\Lambda(\tau)}{1+Y_0\Pi(\tau)}\right)d\tau\right),$$

$$\Pi^{(0)}(t)=\int_0^t\Lambda^{(1)}(x)\gamma_{0,1}(x)dx.$$

At $\alpha_0=const>0$, $\gamma_{0,1}=const\in[-1;1]$ solution (16), (17) has a view:

$$y^{(0)}=\frac{y_0^{(0)}Y_0\left(\frac{1+\nu}{1+\nu\exp(-\alpha_0 t)}\right)^{1-\gamma_{0,1}}}{y_0^{(1)}\exp(-\alpha_0\gamma_{0,1}t)+y_0^{(0)}\left(\frac{1+\nu}{1+\nu\exp(-\alpha_0 t)}\right)^{-\gamma_{0,1}}} \tag{18}$$

$$y^{(1)}=\frac{y_0^{(1)}Y_0\left(\frac{1+\nu}{1+\nu\exp(-\alpha_0 t)}\right)^{1+\gamma_{0,1}}}{y_0^{(0)}\exp(\alpha_0\gamma_{0,1}t)+y_0^{(1)}\left(\frac{1+\nu}{1+\nu\exp(-\alpha_0 t)}\right)^{\gamma_{0,1}}} \tag{19}$$

where $\nu=\alpha_0/Y_0-1>0$ - positive constant. At the positive initial conditions (15), solutions (18), (19) will be positive always at $t>0$. From the analytical view of solutions (18), (19) follows, that type of its asymptotical behavior at $t\to\infty$ is determined only by the sign of coefficient $\gamma_{0,1}$ and has a view :

$$\mathop{Lim}_{t\to\infty} y^{(0)}(t) = \begin{cases} \alpha_0, & for\ \gamma_{0,1} > 0 \\ 0, & for\ \gamma_{0,1} < 0 \end{cases},$$

$$\mathop{Lim}_{t\to\infty} y^{(1)}(t) = \begin{cases} 0, & for\ \gamma_{0,1} > 0 \\ \alpha_0, & for\ \gamma_{0,1} < 0 \end{cases},$$

that corresponds to asymptotical behavior of solution (12) and reflects basic properties of behavior of competitive objects. From here follows also the limitation of solutions of the system (3), (4), (5) at the previous limits of initial data and coefficients of the system at $t > 0$. Lets write the system (6), (7), (8), (9), (10) in a view:

$$y_t^{(00)} = (\alpha_0(t) - (1-\gamma_{0,1}(t))Y(t) + (\gamma_{00,01}(t) - \gamma_{0,1}(t))y^{(0)})y^{(00)} -$$

$$-\gamma_{00,01}(t)y^{(00)2}, \tag{20}$$

$$y_t^{(01)} = (\alpha_0(t) - (1-\gamma_{0,1}(t))Y(t) - (\gamma_{00,01}(t) + \gamma_{0,1}(t))y^{(0)})y^{(01)} +$$

$$+\gamma_{00,01}(t)y^{(01)2}, \tag{21}$$

$$y_t^{(10)} = (\alpha_0(t) - (1+\gamma_{0,1}(t))Y(t) + (\gamma_{10,11}(t) + \gamma_{0,1}(t))y^{(1)})y^{(10)} -,$$

$$-\gamma_{10,11}(t)y^{(10)2} \tag{22}$$

$$y_t^{(11)} = (\alpha_0(t) - (1+\gamma_{0,1}(t))Y(t) - (\gamma_{10,11}(t) - \gamma_{0,1}(t))y^{(1)})y^{(11)} +$$

$$+\gamma_{10,11}(t)y^{(11)2}, \tag{23}$$

$$y^{(00)}(0) = y_0^{(00)},\ y^{(01)}(0) = y_0^{(01)},\ y^{(10)}(0) = y_0^{(10)}, \tag{24}$$

$$y^{(11)}(0) = y_0^{(11)}.$$

Then solution (20), (21), (22), (23), (24) can be written in an analytical view:

$$y^{(00)}(t)=\frac{y_0^{(00)}\Lambda^{(00)}(t)}{1+y_0^{(00)}\Pi^{(00)}(t)},\ \Pi^{(00)}(t)=\int_0^t\Lambda^{(00)}(x)\gamma_{00,01}(x)dx,\quad(25)$$

$$\Lambda^{(00)}(t)=\exp\left(\int_0^t\left(\alpha_0(\tau)-(1-\gamma_{0,1}(\tau))\frac{Y_0\Lambda(\tau)}{1+Y_0\Pi(\tau)}+\right.\right.$$

