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2-Phase model for population growth

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Abstract.
We discuss the 2-component model of population growth when different types of dynamics are attributed to rural and urban population respectively. We suppose the birth rate to be higher for rural areas, and death rate to dominate for cities. The flow between areas compensates these effects. This approach can reduce the number of parameters usually used for population growth behavior. We also discuss the socio-economic aspects of the model: the ability to control population size by adjusting the flow of people from the city to the countryside, and the trends in the urban and rural population. Finally the Earth population model is proposed.

Keywords: population growth, chaotic simulation, modeling of socio-economic processes.

1. Introduction

The problem of numerical simulation of population dynamics attracts a lot of attention in both natural and social sciences. Particularly this issue was supposed to be of great importance when it was proved that the population growth rates were increasing, and the simplest modeling of such dynamics inevitably leads to overpopulation of the planet (eg, a Malthusian crisis). Furthermore, a precise date of overpopulation was esteemed as 2004.

Thus, the “overpopulation problem” initiated a thorough statistical study of the dynamics of population growth, as well as many theoretical works arose, clarifying model representation of such dynamics. Along with Malthus classic work, it should be mentioned Verhulst model [1], Kapitsa model [2], Forrester world-system model [3] and many others. As it turned out, the rate of population growth crucially determines the growth rate of GDP, and this fact has largely spurred interest in the subject [4,5].

The most interesting dynamics is connected with so called “demographic transition point” when population growth sharply decreases and number of people achieves stable value. At the same time, all proposed model are unlikely to describe the entire dynamics of the population: the explosive growth of the initial time, saturation stage (demographic transition point) and then subsequent stabilization of the population. Statistical data are used to adjust

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the model, and often numerical simulation is reduced to finding appropriate coefficients that would fit the observed results (see, e.g. [6]).

We argue that much more productive approach may be based on physical arguments; in particular, we believe that the population should be considered as two-phase system – “rural population” and “urban population”, with each phase to behave under its own laws of growth, having, however, a flow between two phases. This view of the population dynamics growth can decrease the number of arbitrary parameters in the model, and also give additional arguments and levers to control the pace of population growth that is important for countries experiencing problems with overcrowding and/or depopulation challenges. In addition, it becomes possible to construct a universal model describing population dynamics on the Earth.

In this paper we propose and discuss the 2-component model of population growth when different types of dynamics are attributed to rural and urban population respectively. We exclude from consideration the initial stage of human civilization, when the population was too small to introduce residents’ differentiation, and not consider the case of population stabilization.

We assume that the entire population can be divided into two relatively independent groups (phases), focused respectively on the intensive and extensive ways of development - urban and rural areas. Note that these concepts are not geographical, and probably reflect the attitude of the population to the production of wealth and investing in future generations and lifestyles. The most important characteristic that allows extracting these two groups, apparently, is the population density per square kilometer. The problem to calculate/evaluate this value would be another interesting task that we will not consider in our work.

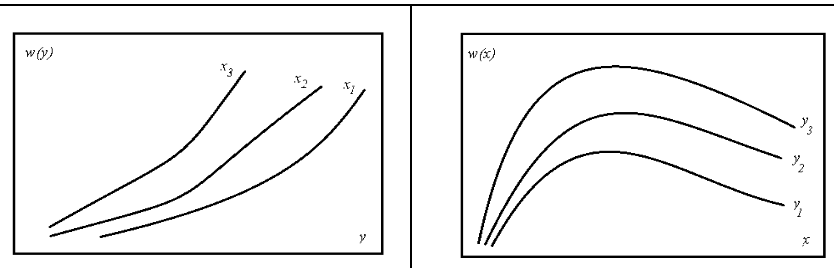
2. Statement of the problem

We suppose there is a closed area (no emigration) with an unlimited resources supply (i.e., country). Let us denote urban population as x, the rural population as y, and the time variable as t.
Then, in the general case, we can write

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,\lambda)+w(x,y,\lambda) \\ \frac{dy}{dt} &= g(x,y,\lambda)-w(x,y,\lambda) \end{aligned} \qquad (1)$$

Here λ is the institutional parameter responsible for the particular worldview of people and their relationship to birth, death, change of residence, and taking into account both objective (laws and restrictions) and subjective (the desire to move to the big city, or, conversely, the nature) factors. The functions describing the change of the urban population and the rural population are f and g respectively. The function of the population flow from one community to another is w. We stress that function w is greatly influenced by authorities, and can differ from country to country. Note that time is not explicitly included in the functional relationship. Our task is to study the possibility of governing the dynamics of the system changing function w (i.e., changes in public policy).

