Chaotic Modeling and Simulation (CMSIM) **3: 377-385, 2016**

Peculiarities of Parametric Resonances in Cross-waves

Tatyana S. Krasnopolskaya[1]
Evgeniy D. Pechuk[1]

[1] Institute of Hydromechanics NAS of Ukraine, Kyiv, Ukraine
(E-mail: t.krasnopolskaya@tue.nl; uzuzun@i.ua)

Abstract: Cross-waves occurs as the realization of parametric resonances, as was written long time ago by M.Faraday. However, derivation the correct equations for the amplitudes of these waves in finite containers for a long time had mathematical problems. All the theoretical works implemented solutions either for an infinitely long or for infinitely deep containers. Appling the method of superposition correct equations are constructed. These equations are such that the nonlinear terms contain the second derivative. For the first time in the case of one cross-wave chaotic regimes were found and their properties were investigated for a rectangular tank when one wall is a flap wavemaker.
Keywords: Cross-waves, Wavemaker, Fluid free surface, Parametric resonance, Chaotic simulation.

1 Introduction

Generation of cross-waves in free-surface of a fluid in various tanks is very well known [4-6, 8]. The waves may be excited by harmonic oscillations of wavemaker and depending on the vibration frequency both longitudinal and transverse patterns may arise. Experimental observations have revealed that waves are excited in two different resonance regimes. The first type of waves corresponds to forced resonance, in which patterns in the direction of wavemaker vibrations are realized with eigenfrequencies equal to the frequency of this excitation. The second kind of waves is parametric resonance waves and in this case the waves are "transverse", with their crests and troughs aligned perpendicular to the vibrating wall of wavemaker. These so-called cross-waves have frequencies equal to half of that of the wavemaker, Faraday, 1831, [2]. To obtain a lucid picture of energy transmission from the wavemaker motion to the fluid free-surface motion the method of superposition, Lamé, 1852, [7], has been used. This method allows to construct a simple mathematical model, which shows how the cross-waves can be generated directly by the oscillations of the wavemaker. All previous theories have considered cross-waves problem applying the Havelock’s solution [1] of the problem for a semi-infinite tank with an infinite depth and a radiation condition instead of a zero velocity condition at the finite depth of the bottom [8].

Received: 8 June 2015 / Accepted: 29 June 2016

ISSN 2241-0503

2 Cross-waves in Rectangular Container

Let us now consider the nonlinear problems of fluid free-surface waves, which are excited by a flap wavemaker at one wall of rectangular tank of a finite length and depth (Fig. 1).

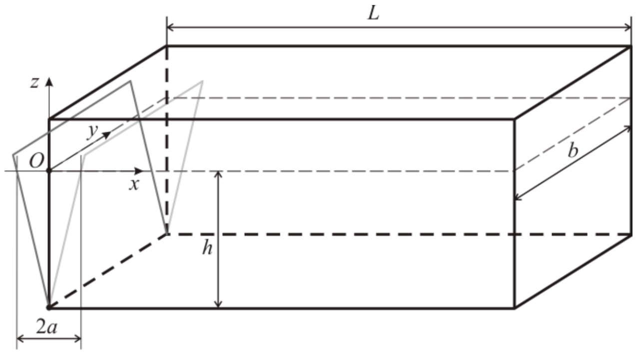


Fig. 1. Schematic of the experimental set-up.

The experiments were performed in a rectangular tank with a length L=50 m, a width b=6.8 m and a depth h=2.5 m. It is convenient to relate the fluid motion to the rectangular coordinate system (Oxyz) with the origin O at the fluid free surface. Motion of the wavemaker in the direction of x can be presented as

$$u(z,t) = F(z)\sin(\omega t) = \left(a + \frac{a}{h}z\right)\sin(\omega t), \tag{1}$$

Where *a* is the amplitude and ω is the frequency of the wavemaker oscillations.

