Chaos in Parametrically Excited Continuous Systems

Tatyana S. Krasnopolskaya[1]
Alexandre A. Gourjii[2]
Viacheslav M. Spektor[1]
Dmytro F. Prykhodko[3]

[1] Institute of Hydromechanics NAS of Ukraine, Kyiv, Ukraine
(E-mail: t.krasnopolskaya@tue.nl; v.m.spektor@gmail.com)
[2] National Technical university of Ukraine "KPI", Kyiv, Ukraine
(E-mail: a.gourjii@gmail.com)
[3] Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
(E-mail: dmitry.prykhodko@gmail.com)

Abstract: Two new mathematical models of cross-waves generation in fluid free surface between two cylindrical shells when the inner wall vibrates radially and parametric oscillations of a cantilever bar with low bending rigidity are worked out. In the cases of internal resonances parametric oscillations of continuous systems are approximated by two eigenmodes with different eigen frequencies. Those two eigen modes are dominant and they are resonant. On the basis of analysis of the largest Lyapunov exponents for a complex system three types of steady-state regimes are found: periodic, quasi-periodic and chaotic regimes. Phase portraits and power spectra are constructed and studied. Attention is concentrated mainly on the properties of chaotic attractors.
Keywords: Waves in fluid free surface, Cross-waves, Cantilever bar, Bending rigidity, Eigenmodes.

1 Introduction

The phenomenon of deterioration of fluid free-surface waves between two cylindrical shells when the inner wall vibrates radially, is rather known, Faraday, 1831, [3]. The waves may be excited by harmonic axisymmetric deformations of the inner shell and depending on the vibration frequency both axisymmetric and non-symmetric wave patterns may arise. Experimental observations have revealed that waves are excited in two different resonance regimes. The first type of waves corresponds to forced resonance, in which axisymmetric patterns are realized with eigenfrequencies equal to the frequency of excitation. The second kind of waves is parametric resonance waves and in this case the waves are "transverse", with their crests and troughs aligned perpendicular to the vibrating wall. These so-called cross-waves have frequencies equal to half of that of the wavemaker, Krasnopolskaya, 1996, [4]. To obtain a lucid picture of energy transmission from the wavemaker motion (inner shell vibrations) to the fluid free-surface motion the method of superposition has been used.

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As the second task oscillation regimes of a cantilever bar with low bending rigidity are studied in the present paper. In the case of internal resonance parametric oscillations of cantilever bar with low bending rigidity are approximated by two eigenmodes with different eigen frequencies, Krasnopolskaya, 2012, [5].

2 Two Mode Model of Cross-waves

Let us theoretically consider the nonlinear problems of fluid free-surface waves which are excited by inner shell vibrations in a volume between two cylinders of finite length. It is useful to relate the fluid motion to the cylindrical coordinate system (r,θ,x). The fluid has an average depth d ; the average position of the free surface is taken as $x=0$, so that the solid tank bottom is at $x=-d$. The fluid is confined between a solid outer cylinder at $r=R_2$ and a deformable inner cylinder (which acts as the wavemaker) at average radius $R_1 = r_1 + a_0(d)^{-1}\int_{-d}^{0}\cos(\eta x)dx = r_1 + 2a_0/\pi$. This inner cylinder vibrates harmonically in such a way that the position of the wall of the inner cylinder is $r = R_1 + \chi_1(x,t) = R_1 - (a_0 + a_1\cos\omega t)\cos\eta x - 2a_0/\pi$,where $\eta = \pi/(2d)$. Assuming that the fluid is inviscid and incompressible, and that the induced motion is irrotational, the velocity field can be written as $\mathbf{v} = \nabla\phi$, with $\phi(r,\theta,x,t)$ the velocity potential. The governing equation is

$$\nabla^2\phi = 0 \qquad \text{on} \qquad (R_1+\chi_1 \le r \le R_2, 0 \le \theta \le 2\pi, -d \le x \le \zeta) \tag{1}$$

where $\zeta(r,\theta,t)$ is free surface displacement.