Using qualitative considerations (based on common approach to describe the dynamics of living organisms like predator-prey system, the game "Life", etc.) we can conclude that for a fixed urban population *x* function *w(y)* is strictly increasing function without saturation (i.e. is convex and the derivative does not change its sign). The qualitative form of the function w(y) is shown in Figure 1a. Similarly, in the case of constant values of rural population y we construct qualitative form of the function w (x), the population flow increases linearly for small values of x and subsequently begins to decrease due to resource constraints and increasing population density in the city (Fig. 1b).



Typical behavior of w(y), $x_3 > x_2 > x_1$.	Typical behavior of w(x), $y_3 > y_2 > y_1$ ..

Fig. 1. The behavior of the function *w*.

Based on the qualitative analysis of the figures 1a,b we propose the following form of the function *w*, satisfying all the properties listed above:

$$w(x, y, \lambda) = \beta \frac{xy}{x^2 + \alpha^2} \qquad (2)$$

Here the numerator *xy* is proportional to the number of meetings between urban and rural residents, that can be treated as a "reassurance" of a resident to change his/her address with probability β. The denominator in the formula (2) imposes restrictions on the movement of villagers into the town with a large urban population. In extreme cases, when the frequency of meetings has no effect on the decision of the villagers move to the city, all of them with some degree of probability take the decision to move. However inures limiting factor associated with the limited space and high population density in urban areas, which is directly proportional to x^2. Term α^2 is introduced to eliminate peculiarity at $x=0$, and reflects the minimum number of people needed the population becomes “city”. Estimates of population dynamics (parameter fitting formula for hyperbolic growth) give us the number of 60-70 thousand people (the size of the human community, when main role play statistical factors rather than personal ones).

For the sake of convenience, we will assume the function *w* always positive, but in general negative function is also possible, and it would represent the flow in reverse (from city to the countryside).

Thus, the proposed model reduces the institutional parameter λ to two parameters α and β. Note that we do not consider the simplest case when *w=const*. Let us choose the functions *f* and *g* in the simplest form:

$$f = -ax, \quad g = by - cy^2 \tag{3}$$

The meaning of this choice is that the urban population decreases, while the rural population is growing, but there is a limit due to the finite resources. In practice, the choice of functions (3) permits us to formulate one of the differences between urban and rural areas: the countryside is a source of replenishment of the population, it possesses a positive population growth, and the city is characterized by a decline in population, as it is more inclined in the production and creation tools/services, both for city's sake and for the village.

Thus, the proposed two-component (two-phase) model can be rewritten in a following form:

$$\begin{aligned} \frac{dx}{dt} &= -ax + \beta\frac{xy}{x^2+\alpha^2} \\ \frac{dy}{dt} &= by - cy^2 - \beta\frac{xy}{x^2+\alpha^2} \end{aligned} \tag{4}$$

These equations however, do not exhibit complex behavior, and only have stable points solutions; the model initially contains no "points of demographic transition" or "overpopulation problem". This is close to the reality, but needs some improvement: we assumed that the response of the system is instantaneous for variables *x* and *y*. To account for the effects associated with the maturation of people, and make the complex dynamics possible, we take into account the presence of delay in the system. Namely, we rewrite system (4) as a

$$\begin{aligned} \frac{dx}{dt} &= -ax(t) + \beta\frac{x(t-\tau)y(t-\tau)}{x^2(t)+\alpha^2} \\ \frac{dy}{dt} &= by(t) - cy^2(t) - \beta\frac{x(t-\tau)y(t-\tau)}{x^2(t)+\alpha^2} \end{aligned} \tag{5}$$

The meaning of the last term in this case is that the move from rural to urban areas is carried out by adults, usually without family. Accordingly, the decision shall be taken in the "meetings" with the same people over the age of the city (almost, it comes to comparing features/lifestyles in the countryside and in the city for people of the same age group). At the same time limiting factor (the denominator in the last term) still depends on the current urban population.

