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Chaotic Mixing Across a Meandering Oceanic Current

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Abstract: A new mathematical model for the transport across a meandering current like the Gulf Stream is suggested. This model is based on a modification of the von Kármán vortex street stream function. The suggested modification allows one to approximate the main patterns in meandering ocean currents as observed by satellites. The inclusion of small perturbations in time (as periodic functions) and in space (in the form of weak eddies) enhance transport and mixing across the current. The mixing across the current is examined by following the deformation of certain well-defined (circular) areas on one side of the current back in time, so that we can determine from which initial part of the current that area is eventually composed.
Keywords: Chaotic mixing, Meandering jet, von Kármán vortex street, Gulf Stream, Chaotic simulation.

1 Introduction

The transport of warm water from the meandering Gulf Stream into the cold water surrounding the jet has been the focus of many recent studies. The distribution of the averaged sea-surface temperature in the Gulf Stream area is visualised by the satellite image shown in Fig.1 [1], in which the variations in the grey tones indicate temperature gradients. Dark grey indicates higher sea-surface temperatures, while light grey corresponds with cold surface water. The meandering Gulf Stream is clearly observed,

Mixing across the jet has been studied by a number of authors, including Bower [2], Bower and Rossby [3], and Samelson [4]. Bower and Rossby performed a field study with RAFOS floats released in the neighbourhood of the Gulf Stream, thus showing that the meanders of the Gulf Stream were responsible for much of the cross-stream motions of their floats released within the jet. However, meanders alone cannot lead to the motion of fluid parcels from one side of the jet to the other. We expect that small temporal and spatial perturbations of the meandering jet could play a crucial role in that respect. Such small perturbations may be caused, for example, by interaction of the jet with neighbouring eddies, as can be observed from the satellite images [1 - 4].

In her study of the mixing process, Bower [2] suggested a simple two-dimensional kinematic model. Her model is based on a simple stream function

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that reproduces the kinematic features of an eastward propagating meandering jet; in this model the meander parameters determine the flow rate of the water propagating downstream. However, Bower's model does not allow for any mixing, any movement of fluid particles across the jet. It is known that the Gulf Stream does not remain invariant in shape due to growth and diminishing of meanders. The effect of time dependence of the meander parameters on the mixing process was studied by Samelson [4]. Still, this model does not describe transport and mixing across the jet. The Gulf Stream may also change shape by interaction with rings in its direct vicinity. The inclusion of eddies in the simple meandering jet model implies additional time-dependence, which could enhance the mixing of fluid parcels within the jet. The Gulf Stream frequently interacts with many rings and it can be expected that these eddies play an essential role in the transport and distribution of tracer properties in the vicinity of the stream [5].

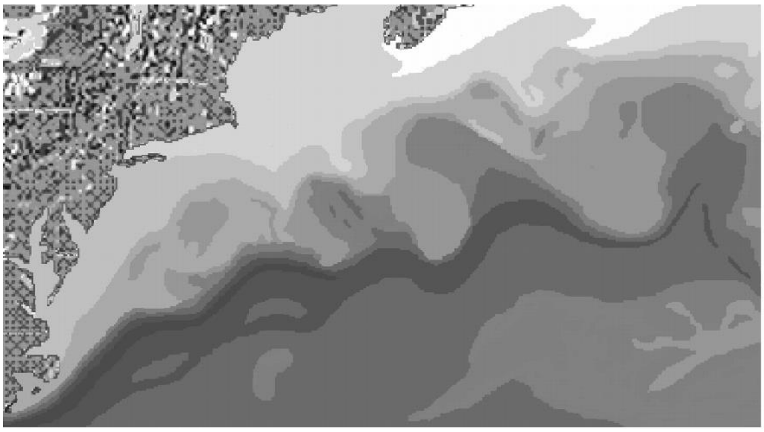


Fig.1. Satellite image of averaged sea-surface temperature in the Gulf Stream region [1].