The dynamic and kinematic free-surface boundary conditions are:

$$\phi_t + 1/2(\nabla\phi)^2 + g\zeta = F(t)$$

$$\phi_x = \nabla\phi\cdot\nabla\zeta + \zeta_t \qquad \text{at} \qquad x = \zeta(r,\theta,t) \tag{2}$$

with g the gravitational acceleration, ρ the fluid density, $F(t)$ is an arbitrary function of time. Here and later the subscripts x,r,θ,t signify partial differentiation.

The kinematic condition at the vibrating inner cylinder is:

$$\phi_r = \chi_t + \nabla\phi\cdot\nabla\chi_1 \qquad \text{at} \qquad r = R_1 + \chi_1(x,t). \tag{3}$$

From the experimental observations we may conclude that the pattern formation has a resonance character, every pattern having its "own" frequency. Assuming that patterns can be described in terms of normal modes with characteristic eigenfrequencies, we expand the potential ϕ and the free-surface displacement ζ in a complete set of eigenfunctions, which are determined by linear theory. The amplitudes of these eigenfunctions are governed by the nonlinear problem (2) - (3). The potential ϕ can be written as the sum of three harmonic functions $\phi = \phi_0 + \phi_1 + \phi_2$, Lamé, 1852, [7]. The solution of the linear problem for ϕ_1 can be written in the form

$$\phi_1 = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \phi_{ij}^{c,s}(t) \frac{\cosh k_{ij}(x+d)}{N_{ij}\cosh k_{ij}d} \psi_{ij}^{c,s}(r,\theta), \quad (4)$$

on the complete systems of azimuthal ($\cos i\theta$, $\sin i\theta$), and radial eigenfunctions $\chi_{ij}(k_{ij}r) = J_i(k_{ij}r) - \frac{J_{i'}(k_{ij}R_1)}{Y_{i'}(k_{ij}R_1)} Y_i(k_{ij}r)$, with some arbitrary amplitudes $\phi_{ij}^{c,s}(t)$. In the solution (4) the notations $\psi_{ij}^{c,s}(r,\theta) = \chi_{ij}(k_{ij}r)(\cos i\theta, \sin i\theta)$ are used, where J_i and Y_i are the i-th order Bessel functions of the first and the second kind, respectively, and N_{ij} is a normalization constant, where the index c (or s) indicates that the eigenfunction $\cos i\theta$ (or $\sin i\theta$) is chosen as the circumferential component; k_{ij} represents eigen wave numbers. The system of functions $\psi_{ij}(r,\theta)$, with $i = 0,1,2,...$ and $j = 1,2,3,...$, is a complete orthogonal system, so any function of the variables r and θ can be represented using the usual procedure of Fourier series expansion. Thus, the free surface displacement $\zeta(r,\theta,t) - \zeta_0(t)$ can be written as ($\zeta_0(t)$ is the mean level of fluid free surface oscillations)

$$\zeta(r,\theta,t) - \zeta_0(t) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \zeta_{ij}^{c,s}(t) \frac{\psi_{ij}^{c,s}(r,\theta)}{N_{ij}}. \quad (5)$$

The velocity potential $\phi_2(r,\theta,x,t)$ can be formulated in terms of an ordinary Fourier series in $\cos\alpha_l x$ with $\alpha_l = l\pi/d$ and in ($\cos i\theta$, $\sin i\theta$), so that the general solution reads, Krasnopolskaya, 1996, [4]

$$\phi_2 = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty} \Phi_{il}^{c,s}(t) \cos\alpha_l x \hat{\chi}_{il}(\alpha_l r)(\cos i\theta, \sin i\theta) \quad (6)$$

with $\hat{\chi}_{il}(\alpha_l r) = I_i(\alpha_l r) - \dfrac{I'_i(\alpha_j R_2)}{K'_i(\alpha_j R_2)} K_i(\alpha_l r)$, where I_i and K_i the i-th order modified Bessel functions of the first and second kind, respectively.