Also to reduce the number of control parameters we use following renormalization.

$$t' = t/\tau, \; x' = x/\alpha, \; y' = y$$

And introducing the notations:

$$A = a\tau, \; B = b\tau, \; C = c\tau, \; D = \beta\tau/\alpha^2$$

Then, omitting the primes in the new variables, we obtain the system:

$$\frac{dx}{dt} = -Ax(t) + D\frac{x(t-1)y(t-1)}{x^2(t)+1}$$
$$\frac{dy}{dt} = By(t) - Cy^2(t) - D\frac{x(t-1)y(t-1)}{x^2(t)+1} \qquad (6)$$

System of equations (6) will be the final mathematical formulation of the proposed model.

From the physical meaning of the system, all variables must be positive (on the parameters *A, B, C, D* this requirement was imposed initially). Lag time now is always 1. In numerical simulation we expect that the most typical regime still to be stationary mode or with mild oscillations near the equilibrium point (the latter is quite typical of early human societies lived slash agriculture, where the population is growing at the beginning, and then, because of the impoverishment of resources began to decrease).

This model contains now modes of complex dynamics, they are possible in this system due to the presence of external feedback. Oscillatory or "chaotic" modes in this system are associated with huge costs (depopulation in towns or villages, the collapse of infrastructure, etc.) In practice, if a management decision could transfer the system from stable stationary mode to complex dynamics, it would mean that the proposed action is a mistake.

Let us estimate the numerical values of the parameters in the system (6). The parameters A, B, C are responsible for the share increase (decrease) of the population in a time where, in comparison with available statistics, we find that they all vary from zero to one. Greatest arbitrariness is related to the choice of the parameter D, because it depends on a set of institutional factors such as persuasion factor β or α **.**

3 The stability points analysis

Let us analyze the system (6) for stability. There are always two trivial solutions: x=y=0, and x=0, $y = B/C$. The first solution corresponds to the lack of human civilization or the initial point, and second corresponds to collapse of urban civilization. For the classification of other solutions we obtain the following equation for fixed points (recall that within the framework of our model x> 0):

$$y = \frac{A}{D}\left(x^2+1\right), \qquad CAx^4 + (2CA - BD)x^2 + D^2x + CA - BD = 0 \qquad (7)$$

To find the exact analytically solution in this case is impossible, however, we can give the following estimates. If CA-BD> 0, the solutions of (7) does not exist. If CA-BD <0, but 2CA-BD> 0, then there is one solution, but if 2CA-BD <0, it is possible one or the two solutions. Type the appropriate parameter plane shown in the figure below. We note that there are 3 areas that can perform different dynamics. For the first area with 2 stationary points one of them is always unstable, thus resulting regime is expected to be constant. In the second area 2 different stable solutions are possible. And in the last area complex dynamic and interchange between different stable stationary points are possible.

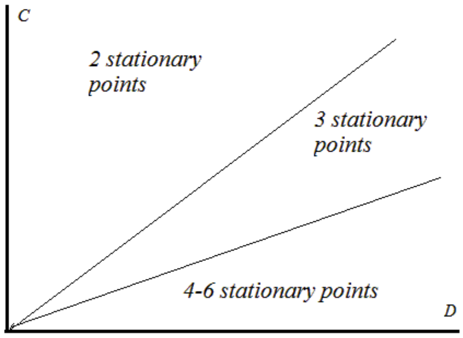


Fig. 2. Parameter plane for stationary points at Ox-axis. Stability analysis.

4. Numerical simulation

Further we will discuss the results of numerical simulations conducted by the Runge-Kutta of 4th order. Time step is $dt = 0.01$, and the parameters values are given in figure captions. Note that at this stage we do not set the task to find the exact quantitative relationship between real data and parameters used for computer modeling, we are focusing on the opportunity to effectively control the dynamics, to change between regimes, etc. In numerical simulations we proceed as follows: for the time 1 (dimensionless time delay) we numerically solve the system (6) without delay, and starting at time 1, we consider the term with delay. Thus we can avoid the need to set the initial conditions in the interval (0, 1).

Below we represent the parameter plane for fixed values of *A*, *B* with varying *D*. We note a good agreement with theoretical calculations (compare fig. 2 and 3). In the lower region, where the number of fixed points is large, it is possible to realize the complex dynamics. Thus, we mark the area of chaotic oscillations with red, and periodic oscillations with different period with green. Unstable (nonphysical) behavior is also present. We note that the region of periodic oscillations is adjacent to the area of chaotic behavior, hence the periodic oscillations in the system can be regarded as precursors of the onset of chaos and/or unstable solutions. The last ones due to the nature of studied system should be avoided in reality.