In the present study we examine the enhancement of tracer transport and mixing caused by the interaction of a two-dimensional meandering jet, modelled by a von Kármán vortex street [6-8], and small time-periodic perturbations modelled in space as a chain of eddies [9]. To observe transport across the jet and the mixing that occur under interaction with perturbations we examine Lagrangian particle dispersion in time [10, 11].

2 Analytical Model of Gulf Stream Eddies

In the Bower's model the stream function has the following form [2]:

$$\psi(x,y,t)=\psi_0\left\{1-\tanh\left[\frac{y-y_C}{\lambda/\cos(\alpha)}\right]\right\} \quad (1)$$

where ψ_0 is the amplitude, which together with the jet width λ, determines the maximum downstream speed. For the case of the Gulf Stream, $\lambda=$ 40 km seems a realistic value. Furthermore, $y_c = A\sin\left[k(x-c_x t)\right]$ defines the centre streamline, *A* is the wave amplitude, $k=2\pi/L$ is the wave number, and $\alpha=\tan^{-1}\left\{Ak\cos\left[k\left(x-c_x t\right)\right]\right\}$, indicates the direction of current. The $\cos(\alpha)$ term is included to ensure that the jet has a uniform width everywhere. It is convenient to consider the flow in a reference frame moving with the phase speed c_x, (as it was done by Bower [2]). In the co-moving frame, the stream function has the following form:

$$\psi'(x',y') = \psi_0\left\{1-\tanh\left[\frac{y'-y'_C}{\lambda/\cos(\alpha')}\right]\right\} + c_x y' \quad (2)$$

with $y'_c = A\sin(kx)$ and $\alpha'=\tan^{-1}\left[Ak\cos(kx')\right]$. The stream function in this frame is independent of time and streamlines can be interpreted as trajectories of fluid parcels relative to the moving frame.

The main coherent structure elements of the Gulf Stream in the moving frame are [10]: (i) an eastward-propagating meandering jet; (ii) regions of fluid recirculation below and above meander crests and troughs; (iii) regions of westward-propagating fluid below and above the jet, and recirculation regions on either side of the jet. We will study the transport of passive particles (tracers) in this co-moving frame.

To study the transport properties of the meandering jet we suggest to use a new mathematical model for the stream function of the Gulf Stream, as introduced in [7]. This new stream function is a modification of the stream function modelling a von Kármán vortex street as encountered in the form of a system of vortices behind a cylinder moving with a constant speed. In the moving coordinate frame, which moves with a constant speed together with vortices, the stream function has the following form [7, 8]:

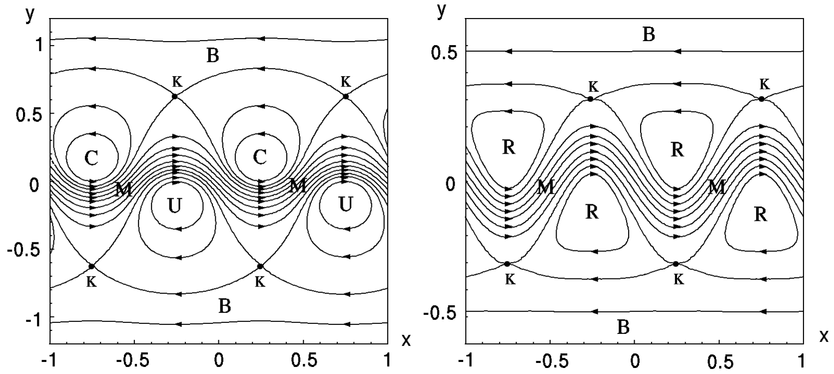


Fig. 2. Coherent structure elements in the von Kármán vortex street (left panel) and in the Samelson model [4] (right panel).

$$\psi(x,y) = -\frac{\Gamma}{4\pi}\ \ln\frac{P(x,y)}{Q(x,y)}\ +\ cy \tag{3}$$

where c is the vortex speed in the x-direction, and *P(x,y)* and *Q(x,y)* are defined as

$$P(x,y) = \cosh\frac{2\pi}{l}\left(y+\frac{h}{2}\right)+\sin\frac{2\pi x}{l};$$

$$Q(x,y) = \cosh\frac{2\pi}{l}\left(y-\frac{h}{2}\right)-\sin\frac{2\pi x}{l} \tag{4}$$

