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Limit Theorems for Compound Renewal Processes: Theory and Application

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Abstract. We consider a few classes of strong limit theorems for compound renewal processes (random sums, randomly stopped sums) $D(t) = \sum_{i=1}^{N(t)} X_i$ under various assumptions on the renewal counting process N(t) and random variables $\{X_i, i \geq 1\}$. First of all we present sufficient conditions for strong (a.s.) approximation of $D(t)$ by a Wiener or α-stable Lévy process under various dependent and moment conditions on summands, mainly focused on the cases of independent, φ-mixing and associated r.v. On the next step the investigation of the rate of growth of the process $D(t)$ and it's increments $D(t+a_t) - D(t)$, when a_t grows, is carried out. Useful applications in risk theory are studied; particularly, non-random bounds for the rate of growth and fluctuations of the risk processes in classical Cramer-Lundberg and renewal Sparre Andersen risk models are discussed as well as the case of risk models with stochastic premiums.
Keywords: Compound Renewal Process, Random Sum, Limit Theorem, Strong Approximation, Integral Tests, Risk Process.

1 Introduction

Let $\{X_i, i \geq 1\}$ be random variables (r.v.), $S(t) = \sum_{i=1}^{[t]} X_i$, $t > 0$, $S(0) = 0$. Also suppose that $\{Z_i, i \geq 1\}$ is a sequence of non-negative i.i.d.r.v., independent of $\{X_i\}$, with common distribution function (d.f.) $F_1(x)$, characteristic function (ch.f.) $f_1(u)$ and $EZ_1 = 1/\lambda > 0$,

$$Z(x) = \sum_{i=1}^{[x]} Z_i, \quad x > 0, \quad Z(0) = 0,$$

and define the *renewal (counting) process*

$$N(t) = \inf\{x \geq 0 : Z(x) > t\}.$$

Compound renewal processes (random sums, randomly stopped sums, compound sums) are defined as

$$D(t) = S(N(t)) = \sum_{i=1}^{N(t)} X_i,$$

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where r.v. $\{X_i, i \geq 1\}$ and renewal process $N(t)$ are given above.

Limit theorems for $D(t) = \sum_{i=1}^{N(t)} X_i$ became rather popular during last 20 years or so (mainly they deal with weak convergence). This topic is interesting not only from theoretical point of view, but also due to numerous practical applications, since mentioned processes often appear in useful applications in queuing theory (accumulated workload input into queuing system in time interval (0,t)), in risk theory (total claim amount to insurance company up to time t), in financial mathematics (total market price change up to time t) and in certain statistical procedures. The most popular example is compound Poisson process, when $N(t)$ is a a homogeneous Poisson process.

This paper presents a few classes of **strong** limit theorems for compound renewal processes (random sums), which summarize authors previous results obtained during last five years. The *first* class is *a strong invariance principle* (**SIP**), other terms are *strong approximation* or *almost sure approximation.*

Definition. We say that a random process $\{D(t), t \geq 0\}$ admits strong approximation by the random process $\{\eta(t), t \geq 0\}$, if $D(t)$ (or stochastically equivalent $D^*(t)$) can be constructed on the rich enough probability space together with $\eta(t)$ in such a way that a.s.

$$|D(t) - \eta(t)| = o(r(t)) \text{ or } O(r(t)) \text{ as } t \to \infty, \tag{1}$$

where approximating error (error term) $r(.)$ is a non-random function.

While week invariance principle provides the convergence of distributions, the strong invariance principle describes how “small” can be the difference between trajectories of $D(t)$ and approximating process $\eta(t)$.

Concrete assumptions on $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ clear up the type of approximating process and the form of error term. Below we mainly focused on the case of i.i.d.r.v. $\{X_i\}$, as well as on φ-mixing and associated summands and present some general results concerning sufficient conditions for strong approximation of $D(t)$ by a Wiener or α-stable Lévy process. Corresponding proofs are based on the rather general theorems about the strong approximation of superposition of càd-làg processes, not obligatory connected with partial sums, Zinchenko ([13], [14]).

SIP-type theorems themselves can serve as a source of a number of interesting strong limit results for compound renewal processes: really, using (1) with appropriate error term one can easily transfer the results about the asymptotic behavior of the Wiener or α-stable Lévy process on the asymptotic behavior of random sums. Thus, the *second class* of limit theorems deal with the rate of growth of $D(t)$ and it's increments. For instance, a number of integral tests for investigation the rate of growth of the process $D(t)$ and it's increments $D(t + a_t) - D(t)$, when $a_t = a(t)$ grows, are proposed. As a consequence various modifications of the LIL and Erdös-Rényi-Csörgő-Révész-type strong law of large numbers (SLLN) are obtained.

2 SIP for compound renewal processes (random sums)

2.1. Independent summands. Next three theorems (Zinchenko[13], [14]) present sufficient conditions on independent summands $\{X_i\}$ and inter-renewal intervals $\{Z_i\}$, which provide a.s. approximation of the random sums of i.i.d.r.v. with finite or infinite variance and clear up the type of approximating process and the form of error term in this case.