We also plotted 3 regions with stable solutions: the first one with 2 stationary points is characterized by the situation when after transient proves x=0, and y=const. It is pure rural community, when all population is concentrated in countryside, AC>BD that means death and restriction of the rural population growth (parameter C) play more important role than birth rate and population flow. People are more likely to die than to survive, and the rest of population settles in rural areas. In the second area stable solution is represented with stable rural and urban population, but rural people dominate.

This situation can be regarded as traditional society when cities are rare and most of population prefer to live in countryside. However with the increase of D and/or decrease of C (the flow grows and/or limitations for rural population become weaker) the stable solution for urban population begins to overcome the rural one. It could occur either because of the cities to become more “popular” (parameter D), or when rural area cannot support more number of people (decrease of C). Both tendencies make people to go away from the countryside. Also near abscissa (low C) there is a narrow area of stable solution when x=0, y=const. It happens when because of the decrease of C the number of people that can survive in rural area becomes too small to be enough for both phases. People thus have to choose where to live, and they prefer to stay where they are born rather than to move away.

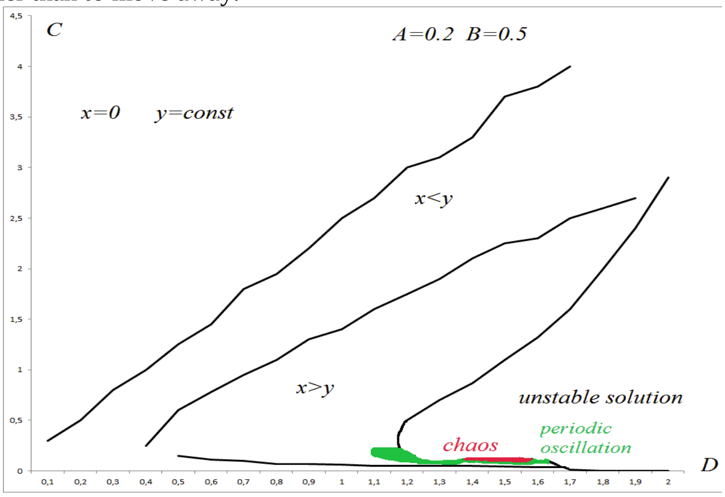


Fig. 3. Parameter region at A = 0.2. B = 0.5

Typical time series are presented in fig. 4-9. Urban population is represented by red, rural – by green, and total number of people – by black solid line. Initial conditions were chosen x=0,1, y=0,9.

Usually after short transient process a stable solution is observed. 2 typical situation are presented in fig. 4 and 5. In the first case there are both rural and urban people, in the second case cities disappear.

Oscillations though being rather rare can also be observed. They are more likely to damp with very long transient process (fig. 7), but also can be stable (fig. 6). This dynamics can be attributed to Neolithic societies when people use the land as much as they could and having depleted it they started to starve and consequently die. However it is not typical for nowadays communities, and thus should be avoided. Complex and even chaotic behavior can also be found, but we present it here as an example of non-physical dynamics, since in usual life this would be the signal of wrong managerial solutions. We should note that area of complex dynamics is very small, and that can also be treated as model adequacy.

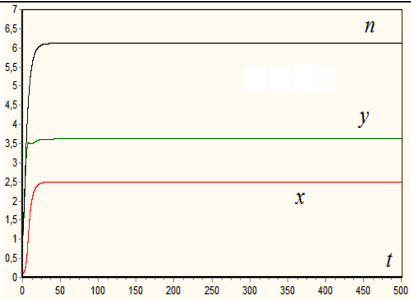


Fig. 4. A=0.2; B=0.5; C=0.1; D=0.4
Transition to stationary mode where rural population exceeds the urban

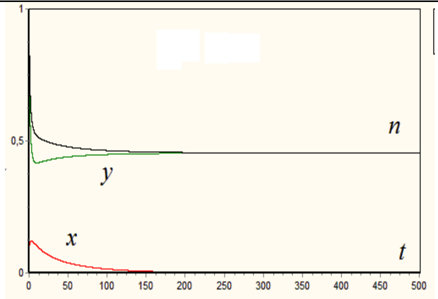


Fig. 5. A=0.2; B=0.5; C=1.1; D=0.4;
Transition to stationary mode, when the urban population disappears.