From the experimental observations, Krasnopolskaya, 2013, [5], we may conclude that the pattern formation has a resonance character, every pattern having its "own" frequency. Assuming that the fluid is inviscid and incompressible, and that the induced motion is irrational, the velocity field can be written as $\mathbf{v} = \nabla\varphi$. Let us consider that patterns can be described in terms of normal modes with own eigenfrequencies. Then we may approximate free surface displacement waves during parametric resonance, when the excitation frequency ω is twice as large as one of the eigenfrequencies ω_{nm}, i.e. $\omega \approx 2\omega_{nm}$ as a function written in the form [4]:

$$\xi \approx \xi_{nm}(t)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{b} + \xi_{00}. \tag{2}$$

When $\xi_{nm}(t) = O(\varepsilon_1^{1/2})$, $\xi_{oo} = O(\varepsilon_1)$ and $\varepsilon_1 = \dfrac{a\omega_{nm}^2}{g}$ is a small parameter. Then a potential of fluid velocity $\varphi = \varphi_1 + \varphi_2 + \varphi_0$ as the solution of the harmonic equation and according to [4] has following components

$$\varphi_0 = \frac{a\,\omega}{4L}\cos\omega t\left[-(L-x)^2 + (z+h)^2\right] =$$

$$= \frac{\varepsilon_1\,\mathrm{g}\cos\omega t}{2\,\omega_{nm}L}\left[(z+h)^2 - (L-x)^2\right];$$

$$\varphi_1 = \varphi_{nm}(t)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{b}\,\frac{\mathrm{ch}[k_{nm}(z+h)]}{\mathrm{ch}(k_{nm}h)}; \qquad (3)$$

$$\varphi_2 = -\varepsilon_1\cos\omega t\,\sum_{j=1}^{\infty}\frac{4g\,h[1-(-1)^j]}{\omega_{nm}\alpha_{0j}(\mathrm{th}\,\alpha_{0j}L)(j\pi)^2}\cos\frac{j\pi z}{h}\,\frac{\mathrm{ch}\,\alpha_{0j}(x-L)}{\mathrm{ch}\,\alpha_{0j}} -$$

$$-\varepsilon_1\dot{\xi}_{nm}\sin\omega t\frac{g\left[\dfrac{3}{2}\left(\dfrac{n\pi}{L}\right)^2 + \dfrac{k_{nm}\,\mathrm{th}\,k_{nm}h}{h}\right]}{\omega_{nm}^2 h\alpha_{0j}\alpha_{m0}\ \mathrm{th}\,\alpha_{m0}L(k_{nm}\,\mathrm{th}\,k_{nm}h)}\cos\frac{m\pi y}{b}\frac{\mathrm{ch}\,\alpha_{m0}(x-L)}{\mathrm{ch}\,\alpha_{m0}} -$$

$$-\varepsilon_1\dot{\xi}_{nm}(t)\sin\omega t\sum_{j=1}^{\infty}\frac{g\left(\dfrac{n\pi}{L}\right)^2[1-(-1)^j]}{\omega_{nm}^2\alpha_{mj}(j\pi)^2\,\mathrm{th}\alpha_{mj}L(k_{nm}\,\mathrm{th}k_{nm}h)}\cos\frac{m\pi y}{b}\cos\frac{j\pi z}{h}\frac{\mathrm{ch}\alpha_{mj}(x-L)}{\mathrm{ch}\alpha_{mj}}.$$

Where $\varphi_{nm}(t) = O(\varepsilon_1^{1/2})$.

Using kinematical free-surface boundary conditions, Krasnopolskaya, 2012, [4],

$$(\varphi_0)_z + (\varphi_1)_z + \xi(\varphi_0)_{zz} + \xi(\varphi_1)_{zz} + \xi^2(\varphi_1)_{zzz} + \xi(\varphi_2)_{zz} =$$

$$= \xi_t + (\varphi_1)_x\xi_x + (\varphi_1)_y\xi_y + (\varphi_0)_x\xi_x + (\varphi_2)_y\xi_y +$$

$+(\varphi_1)_{xz}\xi\xi_x+(\varphi_1)_{yz}\xi\xi_y$,

we may find that the amplitude of the potential $\varphi_{nm}(t)$ is

$$\varphi_{nm}(t)=\frac{\dot{\xi}_{nm}}{k_{nm}\,\mathrm{th}k_{nm}h}-\varepsilon_1\xi_{nm}D\cos\omega t; \tag{4}$$

where

$$D=\frac{1}{k_{nm}\,\mathrm{th}k_{nm}h}\left[\sum_j\frac{8g[1-(-1)^j]}{\omega_{nm}h^2\alpha_{0j}(\mathrm{th}\alpha_{0j}L)}\right]\int_0^L\cos^2\frac{n\pi x}{L}\frac{\mathrm{ch}\alpha_{0j}(x-L)}{\mathrm{ch}\alpha_{0j}L}dx+$$

$$+\frac{g}{\omega_{nm}L}-\frac{g}{\omega_{nm}L}\int_0^L\left(\frac{n\pi}{L}\right)(x-L)\sin 2\frac{n\pi x}{L}dx.$$

The term $-\varepsilon_1\xi_{nm}D\cos\omega t$ expresses the influence of the potentials $\varphi_{o,}\varphi_2$ on the value of the potential φ_1 .