For the dimensionless variables $\tilde{x} = x/l;\quad \tilde{y} = y/l$ the stream function can be written as

$$\psi(\tilde{x},\tilde{y}) = -\frac{1}{2k}\ \ln\frac{P(\tilde{x},\tilde{y})}{Q(\tilde{x},\tilde{y})}\ +\ \tilde{c}\tilde{y} \tag{5}$$

where $P(\tilde{x},\tilde{y}) = \cosh k\left(\tilde{y}+b\right)+\sin k\tilde{x}$;

$$Q(\tilde{x},\tilde{y}) = \cosh k\left(\tilde{y}-b\right)-\sin k\tilde{x}$$

$$\tilde{\psi} = \psi/\Gamma;\quad \tilde{c} = cl/\Gamma;\quad b = h/2l;\quad k = 2\pi$$

Ihe streamlines according to (5) are shown in Fig. 2. In this graph, the meandering jet is marked as M, the recirculation regions of cyclonic and anticyclonic rotation by C and U, and the regions of westward propagating fluid are marked by B. The right panel of Fig. 2 shows the streamlines pattern according to Samelson's model [4]. It can be easily observed that the topological structure of the model flows is the same, containing recirculation cells on either side of the jet, including hyperbolic points (indicated by *K*).

The advection equations for passive tracers in the *x,y*-plane can be written as

$$\dot{x} = u = -\frac{\partial \psi}{\partial y}; \; \dot{y} = v = \frac{\partial \psi}{\partial x} \; . \tag{6}$$

For the stream function (5) these equations become:

$$\begin{cases} \dot{\tilde{x}} = \dfrac{\cosh kb}{PQ}\left(\sinh kb - \sin k\tilde{x} \sinh k\tilde{y}\right) - c \\ \dot{\tilde{y}} = -\dfrac{\cosh kb}{PQ} \cos k\tilde{x} \, \sinh k\tilde{y} \end{cases} \tag{7}$$

In the hyperbolic points $\dot{x} = 0; \quad \dot{y} = 0$, yielding for the coordinates:

$$\tilde{x}_1 = \frac{1}{4}; \; \tilde{x}_2 = \frac{3}{4} \quad \text{and} \quad \tilde{y}_{1,2} = \mp \frac{1}{k} \operatorname{arcsinh}\left(\frac{1}{\tilde{c}} \cosh kb - \sinh kb\right)$$

As mentioned before, Bower's model [2] and the von Kármán vortex street model do not allow for tracer transport from one side of the jet to other. Particles (passive tracers) can exhibit periodic or chaotic trajectories in the recirculation zones or along the meandering jet if we assume that the amplitude of the stream function (5) has a small periodical variation in time proportional to some small parameter ε, for example in the form:

$$\psi(x, y) = -\frac{(1 + \varepsilon \cos \pi t)}{2k} \ln \frac{P(x, y)}{Q(x, y)} + cy \tag{8}$$

(here and in what follows we omit the tilde over dimensionless variables). It may be expected that the perturbations will destroy the transport barriers (separatrices) [1, 10] between the recirculation zones and the westward ambient motion, thus allowing transport of fluid across the jet. However, numerous simulations [8, 10] have shown that transport and extensive mixing may occur only between recirculation zones and that no transport across the jet occurs. Such dynamics remains unchanged even when we add additional perturbations in the form of superimposed velocities. We now assume a time-periodic perturbation velocity field (in some way mimicking a tidal flow) with a velocity component $v_x = A_x \cos \pi t$ in the *x*- direction and $v_y = A_y \cos \pi t$ in the *y*-direction.