$$\left.\left.+(\gamma_{00,01}(\tau)-\gamma_{0,1}(\tau))\frac{y_0^{(0)}\Lambda^{(0)}(\tau)}{1+y_0^{(0)}\Pi^{(0)}(\tau)}\right)d\tau\right),$$

$$y^{(01)}(t)=\frac{y_0^{(01)}\Lambda^{(01)}(t)}{1-y_0^{(01)}\Pi^{(01)}(t)},\ \Pi^{(01)}(t)=\int_0^t\Lambda^{(01)}(x)\gamma_{00,01}(x)dx,\quad(26)$$

$$\Lambda^{(01)}(t)=\exp\left(\int_0^t\left(\alpha_0(\tau)-(1-\gamma_{0,1}(\tau))\frac{Y_0\Lambda(\tau)}{1+Y_0\Pi(\tau)}-\right.\right.$$

$$\left.\left.-(\gamma_{00,01}(\tau)+\gamma_{0,1}(\tau))\frac{y_0^{(0)}\Lambda^{(0)}(\tau)}{1+y_0^{(0)}\Pi^{(0)}(\tau)}\right)d\tau\right),$$

$$y^{(10)}(t)=\frac{y_0^{(10)}\Lambda^{(10)}(t)}{1+y_0^{(10)}\Pi^{(10)}(t)},\ \Pi^{(10)}(t)=\int_0^t\Lambda^{(10)}(x)\gamma_{10,11}(x)dx,\quad(27)$$

$$\Lambda^{(10)}(t)=\exp\left(\int_0^t\left(\alpha_0(\tau)-(1+\gamma_{0,1}(\tau))\frac{Y_0\Lambda(\tau)}{1+Y_0\Pi(\tau)}+\right.\right.$$

$$\left.\left.+(\gamma_{10,11}(\tau)+\gamma_{0,1}(\tau))\frac{y_0^{(1)}\Lambda^{(1)}(\tau)}{1-y_0^{(1)}\Pi^{(1)}(\tau)}\right)d\tau\right),$$

$$y^{(11)}(t)=\frac{y_0^{(11)}\Lambda^{(11)}(t)}{1-y_0^{(11)}\Pi^{(11)}(t)},\ \Pi^{(11)}(t)=\int_0^t\Lambda^{(11)}(x)\gamma_{10,11}(x)dx,\quad(28)$$

$$\Lambda^{(11)}(t) = \exp\left(\int_0^t \left(\alpha_0(\tau) - (1+\gamma_{0,1}(\tau))\frac{Y_0\Lambda(\tau)}{1+Y_0\Pi(\tau)} - .\right.\right.$$

$$\left.\left. - (\gamma_{10,11}(\tau) - \gamma_{0,1}(\tau))\frac{y_0^{(1)}\Lambda^{(1)}(\tau)}{1-y_0^{(1)}\Pi^{(1)}(\tau)}\right)d\tau\right)$$

At $\alpha_0 = const > 0$, $\gamma_{0,1} = const \in [-1;1]$ the analytical solution (25), (26), (27), (28) has a view:

$$y^{(00)}(t) = \frac{Y_0^{\gamma_{0,1}}\alpha_0^{1-\gamma_{0,1}}\xi(t)^{\gamma_{0,1}-1}}{X^{(0)}}\times \tag{29}$$

$$\times\frac{1}{(y_0^{(0)-1} + Y_0^{\gamma_{00,01}/\gamma_{0,1}}\exp(-\gamma_{00,01}\alpha_0 t)X^{(0)-\gamma_{00,01}/\gamma_{0,1}}(y_0^{(00)-1} - y_0^{(0)-1}))}$$

$$y^{(01)}(t) = \frac{Y_0^{\gamma_{0,1}}\alpha_0^{1-\gamma_{0,1}}\xi(t)^{\gamma_{0,1}-1}}{X^{(0)}}\times \tag{30}$$

$$\times\frac{1}{(y_0^{(0)-1} + Y_0^{-\gamma_{00,01}/\gamma_{0,1}}\exp(\gamma_{00,01}\alpha_0 t)X^{(0)\gamma_{00,01}/\gamma_{0,1}}(y_0^{(01)-1} - y_0^{(0)-1}))}$$

$$X^{(0)} = y_0^{(1)}\exp(-\gamma_{0,1}\alpha_0 t) + y_0^{(0)}\xi(t)^{\gamma_{0,1}}Y_0^{\gamma_{0,1}}\alpha_0^{-\gamma_{0,1}},$$

$$y^{(10)}(t) = \frac{Y_0^{-\gamma_{0,1}}\alpha_0^{1+\gamma_{0,1}}\xi(t)^{-\gamma_{0,1}-1}}{X^{(1)}}\times \tag{31}$$