More precise, suppose that $\{X_i, i \geq 1\}$ are i.i.d.r.v., with common distribution function (d.f.) $F(x)$, characteristic function (ch.f.) $f^*(u)$, $EX_1 = m$, $VarX_1 = \sigma^2$ if $E|X_1|^2 < \infty$. Denote $VarZ_1 = \tau^2$, if $E|Z_1|^2 < \infty$, $\nu^2 = \sigma^2\lambda + m^2\tau^2\lambda^3$.

Theorem 1. *(i) Let $E|X_1|^{p_1} < \infty$, $E|Z_1|^{p_2} < \infty$, $p = \min\{p_1, p_2\} > 2$, then $\{X_i\}$ and $N(t)$ can be constructed on the same probability space together with a standard Wiener process $\{W(t), t \geq 0\}$ in such a way that a.s.*

$$\sup_{0 \leq t \leq T} |S(N(t)) - \lambda mt - \nu W(t)| = o(T^{1/p}); \tag{2}$$

(ii) if $p = 2$, then right side of (2) is $o(T \ln\ln T)^{1/2}$;
(iii) if $E\exp(uX_1) < \infty$, $E\exp(uZ_1) < \infty$ for all $u \in (0, u_o)$, then right-hand side of (2) is $O(\ln T)$.

Next suppose that $\{X_i\}$ are attracted to α-stable law with $1 < \alpha < 2$, $|\beta| \leq 1$, then approximating process for $S(t)$ is a stable process $Y_\alpha(t)$ (condition $\alpha > 1$ is needed to have a finite mean). Below we use following

Assumption (C) : *there are $a_1 > 0, a_2 > 0$ and $l > \alpha$ such that for $|u| < a_1$*

$$|f(u) - g_{\alpha,\beta}(u)| < a_2|u|^l, \tag{3}$$

where $f(u)$ is a ch.f. of $(X_1 - EX_1)$ if $1 < \alpha < 2$ and ch.f. of X_1 if $0 < \alpha \leq 1$, $g_{\alpha,\beta}(u)$ is a ch.f. of the stable law.

Assumption (C) not only provides normal attraction of $\{X_i, i \geq 1\}$ to the stable law $G_{\alpha,\beta}(x)$, but also leads to the rather “good” error term $q(t) = t^{1/\alpha - \varrho}$, $\varrho > 0$, in SIP for usual partial sums $S(t)$.

Theorem 2. *Let $\{X_i\}$ satisfy **(C)** with $1 < \alpha < 2$, $|\beta| \leq 1$, $EZ_1^2 < \infty$. Then $\{X_i\}$, $\{Z_i\}$, $N(t)$ can be defined together with α-stable process $Y_\alpha(t) = Y_{\alpha,\beta}(t)$, $t \geq 0$, so that a.s.*

$$\sup_{0 \leq t \leq T} \left|S(N(t)) - m\lambda t - Y_{\alpha,\beta}(\lambda t)\right| = o(T^{1/\alpha - \varrho_1}), \quad \varrho_1 \in (0, \rho_0), \tag{4}$$

for some $\varrho_0 = \varrho_0(\alpha, l) > 0$.

Corollary 1 (SIP for compound Poisson process). *Theorems 1, 2 hold if $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$, in this case $\nu^2 = \lambda EX_1^2$.*

Theorem 3. *Let $\{X_i\}$ satisfy assumption (C) with $1 < \alpha_1 < 2$ and $\{Z_i\}$ satisfy (C) with $1 < \alpha_2 < 2$, $\alpha_1 < \alpha_2$, then*

$$\sup_{0 \leq t \leq T} \left|S(N(t)) - m\lambda t - Y_{\alpha_1,\beta_1}(\lambda t)\right| = o(T^{1/\alpha_1 - \varrho_2}) \text{ a.s.} \tag{5}$$

for some $\varrho_2 = \varrho_2(\alpha_1, l) > 0$.

2.2. SIP for random sums of dependent r.v. Further development is connected with dependent summands: martingales, weakly dependent r.v., mixing and associated sequences. Below we present only few result in this area, connected with φ-mixing and associated summands; more results on this topic as well as detail rigorous proofs and wide bibliography can be find in the author's previous work [15].

Throughout this Section, unless otherwise stated, we suppose that inter-occurrence time intervals $\{Z_i\}$ for renewal process $N(t)$ have finite moments $E|Z_1|^p < \infty$ of order $p > 2$.

2.2.1. **φ-mixing sequences**. Given r.v. $\{X_i, i \geq 1\}$, let F_a^b denote the σ-field generated by $X_a, X_{a+1}, \ldots, X_b$, $a < b < \infty$, and F_b^∞ – the σ-field generated by $X_b, X_{b+1}, \ldots$.

Definition. Sequence $\{X_i, i \geq 1\}$ is said to be φ-mixing if there exist a sequence $\{\varphi(n)\}$ of real numbers, $\varphi(n) \downarrow 0$ as $n \to \infty$, such that for each $t \geq 1$, $n > 0$, $A \in F_1^t$, $B \in F_{t+n}^\infty$

$$|P(AB) - P(A)P(B)| \leq \varphi(n)P(A) \tag{6}$$

Theorem 4. *Let $\{X_i, i \geq 1\}$ be strictly stationary φ-mixing sequence with $EX_1 = m$, $E|X_1|^{2+\delta} < \infty$. Suppose*

$$\sum_{n=1}^{\infty} \phi^{1/2}(n) < \infty \tag{7}$$

and

$$0 < \lim_{n\to\infty} n^{-1} E\left(\sum_{i=1}^{n} (X_i - m)\right)^2 = \sigma_1^2 < \infty. \tag{8}$$

Then $\{X_i\}$ and $N(t)$ can be constructed on the same probability space together with a Wiener process $\{W(t), t \geq 0\}$ in such a way that a.s.

$$\sup_{0\leq t\leq T} |S(N(t)) - mt\lambda - \nu W(t)| = O(T^{1/2-\vartheta_1}), \quad \nu^2 = \sigma_1^2\lambda + m^2\tau^2\lambda^3 \tag{9}$$

for some $\vartheta_1 = \vartheta_1(\delta, p) > 0$.