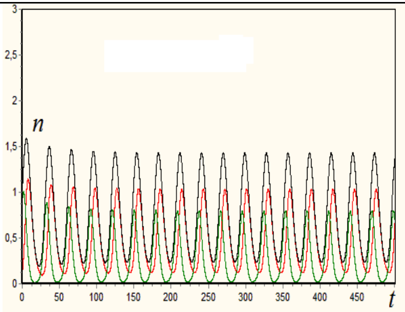


Fig. 6. A=0.2; B=0.5; C=0.3; D=1.1
Periodic oscillations

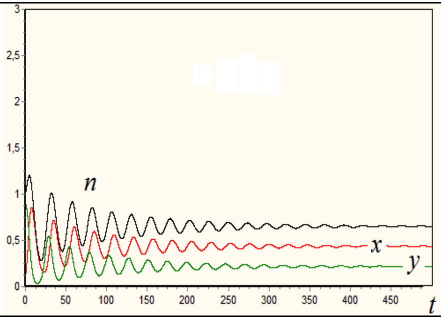


Fig. 7. A=0.2; B=0.5; C=0.45; D=1.1;
Damped oscillations.

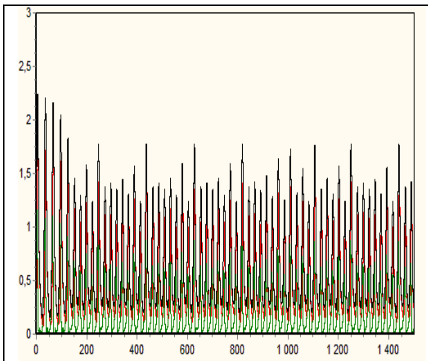


Fig. 8. A=0.2; B=0.5; C=0.1; D=1.6;
Chaotic oscillations

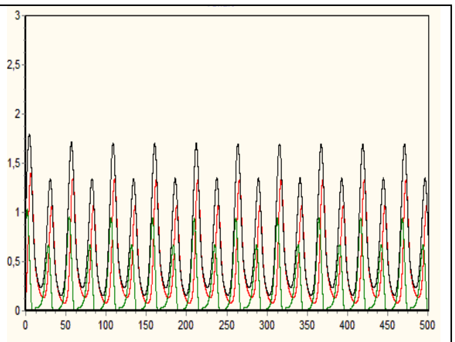


Fig. 9. A=0.2; B=0.5; C=0.19; D=1.5;
Multi periodic oscillations (2-period)

5 The problem of controlling the system dynamics

The most serious problems associated with population growth faced by governments is a overpopulation problem and demographic crisis or extinction of the population. In this connection let us consider how to manage the system dynamics. Suppose the government implements measures that can be reflected as a change in the values of the system parameters *A, B, C, D* and/or the sudden change in the number of *x, y* (artificially people moved from the countryside to the city, or vice versa). Clearly, the parameters *A*, *B*, *C* can not be changed dramatically; they are responsible for the traditional approach to family formation (*B*, *C*) or reflect the outcome of long-term policy in the field of medicine (*A*). Their variations are insignificant and they are extended in time. For these reason we will call this group of parameters as adiabatic.

On the other hand, the parameter *D* can be changed quite drastically: for example, the frequency of "meetings" can be increased or decreased by creating artificial barriers to entry the city or leave the village (strict registration rules), or one can modify α providing residents of the city more "rustic" conditions related to population density (buildings as the private sector). Parameter *D* also could include a variety of unrealistic events (meteor fall, epidemics, etc.).

In the numerical simulation we will implement all the parameter changes at time $t = T/2$. Some results are presented below.
First, we consider additional flow added to the natural one. We studied both constant summand, and summand proportional to rural population. System dynamics does not change, and new system can be reduced to the old one by re-normalization (fig. 10). Then we study the case when there is a 10% shift of the population (people are forced to move from countryside to the cities. The similar situation occurred in collectivization in Russia or during fencing in England). Since mostly solution are stable this momentary shift does not influence system dynamics, and after rather short transient process system returns to its initial state (fig. 11). In both case the system damps the abrupt changes, thus we conclude that such hard measures cannot give expected results.