Figs. 3 and 4 illustrate the chaotic advection in the von Kármán model with a tidal flow velocity in the *y* -direction (Fig. 3) and in the *x* -direction (Fig. 4), both for the case ε = 0.1. The simulations show how an initial circular tracer blob (the shaded circle) is deformed and spread over a larger region, with long filamentary structures. The simulations have shown, however, that the marked fluid parcels will not leave the streamlines of the jet area, in spite of the perturbation of stream function and the additional 'tidal' velocity components.

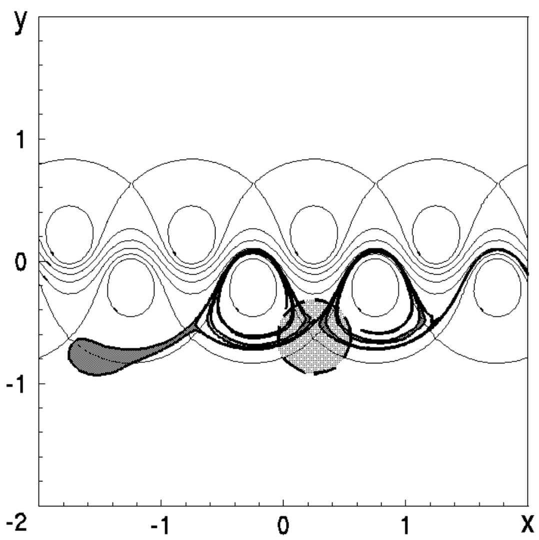


Fig. 3. Chaotic advection pattern in the von Kármán model (8) after 18 periods of the time-periodic (tidal) flow perturbation with $v_y = 0.1 \cos \pi t.$

In order to possibly acquire mixing and transport across the boundaries of the jet, we now apply a small perturbation to the stream function, in such a way that y-component of the velocity is not only periodic in time but also has a more complex spatial structure. We introduce the perturbation in the form of a field of eddies, with locations that are stationary in time. This means that they move westward with a constant speed ***C*** in the moving frame. For such an eddy pattern we use the stream function introduced by Zimmerman [9] in his study of tidal stirring, which in the non-moving Cartesian (x', y') coordinate system is given

$$\psi_z = \frac{1}{\pi\sqrt{2}} \sin \pi x' \sin \pi y' \tag{9}$$

It consists of a square cellular pattern with counter-rotating vortices inside the cells and hyperbolic points in the corner points. The origin of the (x', y') coordinate system (which is a hyperbolic point in the streamline pattern given by (9)) is now put in the point (1, 0) of the co-moving frame and the axes are rotated counter-clockwise over an angle $\pi/4$. In the co-moving frame the stream function (9) then has the following form:

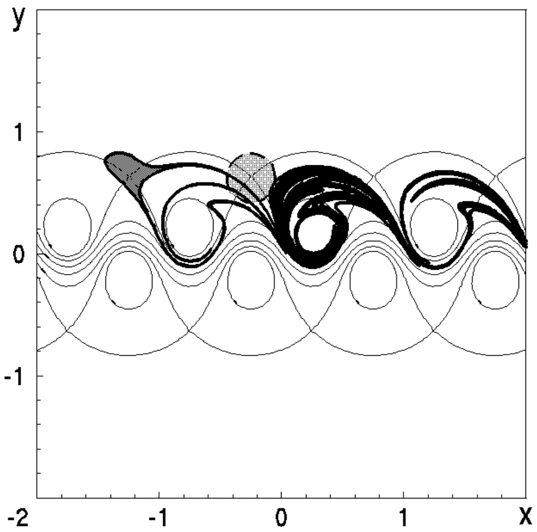


Fig. 4. Mixing pattern of circular blob in the von Kármán model (8) after 18 periods with $v_x = 5\cos \pi t$.

$$\psi_z = \frac{1}{\pi\sqrt{2}} \sin \frac{\pi}{\sqrt{2}}\left[(x-ct)+y-1\right] \sin \frac{\pi}{\sqrt{2}}\left[y-(x-ct)+1\right], \tag{10}$$

where we applied the coordinate transformation $x' = (x+y)/\sqrt{2}; y' = (y-x)/\sqrt{2}.$