Under a parametric resonance, when the excitation frequency is twice as large as one of the eigenfrequencies, i.e. $\omega \approx 2\omega_{nm}$, and according the experimental observations we may assume that the free-surface displacement can be approximated by two resonant modes. So that we may write

$$\zeta \approx \frac{1}{N_{nm}} \zeta_{nm} \psi^c_{nm}(r,\theta) + \frac{1}{N_{0l}} \zeta_{0l} \psi_{0l}(r) + \zeta_0 \tag{7}$$

where ψ_{0l} is the axisymmetric mode which has the eigenfrequency by a value very close to ω, i.e. $\omega_{0l} \approx \omega$. From the experimental observations follows that cross-waves has ampliteds much bigger than the amplitudes of the forced waves with the frequency ω of the wavemaker vibrations. So that we can seek the unknown functions in the form

$$\zeta_{nm}(t) = \varepsilon_1^{1/2} \lambda_1 \left[p_1(\tau_1) \cos\frac{\omega t}{2} + q_1(\tau_1) \sin\frac{\omega t}{2} \right];$$

$$\zeta_{0l}(t) = \varepsilon_1 \lambda_0 \left[p_2(\tau_1) \cos\omega t + q_2(\tau_1) \sin\omega t \right], \tag{8}$$

where $\lambda_1 = k_{nm}^{-1} \,\mathrm{th}(k_{nm} h)$, $\varepsilon_1 = \dfrac{a\omega_{nm}^2}{\mathrm{g}}$ is a small parameter, $\tau_1 = \dfrac{1}{4} \varepsilon_1 \,\omega t$ is a dimensionless slow time, $\lambda_0 = k_{0l}^{-1} \,\mathrm{th}(k_{0l} h)$. By substitution of the expressions (8) into boundary conditions (2)-(3), using (4)-(7) and averaging over the fast time ωt, Krasnopolskaya, 1996, [4], we finally obtain

$$\begin{aligned}
\frac{dp_1}{d\tau_1} &= -\alpha p_1 - \vartheta q_1 + \beta_3 q_1 + \beta(q_1 p_2 - p_1 q_2);\\
\frac{dq_1}{d\tau_1} &= -\alpha q_1 + \vartheta p_1 + \beta_3 p_1 + \beta(p_1 p_2 + q_1 q_2);\\
\frac{dp_2}{d\tau_1} &= -\alpha p_2 - \beta_2 q_2 - 2\beta_4 p_1 q_1;\\
\frac{dq_2}{d\tau_1} &= -\alpha q_2 + \beta_2 p_2 + \beta_4 (p_1^2 - q_1^2) + \beta_5,
\end{aligned} \tag{9}$$

where $\vartheta = \left[\beta_1 + \frac{\beta_6}{2}(p_1^2 + q_1^2) \right]$, $\alpha = \frac{\delta}{\omega_{nm}}$, δ is the ratio of actual to critical damping of the mode, β_i (*i=1,2,…6*) are constant coefficients. The dynamical system (9) is nonlinear, so numerical solutions were obtained. We used the following coefficients (Krasnopolskaya, 1996, [4] – Becker, 1991, [1]) and data:

$\alpha = 0.01$; β_1=0.1; β_2=0.1; β_3=1.3*k*; β_4=-1.2; β_5=0.235*k*;

β_6=1.12; β =-1.531; $p_1(0) = q_1(0) = p_2(0) = q_2(0) = 0.5$.

For these parameters and for different values of *k* (which is dimensionless amplitude of the wavemaker vibrations) extensive numerical calculations were carried out in order to find all steady state regimes. In Figure 1 dependence of the maximum Lyapunov exponent on value *k* is shown.