Quite fast "curb overpopulation" can be achieved by cross-flow of the rural population in the urban population in a city where the birth rate is significantly reduced. Suppose, for example, at some point in mortality fell, reaching 96% of the original level. In this case, there would be an increase in urban population, but at the same time the rural population would slightly decrease, too (that happens because of the limiting factor inversely proportional to the square of the urban population). Total number of people, however, increases. This mode is shown on Fig.12. Similar behavior can be observed and if the birth rate in villages increased by 4% (fig. 13). In that case rural population grows and thus increases the number of citizens. Qualitatively similar behavior can be seen if we increase death rate in cities of decrease birth rate in countryside: both population in rural and urban areas decreases, and total number of people diminishes. It is interesting to note that the change in life conditions in villages influences the system dynamics (and the total number of people) more than similar changes in urban life (we plotted the time series for the change of 4% in comparison to the initial values – both in decrease/increase).

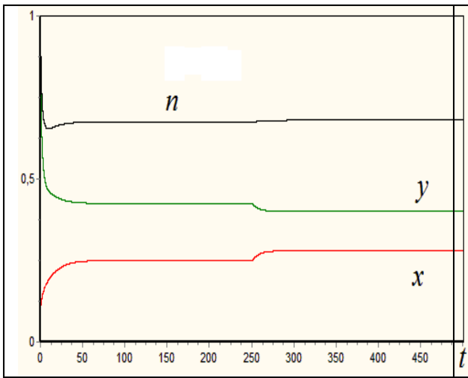


Fig. 10. A=0.2; B=0.5; C=0.9; D=0.5; Shift is 0.1 in dimensionless variables when to the "natural" overflow adds an additional inflow of artificially created

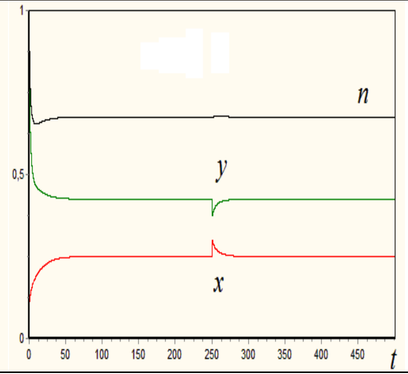


Fig. 11. A=0.2; B=0.5; C=0.9; D=0.5; Shift is 0.1 in dimensionless variables. Sharp one-time change in the number of cities and villages population

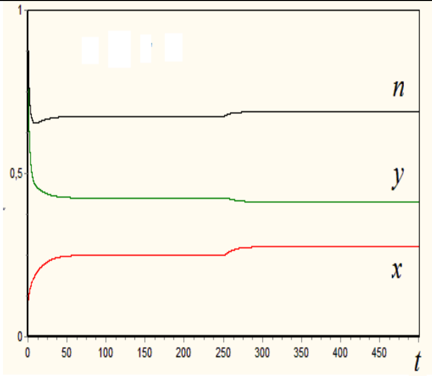


Fig. 12. A=0.2; B=0.5; C=0.9; D=0.5; Mortality in city fell, reaching 96% of the original (parameter *A*)

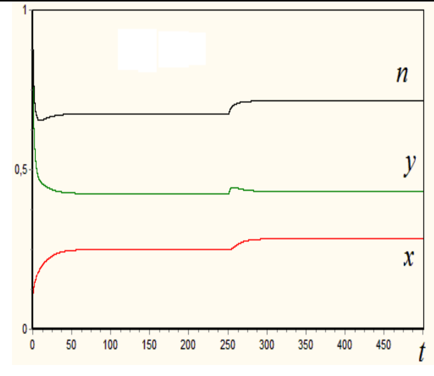


Fig. 13. A=0.2; B=0.5; C=0.9; D=0.5; Birth rate in the countryside is growing at 4% (parameter *B*)

The obtained results may, in particular, explain why the increase in life expectancy in urban areas (ie, a mortality decrease) does not lead to such dramatic changes as a decline in fertility in the village. We stress that the situation in the countryside (source of population) is decisive. At the same time, our results show that the policy can be directed only one of the population groups to achieve the result and do not necessarily affect both the city and the countryside, moreover, it may be advantageous to use that institutional arrangements to only one part of the population.

We also studied the case when the parameters *A*, *B*, *C* remain unchanged, but at time t/2 parameter D varies. As it turned out, the system is very sensitive to changes in this parameter. Especially one can achieve periodic oscillations from stationary ones, or even chaotic. Some examples are presented in fig. 14.