2.2.2. **Associated summands**.

Definition. R.v. $X_1, \ldots, X_n$ are **associated**, if for any two coordinate-wise nondecreasing functions $f, g : R^n \to R^1$,

$$Cov\big(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)\big) \geq 0$$

whenever the covariance is defined. A sequence $\{X_i, i \geq 1\}$ is associated, if every finite sub-collection is associated.

A lot of interesting limit theorems for partial sums of associated summands are presented by Bulinski and Shashkin [1], Yu [17].

Theorem 5. *Let $\{X_i, i \geq 1\}$ be a strictly stationary associated sequence, $EX_1 = m$. Suppose that $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$ and Cox-Grimmett coefficient*

$$u(n) = \sup_{k\geq 1} \sum_{j:|j-k|\geq n} Cov(X_j X_k) = O(e^{-\theta n}) \tag{10}$$

for some $\theta > 0$, inter-renewal intervals $\{Z_i, i \geq 1\}$ are i.i.d.r.v. with $0 < EZ_1 = 1/\lambda < \infty$, $\tau^2 = VarZ_1 < \infty$. Denote

$$E(X_1 - m)^2 + 2\sum_{i\geq 1} E(X_1 - m)(X_i - m) = \sigma_2^2 > 0. \tag{11}$$

Then $\{X_i\}$ and $N(t)$ can be constructed on the same probability space together with a Wiener process $\{W(t), t \geq 0\}$ in such a way that a.s.

$$\sup_{0\leq t\leq T} |S(N(t)) - mt\lambda - \nu W(t)| = O(\varrho(T)), \quad \nu^2 = \sigma_2^2\lambda + m^2\tau^2\lambda^3, \tag{12}$$

where error term is $O(\varrho(T)) = O(T^{1/2-\vartheta_2})$ for some $\vartheta_2 = \vartheta_2(\delta, p) > 0$, when $E|Z_1|^p < \infty$ for $p > 2$, and error term is $o((T \ln\ln T)^{1/2})$, if Z_1 has only second moment.

Corollary 2 (SIP for Poisson random sums). *Theorems 4, 5 hold if $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$.*

3 SIP and rate of growth of compound renewal processes (random sums)

As it was already mentioned, SIP is a nice background for further investigation of the asymptotic behavior of compound renewal processes. Using SIP with appropriate error term one can easily extend the results about the asymptotic behavior of approximating Wiener or stable Levy process on the rate of growth of $D(t)$, when $D(t)$ admits a.s. approximation by one of the mentioned above processes. Formalizing this idea and extending the approach due to Philipp and Stout [9], we formulate rather general theorems (not obligatory connected with random sums). We start with the case, when $D(t)$ admits a.s. approximation by a standard Wiener process.

Definition. Function $f(t)$ is an *upper* function for the process $X(t)$, $t \to \infty$, if $P\{\limsup_{t\to\infty} X(t)/f(t) \leq 1\} = 1$ and $f(t)$ is a *lower* function for $X(t)$, if $P\{\limsup_{t\to\infty} X(t)/f(t) \geq 1\} = 1$.

Theorem 6. *Suppose that random process $D(t)$ admits a.s. approximation by a standard Wiener process $W(t)$ with an error term $O(t^{1/p}), p > 2$, i.e.*

$$\sup_{0\leq t\leq T} |D(t) - Mt - \nu W(t)| = O(T^{1/p}) \text{ a.s. }, \quad M \in R^1, \nu > 0, \tag{13}$$

then function $f(t) = \nu t^{1/2} h(t)$, $h(t) \uparrow \infty$, $\nu > 0$, will be an upper function for centered process $(D(t) - Mt)$, if

$$I_1(h) = \int_1^\infty t^{-1} h(t) \exp\{-h^2(t)/2t\} dt < \infty,$$

and it will be a lower one, if $I_1(h) = \infty$.