Applying the dynamical boundary condition

$$(\varphi_0)_t+(\varphi_1)_t+\xi(\varphi_1)_{tz}+\xi^2(\varphi_1)_{tzz}+(\varphi_2)_t+g\xi+$$
$$+\left[(\varphi_1)_x^2+(\varphi_1)_y^2+(\varphi_1)_z^2\right]+(\varphi_1)_x(\varphi_2)_x+(\varphi_1)_y(\varphi_2)_y+(\varphi_1)_z(\varphi_2)_z+$$

$$+(\varphi_1)_x(\varphi_0)_x+(\varphi_1)_z(\varphi_0)_z+(\varphi_1)_z\xi(\varphi_1)_{zz}+$$

$$+(\varphi_1)_x\xi(\varphi_1)_{xz}+(\varphi_1)_y\xi(\varphi_1)_{yz}=F_0(t),$$

we can get for the resonant amplitude an equation of parametric oscillations

$$\ddot{\xi}_{nm}+\omega_{nm}^2\xi_{nm}+\frac{9}{16}k_{nm}^2\ddot{\xi}_{nm}\xi_{nm}^2+\frac{3}{4}k_{nm}^2\xi_{nm}\dot{\xi}_{nm}^2+$$
$$+\varepsilon_1D_2\xi_{nm}\sin\omega t-\varepsilon_1D_5\dot{\xi}_{nm}\cos\omega t=0. \tag{5}$$

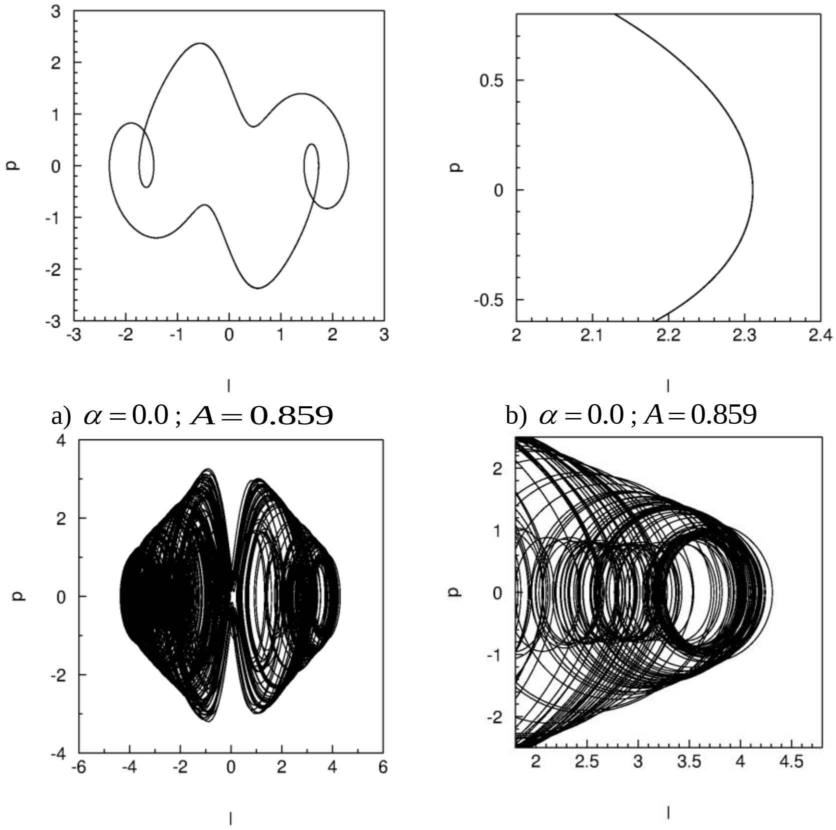
We can rewrite it for the rectangular tank with $L = 50$ m, *h*= 2,5m , $b = 6{,}8$ m and for the wave numbers $n = 40, m = 10$ as the following equation [4,5]:

$$\ddot{\xi}_{nm} + \omega_{nm}^2 \xi_{nm} + \frac{9}{16} k_{nm}^2 \ddot{\xi}_{nm} \xi_{nm}^2 + \frac{3}{4} k_{nm}^2 \xi_{nm} \dot{\xi}_{nm}^2 + \\ +4.78 \omega_{nm}^2 \xi_{nm} \sin \omega t - 2.99 \omega_{nm} \dot{\xi}_{nm} \cos \omega t = 0. \quad (6)$$

Where $k_{nm}^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{b}\right)^2$, the frequency is $\omega_{nm} = 2\pi 1.143$ Hz. We may use the transformation to the dimensionless variables $l = \xi_{nm} / \mu$, p, $\tau = \omega_{nm} t$, and finally get a dynamical system of the third order (when $\omega = 2\pi 2.27$ Hz and $\mu = 0.26$ m) in the following form

$$\dot{l} = p;$$
$$\dot{p} = -\frac{l + \alpha p + 1.4003 l p^2 + 4.78 A l \sin(2\tau - \beta\tau) - 2.99 A p \cos(2\tau - \beta\tau)}{1 + 1.0504 l^2};$$
$$\dot{\tau} = 1.$$

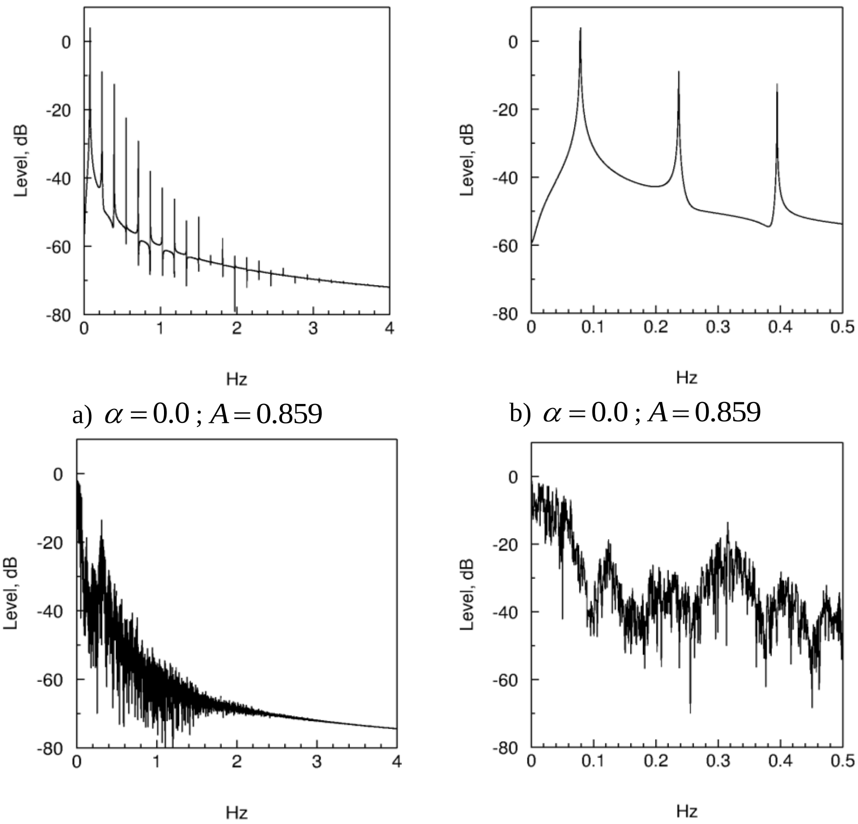
This system (at $\beta = 2 - \omega / \omega_{nm}$ =0.014 and without damping forces when $\alpha = 0.0$) has for the initial conditions *l(0)=0.3456*, *p(0)=1.104* very rich dynamics with regular and chaotic solutions. As an example of regular solutions in the Fig.2 a) and b) the projections on the plane *l*,*p* of phase portraits for $A = 0.859$ (the value of A is proportional to the amplitude of wavemaker oscillations) are shown in different scales. The cross-sections in the phase portrait are due to the projection on the plane [3]. The system of equations under consideration has the unique solutions. Power spectral densities of the solution are presented in Fig.3 a) and b). They have discrete peaks and are equidistant in frequency. The main frequency of regular cross-wave amplitude oscillations is around *0,2 Hz* which is close to a half of the frequency of the wavemaker oscillations *f=1/π Hz*. Firstly, for one mode approximation of the cross-waves on fluid free surface chaotic oscillation regimes were found [6] in the system under consideration without damping forces when $\alpha = 0.0$; $A = 1.759$ at $\beta = 2 - \omega / \omega_{nm}$ =0.014 as could be seen in Fig.2 c) and d).