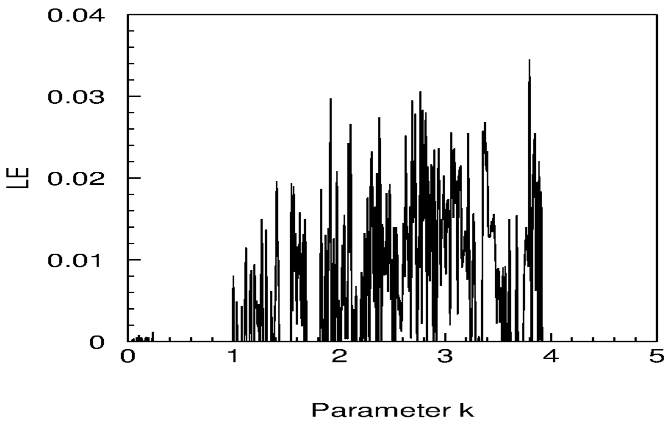
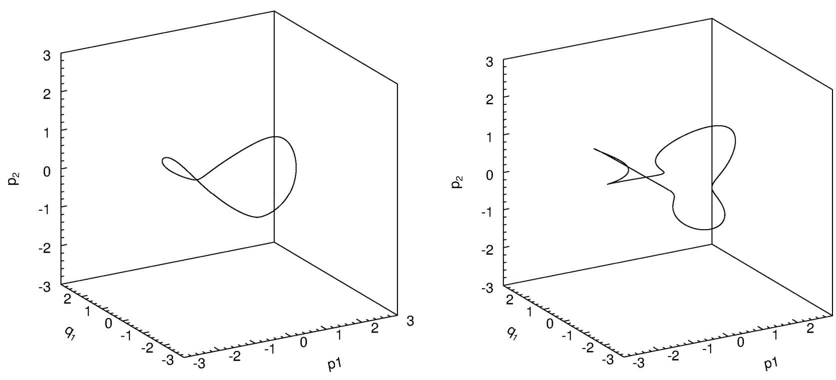
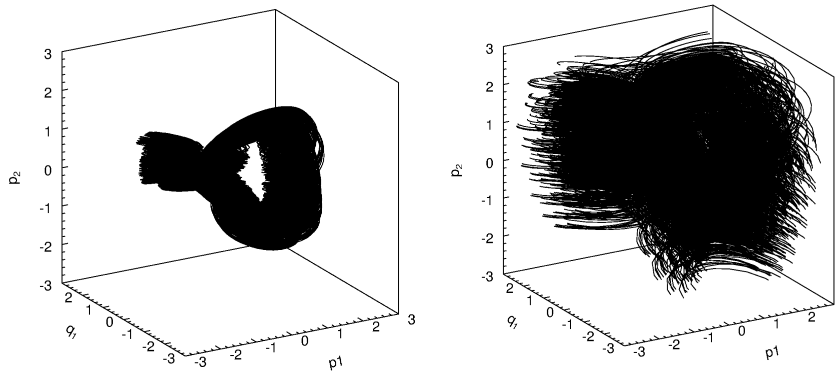


Fig. 1. The dependence of the maximum Lyapunov exponent on value *k.*

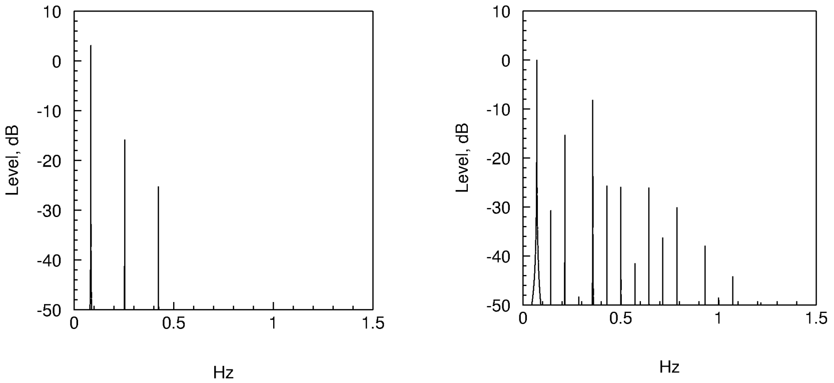


a) $k = 0.5$ b) $k = 0.8$



c) $k=1$ d) $k=3$

Fig.2. Phase portraits for regular (cases a, b) and chaotic regimes (cases c, d).