Theorem 7. *If random process $D(t)$ admits a.s. approximation (13) by a standard Wiener process $W(t)$ with an error term $O(t^{1/p}), p > 2$, then a.s.*

$$\limsup_{t\to\infty} \frac{|D(t) - Mt|}{\sqrt{2t \ln\ln t}} = \nu. \tag{14}$$

The proofs of these theorems easily follows from the famous Kolmogorov-Petrovski test and classical LIL for a Wiener process and form of error term in (13). For details see Zinchenko ([13] - [16]). Similarly, Chung's LIL for Wiener process obviously provides

Theorem 8. *Let $D(t)$ be as in previous Theorem, then a.s.*

$$\liminf_{T\to\infty} \left(\frac{8 \ln\ln T}{\pi^2 T}\right)^{1/2} \sup_{0\le t\le T} |D(t) - Mt| = \nu. \tag{15}$$

Now consider the case of ***i.i.d.*** summands. Obviously Theorems 6 – 8 immediately yield following statements:

Corollary 3. *Let $E|X_1|^{p_1} < \infty$, $E|Z_1|^{p_2} < \infty$ for some $p_1 > 2$, $p_2 > 2$, $EX_1 = m$, $0 < EZ_1 = 1/\lambda < \infty$, then $f(t) = \nu t^{1/2} h(t)$, $h(t) \uparrow \infty$ will be an upper function for centered process $(D(t) - m\lambda t) = (S(N(t)) - m\lambda t)$, if $I_1(h) < \infty$, and lower function, if $I_1(h) = \infty$.*

Corollary 4 (Classical LIL for random sums of i.i.d.r.v.). *Let $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ be independent sequences of i.i.d.r.v., $EX_1 = m$, $0 < EZ_1 = 1/\lambda < \infty$, $\sigma^2 = VarX_1 < \infty$, $\tau^2 = VarZ_1 < \infty$. Then a.s.*

$$\limsup_{t\to\infty} \frac{|S(N(t)) - m\lambda t|}{\sqrt{2t \ln\ln t}} = \nu, \quad \nu^2 = \lambda\sigma^2 + \lambda^3 m^2 \tau^2. \tag{16}$$

Corollary 5 (Chung's LIL for random sums)). *Let $\{X_i\}$ and $\{Z_i\}$ be as in Corollary 4, then a.s.*

$$\liminf_{T\to\infty} \left(\frac{8 \ln\ln T}{\pi^2 T}\right)^{1/2} \sup_{0\le t\le T} |S(N(t)) - m\lambda t| = \nu, \quad \nu^2 = \lambda\sigma^2 + \lambda^3 m^2 \tau^2. \tag{17}$$

Since random sums $S(N(t))$ of ***dependent r.v.***, introduced in sub-sections 2.2.1 and 2.2.2, also satisfy (13) with $M = \lambda m$, $\nu^2 = \sigma_i^2 \lambda + m^2 \tau^2 \lambda^3$, $i = 1, 2$, $1/p = (1/2) - \vartheta$ for some $\vartheta > 0$, Theorems 6– 8 yield following Corollaries:

Corollary 6 (Classical LIL for random sums, associated summands). *Let $\{Z_i\}$ be i.i.d.r.v. with $0 < EZ_1 = 1/\lambda < \infty$, $\tau^2 = VarZ_1 < \infty$, $\{X_i\}$ constitute the strictly stationary associated sequence with mean $EX_1 = m$ and covariance, satisfying sufficient conditions for SIP (Theorem 5), then a.s.*

$$\limsup_{t\to\infty} \frac{|S(N(t)) - m\lambda t|}{\sqrt{2t \ln\ln t}} = \nu, \quad \nu^2 = \lambda\sigma_2^2 + \lambda^3 m^2 \tau^2. \tag{18}$$

Corollary 7 (Classical LIL for random sums, ϕ-mixing summands). *Statement analogous to (18) holds with corresponding σ and ν for strictly stationary ϕ-mixing summands satisfying all conditions of Theorem 4.*

On the other hand, when ***independent*** summands are attracted to the stable distribution $G_{\alpha,\beta}$, which is not concentrated on the half of the axe, from Theorem 2 and results for a stable processes (Donsker and Varadhan [4]) follows

Corollary 8. *Let $\{X_i, i \geq 1\}$ satisfy **(C)** with $1 < \alpha < 2$ and $\{Z_i, i \geq 1\}$ be as in Corollary 4, then a.s.*

$$\liminf_{T\to\infty} \left(\frac{\ln\ln T}{T}\right)^{1/\alpha} \sup_{0\leq t\leq T} |S(N(t)) - m\lambda t| = C_{\alpha,\beta}\lambda^{1/\alpha}, \quad C_{\alpha,\beta} > 0. \tag{19}$$

When summands $\{X_i, i \geq 1\}$ are attracted to the asymmetric stable law $G_{\alpha,-1}$, then the approximating process for $D(t) = S(N(t))$ is a stable process $Y_{\alpha,-1}(t)$ without positive jumps, whose rate of growth can be successfully investigated with the help of certain integral test. Combining this fact with the SIP-type Theorem 2 or Theorem 3, we get

Theorem 9. *Let $\{X_i, i \geq 1\}$ satisfy **(C)** with $1 < \alpha_1 < 2$, $\beta = -1$ and $EZ_1^2 < \infty$ or $\{Z_i, i \geq 1\}$ satisfy **(C)** with $1 < \alpha_2 < 2$, $\alpha_1 < \alpha_2$, $|\beta| \leq 1$. Then $f(t) = t^{1/\alpha}h(t)$, where regular $h(t) \uparrow \infty$, will be an upper function for centered process $(D(t) - m\lambda t)$, if*

$$I_2(h) = \int_1^\infty t^{-1}h^{-\theta_1/2}(t)\exp\{-B_1 h^{\theta_1}(t)\}dt < \infty,$$

where

$$B_1 = B(\alpha_1) = (\alpha_1 - 1)\alpha_1^{-\theta_1}|\cos(\pi\alpha_1/2)|^{1/(\alpha_1-1)}, \quad \theta_1 = \alpha_1/(\alpha_1 - 1), \tag{20}$$

and $f(t)$ will be a lower function, if $I_2(h) = \infty$.