The equations for fluid trajectories in this flow field, which is described by the superposition of the two stream functions (7) and (10), have the following form:

$$\begin{cases} \dot{x} = \Gamma_1 \dfrac{\cosh kb}{PQ}\left(\sinh kb - \sin kx \sinh ky\right) - c - \sin \pi\sqrt{2}y \\ \dot{y} = -\Gamma_1 \dfrac{\cosh kb}{PQ} \cos kx \sinh ky + \sin \pi\sqrt{2}\left(x - ct - 1\right) \end{cases} \qquad (11)$$

Here $\Gamma_1 = 1 + \varepsilon \cos \omega t$ is the time-dependent amplitude of the von Kármán vortex street function, containing a small disturbance $\varepsilon \cos \omega t$, with $\varepsilon \geq 0$ and ω the frequency of 'tidal' perturbation. In order to enhance influence only rings located in the area of the von Kármán street and suppress an influence of other rings in the y-direction we suppose that the amplitude of the stream function (10) is exponentially decaying in y-direction. Consider, for example,

$$\psi_z = \frac{1}{\pi\sqrt{2}} \frac{A(t)}{B(y)} \left(\cos \pi\sqrt{2}y - \cos \pi\sqrt{2}\left(1 - x + ct\right)\right) \qquad (12)$$

where $A(t) = \varepsilon_1(\gamma_1 + \gamma_2 \cos \omega t)$; $B(y) = \exp\left(C_2 y\right)^2$, and ε_1 is the new positive small parameter, γ_1, γ_2, C_2 are constants. In this case the streamlines correspond to those shown in Fig. 5, and the equations for fluid trajectories have the following form:

$$\begin{cases} \dot{x} = \Gamma_1 \dfrac{\cosh kb}{PQ}\left(\sinh kb - \sin kx \sinh ky\right) - c - \dfrac{1}{\pi\sqrt{2}} \dfrac{A(t)}{B(y)} \sin \pi\sqrt{2}y + \\ \qquad + \dfrac{C_2^2 y}{\pi\sqrt{2}} \dfrac{A(t)}{B(y)} \left(\cos \pi\sqrt{2}y - \cos \pi\sqrt{2}\left(1 - x + ct\right)\right) \\ \dot{y} = -\Gamma_1 \dfrac{\cosh kb}{PQ} \cos kx \sinh ky + \sin \pi\sqrt{2}\left(1 - x + ct\right) + \\ \qquad + \dfrac{1}{\pi\sqrt{2}} \dfrac{A(t)}{B(y)} \sin \pi\sqrt{2}\left(1 - x + ct\right) + 0.05 \cos\left(\omega_1 t\right) \end{cases}$$

(13)

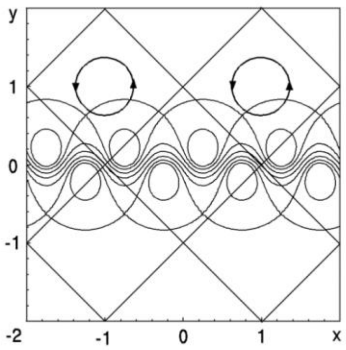


Fig.5. Streamline pattern of the flow according with stream function (5) with perturbations (12).