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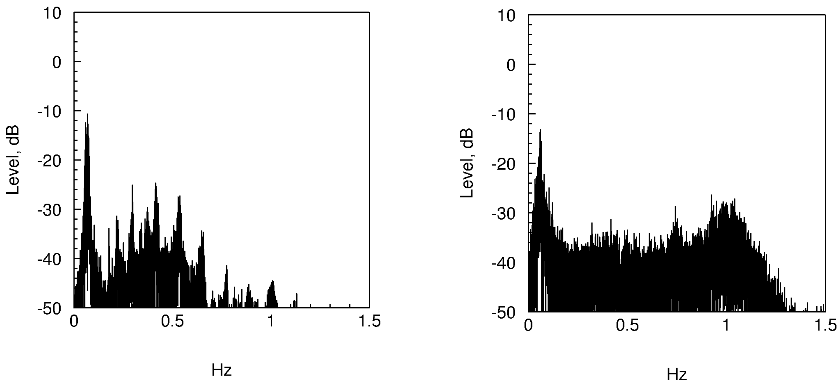


c) $k=1$ d) $k=3$

Fig. 3. Power spectra computed for p_1 data (cases a, b, c and d).

As we may conclude from numerical data and graphs in Figures 1-3 the dynamical system (9) has both regular (*k*=0.5; *k*=0.8) and chaotic regimes (*k*=1; *k*=3). The chaotic regimes could be realized when $k \geq 1$. For such values of corresponding amplitudes of wavemaker oscillations the largest Lyapunov exponents are positive, phase portraits have complicated structures of trajectory sets and power spectra are continuous ones.

3 Two Mode Approximation of Vibrations of Cantilever Bar with Low Bending Rigidity

It has been known that it is possible to stabilize a rigid pendulum and a flexible cantilever bar with very low bending rigidity vertically upwards under harmonic oscillations, Champneys, 2000, [2]. The nonlinear equation for flexible vibrations $\eta(x,t)$ of the cantilever bar can be written in the following form, Krasnopolskaya, 2013, [6]:

$$EJ_0 \frac{\partial^4 \eta}{\partial x^4} + \rho F g \frac{\partial}{\partial x}\left(\left[(l-x)+(l-x)\frac{\partial^2}{\partial t^2}\left(\frac{a}{g}\cos\omega t\right)\right]\frac{\partial \eta}{\partial x}\right) - \\ -3\alpha_3 E^3 J_2 \left[\frac{\partial^4 \eta}{\partial x^4}\frac{\partial^2 \eta}{\partial x^2} + 2\left(\frac{\partial^3 \eta}{\partial x^3}\right)^2\right]\frac{\partial^2 \eta}{\partial x^2} + \rho F \frac{\partial^2 \eta}{\partial t^2} = 0. \tag{10}$$

In this equation EJ_0 is bending rigidity, ρ is the bar density, *F* is cross section area, *a* is an amplitude (ω is a frequency) of a clamped base oscillations, *l* is a bar length, $\alpha_3 E^3 J_2$ is a constant coefficient due to nonlinear stiffness of the bar. Our experiments revealed that oscillations of the bar can be approximated by two eigenmode oscillations, namely, by the second and the third eigen modes, Krasnopolskaya, 2012, [5] when the second eigen frequency is close to a half of ω and in three times smaller than the third eigenfrequency. In this case we may write

$$\eta = \varepsilon\left[A_2(\tau)\cos\frac{\omega t}{2} + B_2(\tau)\sin\frac{\omega t}{2}\right]\varphi_2(x) + \\ \varepsilon\left[A_3\cos\frac{3\omega t}{2} + B_3\sin\frac{3\omega t}{2}\right]\varphi_3(x) \tag{11}$$

Here ε is the small parameter, $\varphi_2(x)$ is the second eigenmode, $\varphi_3(x)$ is the third eigenmode, Krasnopolskaya, 2012, [5].