As a consequence we easily obtain following modification of the LIL:

Corollary 9. *Let $\{X_i, i \geq 1\}$ satisfy **(C)** with $1 < \alpha < 2$, $\beta = -1$. Assume that $EZ_1^2 < \infty$. Then a.s.*

$$\limsup_{t\to\infty} \frac{S(N(t)) - m\lambda t}{t^{1/\alpha}(B^{-1}\ln\ln t)^{1/\theta}} = \lambda^{1/\alpha}, \tag{21}$$

$$B = B(\alpha) = (\alpha - 1)\alpha^{-\theta}|\cos(\pi\alpha/2)|^{1/(\alpha-1)}, \quad \theta = \alpha/(\alpha - 1). \tag{22}$$

Corollary 10. *Corollaries 3 – 9 are true, when $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$.*

4 How big are increments of the random sums?

Partial answer on this question also can be obtained with the help of the SIP-type results for compound renewal processes (as it will be demonstrated below). More precisely, we consider increments $D(T + a_T) - D(T) = S(N(T + a_T)) - S(N(T))$ and study its' asymptotics, when a_T grows as $T \to \infty$, but not faster then T. A number of results in this area (but only for *independent summands*) were obtained by Zinchenko and Safonova [10], who proved various modifications of Erdös-Rényi-Csörgő-Révész law [3] for increments of random sums using appropriate SIP-type results. Remarkable progress in studying the magnitude of increments of compound renewal processes was achieved by Frolov [6], Martikainen and Frolov [8] with the help of other methods. The case of *dependent* summands was studied in [16], where the detail proofs of the following theorems are presented. Notice that assumptions on $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$, which determine the form of approximating process and error term, have impact on the possible length of intervals a_T under consideration.

4.1. Summands with finite variance. We start with the case of i.i.d. r.v. with "light tails" , when $\{X_i\}$ and $\{Z_i\}$ satisfy Cramer's condition. In this case the approximating process is a standard Wiener process and error is the smallest, i.e. $O(\ln T)$, so a_T may increase in a slowest rate.

Theorem 10. *Let $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ be independent sequences of i.i.d.r.v., $EX_1 = m$, $varX_1 = \sigma^2$, $EZ_1 = 1/\lambda > 0$, $varZ_1 = \tau^2$,*

$$E\exp(uX_1) < \infty, \quad E\exp(uZ_1) < \infty, \text{ as } |u| < u_0, u_0 > 0, \tag{23}$$

function $a_T, T \geq 0$ satisfies following conditions: $0 < a_T < T$ and T/a_T does not decrease in T. Also assume that

$$a_T/\ln T \to \infty \text{ as } T \to \infty. \tag{24}$$

Then a.s.

$$\limsup_{T\to\infty} \frac{|D(T + a_T) - D(T) - m\lambda a_T|}{\gamma(T)} = \nu, \tag{25}$$

where $\nu^2 = \lambda\sigma^2 + \lambda^3 m^2 \tau^2$, $\gamma(T) = \{2a_T(\ln\ln T + \ln T/a_T)\}^{1/2}$.

For concrete $a_T = T^\rho, 0 < \rho < 1$ or $a_T = (\ln T)^\rho$, $\rho > 1$ we have:

Corollary 11. *Let $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ be the same as in Theorem 10. Then a.s.*

$$\limsup{}_{T\to\infty} \frac{|D(T + T^\rho) - D(T) - m\lambda T^\rho|}{(2(1-\rho)T^\rho \ln T)^{1/2}} = \nu, \quad 0 < \rho < 1,$$

$$\limsup{}_{T\to\infty} \frac{|D(T + (\ln T)^\rho) - D(T) - m\lambda(\ln T)^\rho|}{\left(2\ln^{(\rho+1)} T\right)^{1/2}} = \nu, \quad \rho > 1.$$

The weaker moment conditions lead to more restrictive conditions on the rate of growth of a_T.

Theorem 11. *Let $\{X_i, i \geq 1\}$, $\{Z_i, i \geq 1\}$ and a_T satisfy all conditions of previous theorem with following assumptions used instead of (23)*

$$EX_1^{p_1} < \infty, \quad p_1 > 2, \quad EZ_1^{p_2} < \infty, \quad p_2 > 2.$$

Then (25) is true if $a_T > c_1 T^{2/p}/\ln T$ for some $c_1 > 0$, $p = \min\{p_1, p_2\}$.

Auxiliary SIP-type theorems are also useful in the case of *dependent* summands (discussed in Section 2). For example, Theorem 5 or Corollary 2 yield

Theorem 12. *Let $N(T)$ be homogeneous Poisson process with intensity $\lambda > 0$ and let $\{X_i\}$ be the strictly stationary associated sequence with mean $EX_1 = m$ and covariance, satisfying sufficient conditions for SIP (Theorem 5). Suppose that function $a_T, T \geq 0$ satisfies all conditions of Theorem 10 and $a_T > c_1 T^{2/p}/\ln T$ for some $c_1 > 0$, $1/p = (1/2) - \vartheta$, $0 < \vartheta < 1/2$. Then a.s.*

$$\limsup_{T\to\infty} \frac{|S(N(T+a_T)) - S(N(T)) - m\lambda a_T|}{\gamma(T)} = \nu, \tag{26}$$

where $\nu^2 = \lambda(\sigma_2^2 + m^2)$, $\gamma(T) = \{2a_T(\ln\ln T + \ln T/a_T)\}^{1/2}$.