$$\times\frac{1}{(y_0^{(1)-1} + Y_0^{-\gamma_{10,11}/\gamma_{0,1}}\exp(-\gamma_{10,11}\alpha_0 t)X^{(1)\gamma_{10,11}/\gamma_{0,1}}(y_0^{(10)-1} - y_0^{(1)-1}))}$$

$$y^{(11)}(t) = \frac{Y_0^{-\gamma_{0,1}}\alpha_0^{1+\gamma_{0,1}}\xi(t)^{-\gamma_{0,1}-1}}{X^{(1)}}\times \tag{32}$$

$$\times \frac{1}{(y_0^{(1)-1} + Y_0^{\gamma_{10,11}/\gamma_{0,1}} \exp(\gamma_{10,11}\alpha_0 t) X^{(1)-\gamma_{10,11}/\gamma_{0,1}} (y_0^{(11)-1} - y_0^{(1)-1}))},$$

$$X^{(1)} = y_0^{(0)} \exp(\gamma_{0,1}\alpha_0 t) + y_0^{(1)} \xi(t)^{-\gamma_{0,1}} Y_0^{-\gamma_{0,1}} \alpha_0^{\gamma_{0,1}}, \qquad (33)$$

$$\xi(t) = 1 + (\alpha_0 / Y_0 - 1)\exp(-\alpha_0 t). \qquad (34)$$

At the positive initial conditions (24), solutions (29), (30), (31), (32), (33), (34) are positive always at $t > 0$. Taking into account that at $0 < Y_0 < \alpha_0$, limit $\lim_{t\to\infty} \xi(t) = 1$, the type of solutions (29), (30), (31), (32), (33), (34) asymptotical behavior at $t \to \infty$ is determined only by the signs of coefficients $\gamma_{s,k}$:

$$\lim_{t\to\infty} y^{(00)}(t) = \begin{cases} \alpha_0, & for\ (\gamma_{0,1} > 0\ and\ \gamma_{00,01} > 0); \\ 0, & for\ (\gamma_{0,1} > 0\ and\ \gamma_{00,01} < 0)\ or\ (\gamma_{0,1} < 0); \end{cases}$$

$$\lim_{t\to\infty} y^{(01)}(t) = \begin{cases} \alpha_0, & for\ (\gamma_{0,1} > 0\ and\ \gamma_{00,01} < 0); \\ 0, & for\ (\gamma_{0,1} > 0\ and\ \gamma_{00,01} > 0)\ or\ (\gamma_{0,1} < 0); \end{cases}$$

$$\lim_{t\to\infty} y^{(10)}(t) = \begin{cases} \alpha_0, & for\ (\gamma_{0,1} < 0\ and\ \gamma_{10,11} > 0); \\ 0, & for\ (\gamma_{0,1} < 0\ and\ \gamma_{10,11} < 0)\ or\ (\gamma_{0,1} > 0); \end{cases}$$

$$\lim_{t\to\infty} y^{(11)}(t) = \begin{cases} \alpha_0, & for\ (\gamma_{0,1} < 0\ and\ \gamma_{10,11} < 0); \\ 0, & for\ (\gamma_{0,1} < 0\ and\ \gamma_{10,11} > 0)\ or\ (\gamma_{0,1} > 0). \end{cases}$$

Continuing the process of decomposition by the scheme, shown on Fig.1, we get the system of 2^N equations at the N-level of hierarchy. Each equation of system is reduced to equation of Bernoulli like (13), (14), (20), (21), (21), (22), (23). A common solution of this type equation is written in form:

$$y^{(s)}(t) = \frac{y_0^{(s)} \Lambda^{(s)}(t)}{1 + y_0^{(s)} \Pi^{(s)}(t)}, \quad \Pi^{(s)}(t) = \int_0^t \Lambda^{(s)}(x) \gamma_{s,k}(x) dx,$$

($s, k \in D_N$), $|s - k| = 1$,

where $\Lambda^{(s)}$ depends from the known elementary functions, got on the previous levels of hierarchy model.

4. Conclusions

Thus, using the procedure of spliting (decomposition) for the logistic equation of population dynamics we got the hierarchical 2^N system of Lotka-Volterra for separate competitive objects and reduced this system to the set of independent equations of Bernoulli. It allowed us to get analytical solution of the Lotka-Volterra system for 2^N equations and to show asymptotical behavior of solution at $t \to \infty$ for the particular cases - constant coefficients of the system. The analytical solutions of the Lotka - Volterra system help to investigate behavior of competitive biological objects for the age-structured models of living cells competition with viruses in the problems of model identification and optimal control [11], [12].

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