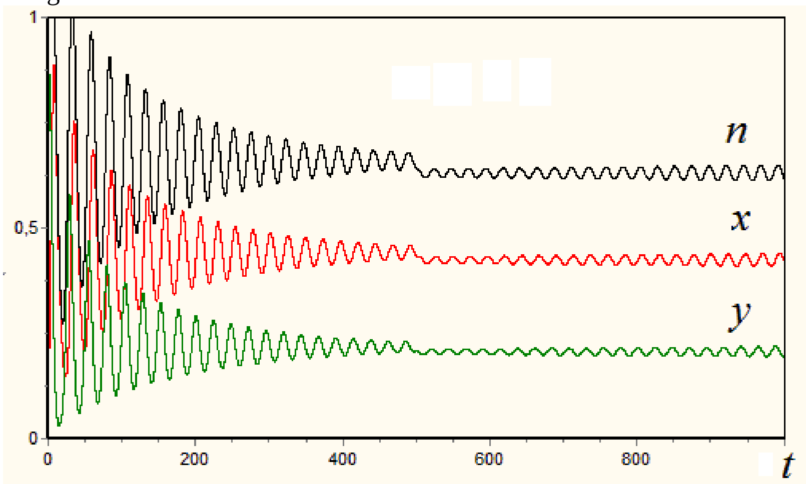


Fig. 14. A=0.2; B=0.5; C=0.43; D=1.1; An example when the parameter D varies by only 4%, but damped oscillations become stable.

The changes in institutional parameters can be implemented easier, and can be performed in a short period of time. However adiabatic parameters evolve over time, allowing the system to adapt to the new situation. They are thus highly rigid. Obviously, the best way to influence the dynamics of the system lies in the combination of exposure to adiabatic and institutional parameters.

6 Two-component model of population growth of the Earth

On a base of the foregoing observations, it is possible to formulate a model of population growth of the Earth in the form of a chain of equations describing each country separately. Obviously, just scaling the resulting model is not possible, since the processes of flow of the population in different countries has quite different nature. In addition we have the processes of migration between countries. Apart from introducing such a model term describing the flow of rural population in the city, we must also take into account immigration, i.e. overflow of the population of one country to another. For some countries, this flow is the main source of population growth (e.g. the U.S.). This term is logical to take in a similar term describing the flow of rural population in the city as in a single country.

We assume the process of emigration occurs regularly, and go into exile in the first place for economic reasons: travel for higher wages, better living conditions, the medicine. And emigration has its source mainly in urban residents who know foreign languages, and have better opportunity to travel.

Besides these processes emigration can occur because of war, natural disasters and other force majeure, but they will not be taken into account at this stage. Given all this system of equations will have the following form:

$$\begin{aligned}\frac{dx_i}{dt} &= -a_i x_i(t) + \beta_i \frac{x_i(t-\tau_i) y_i(t-\tau_i)}{x_i^2(t)+\alpha_i^2} + \sum_k \gamma_{ik} \frac{x_i(t-\tau_i) x_k(t-\tau_k)}{x^2{}_i(t) + x^2{}_k(t) + \alpha^2{}_i + \alpha^2{}_k} \\ \frac{dy_i}{dt} &= b_i y_i(t) - c_i y_{i_i}{}^2(t) - \beta_i \frac{x_i(t-\tau_i) y_i(t-\tau_i)}{x_i^2(t)+\alpha_i^2}\end{aligned} \quad (8)$$

Here i = 1,2,3, ... N denotes the total number of countries in the world. Totally turns out 2N differential equations. The Earth's population will be calculated as a simple sum of all (x_i +y_i). Analysis of the system (8) can be carried out numerically, similar to the analysis of the system (6). Detailed analysis of this system will be presented in further publications.

The proposed model of the Earth's population growth (8) has obvious advantages compared with other models and can be a good basis for the calculation of the realistic medium-term and long-term forecast population growth of the Earth, which is important to many international organizations such as UNESCO or the United Nations. This work however is very time-consuming because of the need to analyze a large amount of statistical data on population growth in the countries and on this basis to determine the model parameters, such as the rate of flow of the villagers in the city and residents of one country to another.

We stress that discussed approach to study population dynamics allows introducing less numbers of variables to describe population growth. It also can explain some peculiarities in population dynamics and can be used for more effective managerial solutions in social aspects of human life.

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