4.2. Summands attracted to the stable law. When i.i.d.r.v. $\{X_i, i \geq 1\}$ are attracted to an asymmetric stable we have

Theorem 13. *Suppose that $\{X_i, i \geq 1\}$ satisfy **(C)** with $1 < \alpha < 2$, $\beta = -1$, $EX_1 = m$, $EZ_1 = 1/\lambda > 0$, $EZ_1^2 < \infty$. Function a_T is non-decreasing, $0 < a_T < T$, T/a_T is also non-decreasing and provides ${d_T}^{-1}T^{1/\alpha - \varrho_2} \to 0$ for certain $\varrho_2 > 0$ determined by the error term in SIP-type Theorem 2. Then a.s.*

$$\limsup_{T\to\infty} \frac{D(T+a_T) - D(T) - m\lambda a_T}{d_T} = \lambda^{1/\alpha}, \tag{27}$$

where normalizing function $d_T = a_T^{1/\alpha}\{B^{-1}(\ln\ln T + \ln T/a_T)\}^{1/\theta}$, constants B, θ are defined in (22).

5 How small are increments of the random sums?

SIP-type results can help in solution of this problem too. For instance, combining conclusions of Theorem 1 and corresponding facts for a Wiener process (Csörgő and Révész[3]), we have following statement, which holds when summands $\{X_i\}$ as well as inter-occurrence times $\{Z_i\}$ satisfy the Cramer's condition:

Corollary 12. *Assume that i.i.d.r.v. $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ satisfy all conditions of the Theorem 10, $\nu^2 = \lambda\sigma^2 + \lambda^3 m^2\tau^2$and $a_T(\ln T)^{-3} \to \infty$ as $t \to \infty$, then a.s.*

$$\lim_{T\to\infty} \gamma(T, a_T) \inf_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} |D(t+s) - D(t) - m\lambda a_T| = \nu. \tag{28}$$

6 Applications in risk theory

6.1. Sparre-Anderssen collective risk model. Within this model (rather popular in the actuarial mathematics) the risk process, which describes the evolution of reserve capital, is defined as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i = u + ct - S(N(t)), \tag{29}$$

where: $u \geq 0$ denotes an initial capital; $c > 0$ stands for the gross premium rate; renewal (counting) process $N(t) = \inf\{n \geq 1 : \sum_{i=1}^{n} Z_i > t\}$ counts the number of claims to insurance company in time interval [0,t]; positive i.i.d.r.v. $\{Z_i, i \geq 1\}$ are time intervals between claim arrivals; positive i.i.d.r.v.$\{X_i\}$ with d.f. $F(x)$ denote claim sizes; the sequences $\{X_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$ are independent; $EX_1 = m$, $EZ_1 = 1/\lambda > 0$.

Classical Cramér-Lundberg risk model is model (29), where $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$.

In the framework of collective risk model random sum $D(t) = \sum_{i=1}^{N(t)} X_i = S(N(t))$ can be interpreted as a total claim amount arising during time interval $[0, t]$, and increments

$$D(T + a_T) - D(T) = \sum_{i=N(T)+1}^{N(T+a_T)} X_i$$

as claim amounts during the time interval $[T, T + a_T]$.

Since process $D(t)$ is a typical example of the compound renewal process (compound Poisson process in Cramér-Lundberg model), main results of the Sections 2 – 5 can be applied to investigation of the risk process $U(t)$. Fist of all, Theorems 1 – 3 yield the SIP-type results for $D(t)$ and $U(t)$ under various assumptions on the claim sizes $\{X_i, i \geq 1\}$ and inter-arrival times $\{Z_i, i \geq 1\}$.

For small claims and $\{Z_i\}$ satisfying Cramér's condition, processes $D(t)$ and $U(t)$ admit strong approximation by a Wiener process with the error term $O(\ln t)$; for large claims with finite moments of order $p > 2$ the error term is $o(t^{1/p})$, if $p = 2$, then error term is $o((t \ln \ln t)^{1/2})$. For catastrophic events claims can be so large that their variance is infinite. In this case we assume that $\{X_i\}$ are in domain of normal attraction of asymmetric stable law $G_{\alpha,1}$ with $1 < \alpha < 2$, $\beta = 1$, and additionally satisfy condition (C). Then by Theorems 2 and 3 an approximating process for $D(t)$ is α-stable process $Y_{\alpha,1}$ with $1 < \alpha < 2$, $\beta = 1$, and risk (reserve) process $U(t)$ admits a.s. approximation by α-stable process $Y_{\alpha,-1}$, $1 < \alpha < 2$, $\beta = -1$, which has only negative jumps; the error term is presented in mentioned theorems.