a) $\alpha = 0.0$; $A = 0.859$ b) $\alpha = 0.0$; $A = 0.859$

c) $\alpha = 0.0$; $A = 1.759$ d) $\alpha = 0.0$; $A = 1.759$

Fig. 2. Phase portraits for regular (cases a, b) and chaotic regimes (cases c, d).



a) $\alpha = 0.0$; $A = 0.859$ b) $\alpha = 0.0$; $A = 0.859$

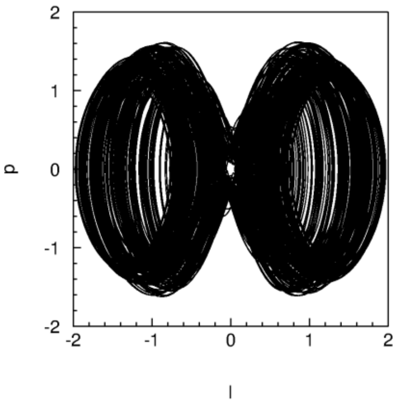
c) $\alpha = 0.0$; $A = 1.759$ d) $\alpha = 0.0$; $A = 1.759$

Fig. 3. Power spectral density computed for p data.

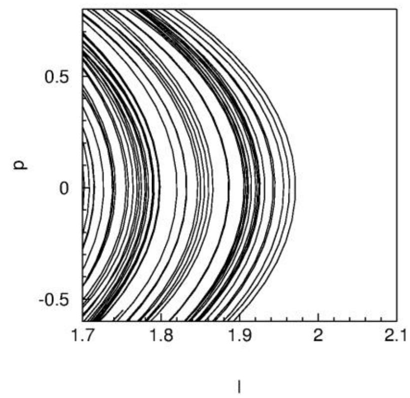
This dynamical system with additional damping forces, when $\alpha = 0.01$ and for the same initial conditions has also both regular solutions and chaotic ones as shown in Fig.4 Comparing the graphs of phase portraits we may conclude that the dynamical systems have similar behaviors. For the dynamical system with a damping force the power spectral density for the regular regimes (Fig.5 c) and d)) has more peaks then for dynamical system without damping (Fig.3 a) and b)) because of more turns in the limit cycle. The power spectral densities for chaotic regimes are continuous.

3 Conclusions

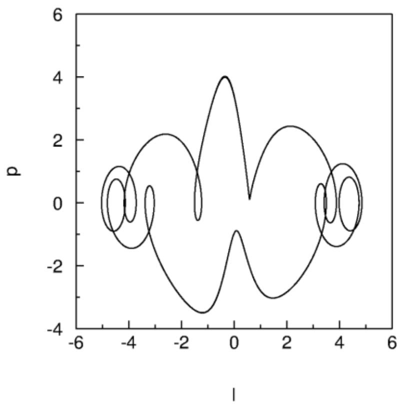
A new model for the cross-waves of fluid free surface oscillations is worked out for a rectangular tank of finite sizes. This mathematical model shows how the cross-waves are generated directly by the oscillations of the wavemaker. Existence of chaotic attractors was established for the dynamical system which,