3 Numerical Experiment

In order to gain insight in the transport and mixing across the jet based on tracing the paths of individual particles (initially in a blob of marked tracers), we will use a Lagrangian description. We will investigate the motion of mathematical points that move at each instant with the velocity corresponding to their instant positions. Thus, the dyed particles are supposed to be inertialess, and not subjected to diffusion. We will examine the deformation of some circular area back in time, in order to determine from which part of the flow domain that area originates. For this purpose we use a contour tracking method conserving all topological properties in 2D flows [8, 11]. Any algorithm of contour tracking is based on the tracking of points distributed along the boundary of the initial blob and after each time step connecting them with neighbouring points. Because of non-uniform stretching and folding of the contour, after some time two neighbouring points may drift far apart, thus making the contour tracking less accurate and hence unreliable. The obvious way to overcome this problem is to increase the number of points on the contour. It not required to do this uniformly,but only in those parts of the initial contour where considerable stretching or folding will occur at later times. The essence of our algorithm is clear: (i) if it appears that some distance Δl_k between two neighbouring points becomes larger than some initially prescribed value l_{dis} , an additional point is inserted on the initial contour in the middle between points *k* and *k*+1; the equations (13) are then again numerically integrated for that newly added point, after which the points are renumbered. (ii) At each time step all sets of three neigbouring points are checked to see if they are still reasonablye well aligned: if the angle γ_{m} between three points m-1, m and m+1 appears to be smaller than some prescribed value γ (usually $\gamma = 120^{\circ}$), additional points are inserted on the initial contour line between points m-1, m and m+1 in such a way that, finally, the distances between all old and new neighbouring points do not exceed the value l_{cur} and all vertices of the polygon approximating the contour are larger than γ. An additional and important check of the proposed algorithm is the accuracy of fulfilling the area conservation condition.

The results presented have been obtained by numerical integration of the advection equations (13) with the following parameter values:

$\varepsilon = 0; \varepsilon_1 = 0.1; \gamma_1 = 2.0; \gamma_2 = 0.33; \quad C_2 = 2.0; \ \omega = 39.77; \ \omega_1 = \frac{\omega}{120} \approx 0.33$

In these simulations, a circular marker blob (dark grey; radius 0.4) was centred in the point (0.45, 0.95). Fig. 6 shows the satellite image of averaged sea-surface temperature in the Gulf Stream region (also shown in Fig. 1), with the streamline pattern of the von Kármán vortex stress (see Fig. 5) superimposed.

One clearly observes (both in Fig. 1 and Fig. 6) a large area of relatively warm fluid (visible as a darker shade of grey) north of the crest of the fourth meander in the Gulf Stream. Obvious questions are: how was created, from which part of the jet does it originate? To answer these questions we examine the deformation of a circular area in our model flow (13) located in that area (indicated by the broken contour), by integrating backwards in time. The result of these numerical integrations is shown in Fig. 7, with the initial position of the marker circle indicated by the dashed contour. The first panel in Fig. 7 shows the location of the deformed blob of fluid parcels (shown in black) at approximately 15 days backwards in time. The following panels in Fig. 7 show the marker blob at three-days intervals, namely at 12, 9, 6, 3, and 0 days backwards in time. Obviously, in the last panel the marker blob occupies the initial circular area. In order to estimate the cross-jet transport of fluid parcels we may compare the area of the cold fluid parcel (represented by the dark spot above the streamlines in the first panel in Fig. 7 and the area of the initial circular blob. The dark-grey patch is approximately a factor of two smaller, which indicates that in this specific case after 15 days half of the area of the circular blob is occupied by warm fluid, and half of it by cold fluid.