By substitution of the expressions (11) into the equation (10) and averaging over the fast time we get

$$\frac{dA_2}{d\tau}=-\frac{\xi_2}{2\omega}A_2+\frac{1}{\omega^2}\left\{\frac{\gamma_1}{2}B_2-\frac{\gamma_2}{2}B_3-\xi(\omega)B_2-2I_3\gamma_0\left[\frac{3}{8}\alpha_5B_2\left(A_2^2+B_2^2\right)\right.\right.$$

$$\left.\left.+\frac{\alpha_6}{8}\left(A_2^2B_3-B_2^2B_3+2A_2B_2A_3\right)+\frac{\alpha_7}{4}B_2(A_3^2+B_3^2)\right]\right\};$$

$$\frac{dB_2}{d\tau}=-\frac{\xi_2}{2\omega}B_2+\frac{1}{\omega^2}\left\{\frac{\gamma_1}{2}A_2+\frac{\gamma_2}{2}A_3+\xi(\omega)A_2+2I_3\gamma_0\left[\frac{3}{8}\alpha_5A_2\left(A_2^2+B_2^2\right)\right.\right.$$

$$\left.\left.+\frac{\alpha_6}{8}\left(A_2^2A_3-B_2^2A_3+2A_2B_2B_3\right)+\frac{\alpha_7}{4}A_2(A_3^2+B_3^2)\right]\right\};$$

$$\frac{dA_3}{d\tau}=-\frac{\xi_2}{2\omega}A_3+\frac{1}{\omega^2}\left\{-\frac{\gamma_3}{6}B_2-\xi_3(\omega)B_3-\frac{2}{3}I_3\gamma_0\left[\frac{\alpha_9}{8}\left(3A_2^2B_2-B_2^3\right)+\right.\right.$$

$$\left.\left.+\frac{\alpha_{10}}{8}B_3\left(A_2^2+B_2^2\right)+\frac{3\alpha_{12}}{8}B_3(A_3^2+B_3^2)\right]\right\};$$

$$\frac{dB_3}{d\tau}=-\frac{\xi_2}{2\omega}B_3+\frac{1}{\omega^2}\left\{+\frac{\gamma_3}{6}A_2+\xi_3(\omega)A_3+\frac{2}{3}I_3\gamma_0\left[\frac{\alpha_9}{8}A_2\left(A_2^2-3B_2^{\,2}\right)+\right.\right.$$

$$\left.\left.+\frac{\alpha_{10}}{8}A_3\left(A_2^2+B_2^2\right)+\frac{3\alpha_{12}}{8}A_3(A_3^2+B_3^2)\right]\right\};$$