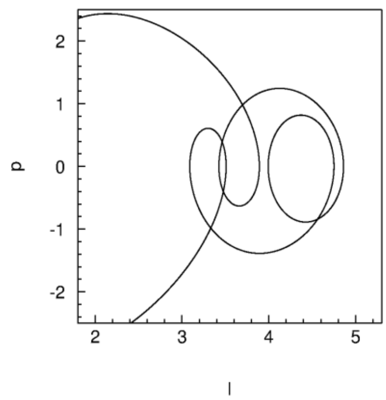
a) $\alpha = 0.01,\ A = 1.07$



b) $\alpha = 0.01,\ A = 1.07$

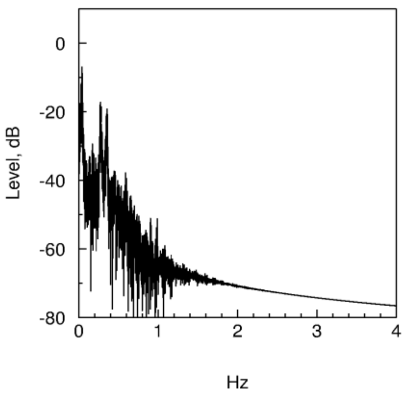


c) $\alpha = 0.01,\ A = 1.91$

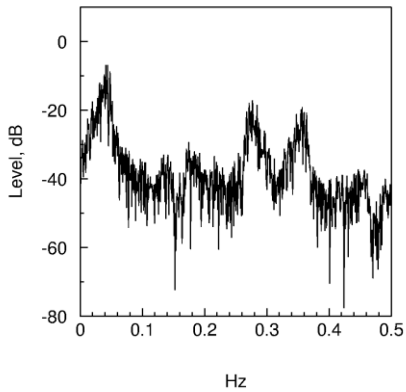


d) $\alpha = 0.01,\ A = 1.91$

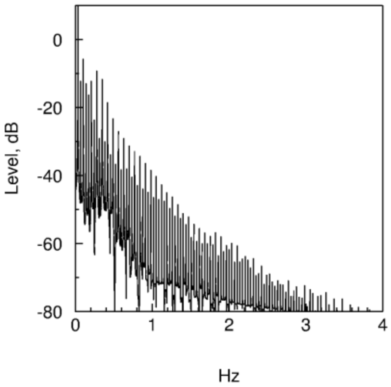
Fig.4. Phase portrait projections for chaotic (a, b) and regular (c, d) regimes.



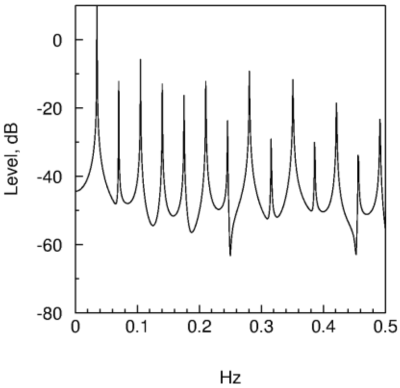
a) $\alpha = 0.01,\ A = 1.07$



b) $\alpha = 0.01,\ A = 1.07$



c) $\alpha = 0.01,\ A = 1.91$



d) $\alpha = 0.01,\ A = 1.91$

Fig. 5. Power spectral densities computed for p data.

describes one resonant cross-wave at fluid free-surface. For this system chaotic regimes were found for the first time and investigated.

References

1. T.H. Havelock. Forced surface waves on water. *Philosofical Magazine*, vol. 8(7), 569-576, 1929.
2. M. Faraday. On a peculiar class of acoustical figures; and on certain forms assumed by groups of particles upon vibrating elastic surface, *Philosophical Transactions of Royal Society of London A*, vol. 121, 299-340, 1831.
3. J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* Springer, New York, 1986.
4. T.S. Krasnopolskaya, V.M. Spektor. Cross-waves in a rectangular channel of finite dimansions. *Bulletin of Taras Shevchenko National University of Kiev, Series Physics & Mathematics,* no. 1, 93-100, 2012.
5. T.S. Krasnopolskaya, V.M. Spektor. Experimental study of cross-waves excitation *Bulletin of Taras Shevchenko National University of Kiev, Series Physics & Mathematics,* no. 4, 55 -58, 2013.
6. T.S. Krasnopolskaya, A.A. Gourgjii, V.M. Spektor, D.F. Prykhodko. Chaos in parametrically excited continuous systems. *Chaotic Modeling and Simulation,* no. 3, 413-422, 2013.
7. G. Lamé. *Leçons sur la Théorie Mathématique de l'Élasticité des Corps Solids,* C. Bachelier, Paris, 1852.
8. J.W. Miles and D. Henderson, Parametrically forced surface waves, *Annu. Rev. Fluid Mech.* Vol. 22, 143-165, 1990.