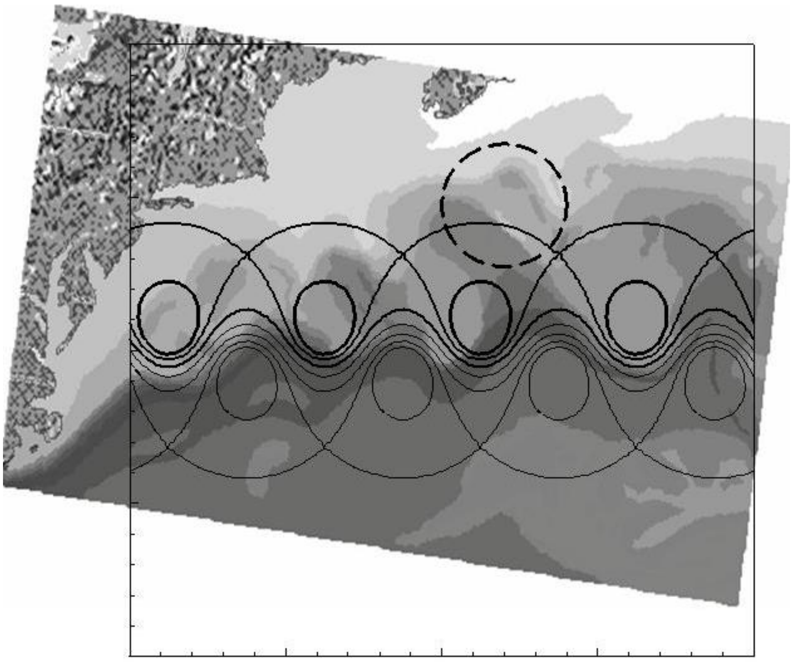
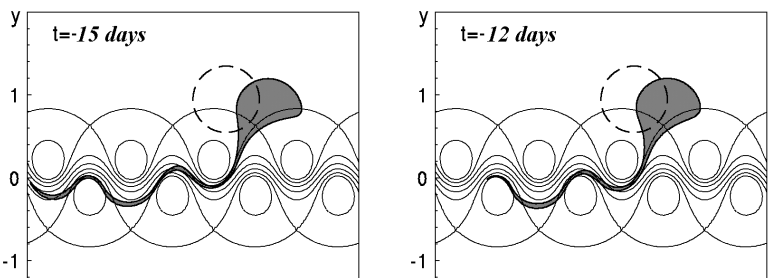


Fig.6. Satellite image of the averaged sea-surface temperature in the Gulf Stream region (see Fig. 1) with the streamlines of the von Kármán vortex street model superimposed.



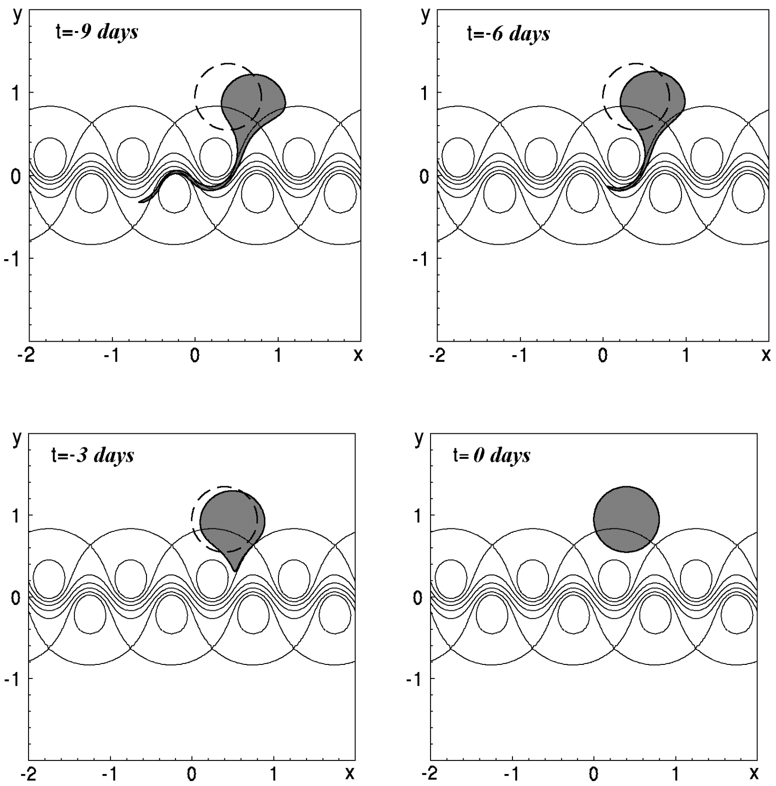


Fig.7. Deformation of a circular marker area obtained by integrating the model flow given by (13) backwards in time.

4 Conclusions

As a model for cross-current transport in the meandering Gulf Stream we have considered the von Kármán vortex street model of a meandering jet when interacting with a stationary pattern of eddies [5]. The numerical simulations have shown that such a model is able to show cross-jet transport of fluid parcels and intensive chaotic mixing.

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