The form of error term in SIP is "good" enough for investigation the rate of growth of total claims and asymptotic behavior of the reserve process. Due to results of Section 3 various modifications of the LIL for $D(t)$ can be obtained almost without a proof. So, in the case of small claims (satisfying Cramér's condition) or large claims (but with finite moments of order $p \geq 2$) for large t we can a.s. indicate upper/lower bounds for growth of total claim amounts $D(t)$ as $m\lambda t \pm \nu\sqrt{2t \ln \ln t}$ and for reserve capital $U(t)$ as $u + t\rho m\lambda \pm \nu\sqrt{2t \ln \ln t}$, where $\sigma^2 = VarX_1$, $\tau^2 = VarZ_1$, $\nu^2 = \lambda\sigma^2 + \lambda^3 m^2 \tau^2$, $\rho = (c - \lambda m)/\lambda m > 0$ is a safety loading.

For large claims in domain of normal attraction of asymmetric stable law $G_{\alpha,1}$ with $1 < \alpha < 2$, $\beta = 1$ (for instance, Pareto type r.v. with $1 < \alpha < 2$) Corollary 9 for large t provides a.s. upper bound for the risk process

$$U(t) \leq u + \rho m\lambda t + \lambda^{1/\alpha} t^{1/\alpha} (B^{-1} \ln \ln t)^{1/\theta}.$$

SIP-type results also help to answer on the question: how large can be fluctuations of the total claims/payments on the intervals whose length a_T increases as $T \to \infty$. Indeed, under appropriate conditions on claim size distributions and for rather " large" intervals a_T (but growing not faster then T) increments $D(T + a_T) - D(T)$ satisfy variants of Erdös-Rényi-Csörgő-Révész LLN similarly to (25) or (27). More results in this direction are presented in [12], [14].

Until recently, main known results concerning $U(t)$ and $D(t)$ were focused on the case of *independent* claim sizes $\{X_i, i \geq 1\}$. Our approach allows to study the case of *dependent* claims too. Thus, certain results about strong approximation of the risk process and approximation of ruin probabilities, bounds for rates of growth and fluctuations of total claim amounts in the case of weakly φ-mixing and associated r.v. (studied in Section 2) can be obtained similar to how it was done for independent summands. Our general approach also gives a possibility to study more complicated risk models with stochastic premiums.

6.2. Risk process with stochastic premiums. Within the *risk model with stochastic premiums* the risk process $U(t)$, $t \geq 0$, is defined as

$$U(t) = u + Q(t) = u + \Pi(t) - S(t) = u + \sum_{i=1}^{N_1(t)} y_i - \sum_{i=1}^{N(t)} x_i, \qquad (30)$$

where: $u \geq 0$ is an initial capital; point process $N(t)$ models the number of claims in the time interval $[0, t]$; positive r.v. $\{x_i : i \geq 1\}$ are claim sizes; $Ex_1 = \mu_1$; point process $N_1(t)$ is interpreted as a number of polices bought during $[0, t]$; r.v. $\{y_i : i \geq 1\}$ stand for sizes of premiums paid for corresponding polices, $Ey_1 = m_1$.

We call $U(t)$ (or $Q(t)$) the ***Cramér-Lundberg risk process with stochastic premiums*** **(CLSP)** if $N(t)$ and $N_1(t)$ are two independent *Poisson processes* with intensities $\lambda > 0$ and $\lambda_1 > 0$; $\{x_i\}$ and $\{y_i\}$ are two sequences of positive i.i.d.r.v. independent of the Poisson processes and of each other with d.f. $F(x)$ and $G(x)$, respectively, $\lambda_1 Ey_1 > \lambda Ex_1$.

This model, being a natural generalization of the classical Cramér-Lundberg risk model, was studied by Zinchenko and Andrusiv [11]. Korolev *et al.* [7] present an interesting example of using (30) for modeling the speculative activity of money exchange point and optimization of its profit.

Notice that process $Q(t) = \Pi(t) - S(t)$ is again a compound Poisson process with intensity $\lambda^* = \lambda + \lambda_1$ and d.f. of the jumps $G^*(x) = \frac{\lambda_1}{\lambda^*} G(x) + \frac{\lambda}{\lambda^*} F^*(x)$, where $F^*(x)$ is a d.f. of the random variable $(-x_1)$. In the other words

$$Q(t) = \sum_{i=1}^{N^*(t)} \xi_i, \qquad (31)$$

where $N^*(t)$ is homogeneous Poisson process with intensity $\lambda^* = \lambda + \lambda_1$ and i.i.d.r.v. ξ_i have d.f. $G^*(x)$.

Thus, all results for compound Poisson process, obtained in Sections 2 – 5, are applicable to $Q(t)$. For instance, we have following SIP-type results:

Theorem 14 (SIP for CLSP, finite variance case). *(I)If in model (30) both premiums $\{y_i\}$ and claims $\{x_i\}$ have moments of order $p > 2$, then there is a standard Wiener process $\{W(t), t \geq 0\}$ such that a.s.*

$$\sup_{0\leq t\leq T} |Q(t) - (\lambda_1 m_1 - \lambda\mu_1)t - \tilde{\sigma}W(t)| = o(T^{1/p}), \quad \tilde{\sigma}^2 = \lambda_1 m_2 + \lambda\mu_2, \tag{32}$$

where $\mu_2 = Ex_1^2$, $m_2 = Ey_1^2$.