In our calculations the following parameters have been used

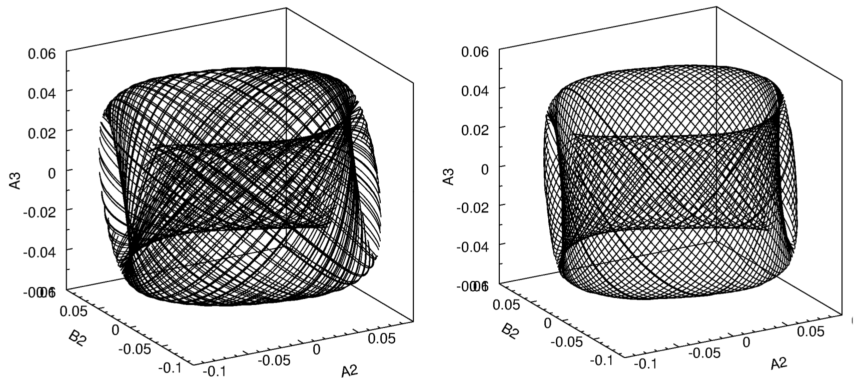
ρ = 1.7*10^{-3} *kg/см*^{3}; g = 980 *см/sec*^{2}; *a* = 0.9*см*; B =0.055; *l* = 26.7*см*;
r = 0.15 *см*; G = 0.1398*10^9 kg/(см g sec^2); g_2 = 0.0547 g10^6;
λ_2 = 18.031; λ_3 = 184.32; *E*=1.4227*10^5*kg/(см gsec*^{2}),

$$\xi_2=\frac{8.0\cdot10^{-5}}{\rho Fl}=0.17\sec^{-1},\quad \gamma_0=\frac{g}{l},\quad \alpha_3^*=\frac{2}{27}\frac{Eg_2}{G^3},$$

$$I_3=5B\alpha_3^*E^2\cdot10^{-5},\quad \xi(\omega)=\frac{g}{2A}-\frac{g}{A\omega}\sqrt{\frac{g\lambda_2}{l}},$$

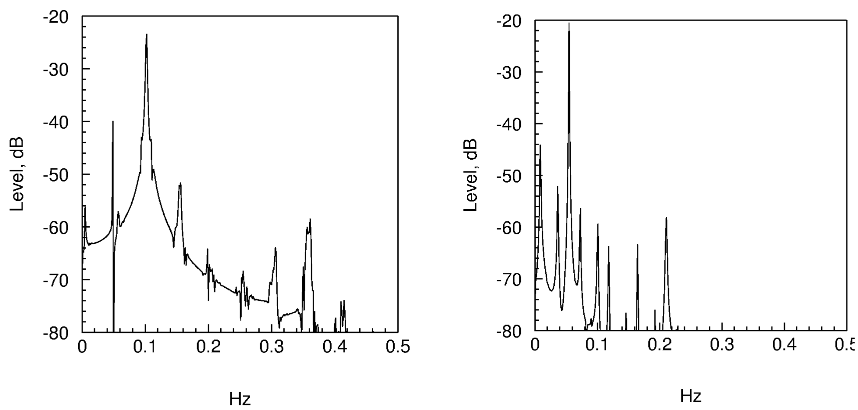
$$\xi_3(\omega)=\frac{3g}{2A}-\frac{g}{A\omega}\sqrt{\frac{g\lambda_3}{l}}.$$

Only regular regimes were found for different initial conditions as steady state regimes. The maximum Lyapunov exponents were not positive for all of them. In Figure 4 phase portrait projections are shown for quasi-periodic (ω =40 *rad/sec*) and periodic (ω =60 *rad/sec*) regimes. Power spectra for these regimes are presented in Figure 5, where only several spikes are visible. Quasi-periodic and periodic regimes are typical for the above-mentioned dynamic system which has a symmetry relatively unknown variables.



a) $\omega = 40$ b) $\omega = 60$

Fig. 4. Phase portraits for different excitation frequencies.



a) $\omega = 40$ b) $\omega = 60$

Fig. 5. Power spectra computed for A_2 time realization for different frequencies of clamped base oscillations.

4 Conclusions

Two new models expressing interaction of two eigenmodes at the condition of internal resonances and parametric oscillations of continuous systems are developed. Models are simulated. The existence of chaotic attractors was established for the dynamical system presenting cross-waves and forced waves interaction at fluid free-surface in a volume between two cylinders of finite length. For averaged symmetric systems describing two parametric eigen modes of a flexible cantilever bar with very low bending rigidity no chaotic regimes were found.

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