(II) If premiums $\{y_i\}$ and claims $\{x_i\}$ are light-tailed with finite moment generating functions in some positive neighborhood of zero, then a.s.

$$\sup_{0\leq t\leq T} |Q(t) - (\lambda_1 m_1 - \lambda\mu_1)t - \tilde{\sigma}W(t)| = O(\log T), \tag{33}$$

Proof immediately follows from Corollary 1 since $Q(t)$ is a compound Poisson process (see (31)) with intensity $\lambda^* = \lambda + \lambda_1$, whose jumps have mean $\frac{\tilde{a}}{\lambda^*} = \frac{\lambda_1}{\lambda^*}m_1 - \frac{\lambda}{\lambda^*}\mu_1$, and second moment $\frac{\tilde{\sigma}^2}{\lambda^*} = \frac{\lambda_1}{\lambda^*}m_2 + \frac{\lambda}{\lambda^*}\mu_2$.

For catastrophic accidents claims can be so large that they have infinite variance, i.e. belong to the domain of attraction of a certain stable law. Thus, due to Theorem 2, for Cramér-Lundberg risk process with stochastic premiums we have:

Theorem 15 (SIP for CLSP, large claims attracted to α-stable law). *Suppose that claim sizes $\{x_i\}$ satisfy **(C)** with $1 < \alpha < 2$, $\beta \in [-1, 1]$, premiums $\{y_i\}$ are i.i.d.r.v. with finite variance, then a.s.*

$$\left|Q(t) - (\lambda_1 m_1 - \lambda\mu_1)t - (\lambda + \lambda_1)^{1/\alpha} Y_{\alpha,\beta}(t)\right| = o(t^{1/\alpha - \varrho_2}), \quad \rho_2 \in (0, \rho_0), \tag{34}$$

for some $\varrho_0 = \varrho_0(\alpha, l) > 0$.

On the next step we focus on investigation the rate of growth of risk process $Q(t)$ as $t \to \infty$ and its increments $Q(t + a_t) - Q(t)$ on intervals, whose length a_t grows, but not faster than t. Again the key moments are representation of $Q(t)$ as compound Poisson process (31), Theorems 14, 15 and application of the results obtained in Sections 3–5, namely, various modifications of the LIL and Erdös-Rényi-Csörgő-Révész law for compound Poisson processes.

Corollary 13 (LIL for CLSP). *If in model (30) both premiums $\{y_i\}$ and claims $\{x_i\}$ have moments of order $p \geq 2$, then*

$$\limsup_{t\to\infty} \frac{|Q(t) - \tilde{a}t|}{\sqrt{2t\ln\ln t}} = \tilde{\sigma}, \text{ where } \tilde{a} = \lambda_1 m_1 - \lambda\mu_1, \quad \tilde{\sigma}^2 = \lambda_1 m_2 + \lambda\mu_2.$$

Next we shall consider the case when r.v. $\{x_i, i \geq 1\}$ in CLSP-model (30) are attracted to an asymmetric stable law $G_{\alpha,1}$, but premiums have $Ey_1^2 < \infty$. Theorem 9 and Corollary 9 yield following statement:

Corollary 14. *Let $\{x_i, i \geq 1\}$ satisfy condition **(C)** with $1 < \alpha < 2$, $\beta = 1$ and $Ey_1^2 < \infty$. Then a.s.*

$$\limsup_{t\to\infty} \frac{Q(t) - (\lambda_1 m_1 - \lambda\mu_1)t}{t^{1/\alpha}(B^{-1}\ln\ln t)^{1/\theta}} = (\lambda + \lambda_1)^{1/\alpha},$$

where $B = B(\alpha) = (\alpha - 1)\alpha^{-\theta}|\cos(\pi\alpha/2)|^{1/(\alpha-1)}, \quad \theta = \alpha/(\alpha - 1)$.

Corollary 15 (Erdös-Rényi-Csörgő-Révész law for CLSP-model). *Let in CLSP-model claims* $\{x_i, i \geq 1\}$ *and premiums* $\{y_i, i \geq 1\}$ *be independent sequences of i.i.d.r.v. with* $Ex_1 = \mu_1$, $Ex_1^2 = \mu_2$, $Ey_1 = m_1$, $Ey_1^2 = m_2$ *and finite moment generating functions*

$$E\exp(ux_1) < \infty, \quad E\exp(uy_1) < \infty \text{ as } |u| < u_0, \quad u_0 > 0.$$

Assume that non-decreasing function a_T, $T \geq 0$, *satisfies all conditions of Theorem 13, then a.s.*

$$\limsup_{T\to\infty} \frac{|Q(T + a_T) - Q(T) - a_T(\lambda_1 m_1 - \lambda\mu_1)|}{\gamma(T)} = \tilde{\sigma},$$

where $\gamma(T) = \{2a_T(\ln\ln T + \ln T/a_T)\}^{1/2}$, $\quad \tilde{\sigma}^2 = \lambda_1 m_2 + \lambda\mu_2$.

Remark. General Sip-type theorems also give the possibility to investigate more general cases, when $\{y_i\}$ and $\{x_i\}$ are sequences of dependent r.v., for example, associated or weakly dependent, $N(t)$ and $N_1(t)$ can be renewal processes, Cox processes, ets. Partly, such problems were solved in [12].

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