

# Stochastic Properties of Dynamical Systems Arising from (quantum) Spaces and Actions of (quantum) Groups

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**Abstract.** We review novel results and investigate actions and transformations of (quantum) groups and semigroups on (quantum) spaces, present dynamical systems and zeta functions arising from these spaces, actions and transformations, discuss their stochastic properties.

**Keywords:** Dynamical System, Ergodic Transformation, Group Action, Equidistribution, Zeta function, Arithmetic Surface.

## 1 Introduction

A history of a semigroup and a group action on tori and projective spaces can be found among other in the book by A.G. Postnikov [1], in the paper by I.Ya. Gol'dsheid, G.A. Margulis [2] and in the supplement by B.M. Gurevich, Ya. G. Sinai [3] to the Russian translation of the English edition of the book by P. Billingsley [4].

Here we review novel results and investigate actions and transformations of (quantum) groups and semigroups on (quantum) spaces, present dynamical systems and zeta functions arising from these spaces, actions and transformations, discuss their stochastic properties.

## 2 Dynamical systems from spaces

It is well known that one-dimensional projective space  $\mathbf{P}^1(\mathbf{Q})$  parametrize the set of dynamical systems in such a way that for any rational point  $Q \in \mathbf{P}^1(\mathbf{Q})$ ,  $Q = (\frac{a}{b}, 1)$ ,  $a, b \in \mathbf{Z}$ ,  $(a, b) = 1$  we naturally associate dynamical system  $(\mathbf{T}, T_Q)$ . Here  $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ ,  $\mathbf{T}^{\mathbf{Z}} = (\dots, x_{-1}, x_0, x_1, \dots)$ ,  $x_i \in \mathbf{T}$ ,  $X = \{\mathbf{x} = (x_k) : bx_{k+1} = ax_k \text{ for all } k \in \mathbf{Z}\}$ ,  $T_Q : X \rightarrow X$ . More generally, for any primitive polynomial  $g(x) \in \mathbf{Z}[x]$  of degree  $d \geq 1$  it is possible to construct its Frobenius and companion matrices and define a homeomorphism  $T_F$  of a compact  $d$ -dimensional subgroup of  $\mathbf{T}^d$ . These considerations can be extended to elliptic curves [5] and to abelian varieties. For elliptic curves authors of the paper



[5] implement these by the following way. Let  $q \in \mathbf{Q}_p$  and  $\log^+ x$  denotes  $\max\{\log x, 0\}$ . For a generic element  $x$  of  $\mathbf{Z}_p$  authors define  $q$ -transformation  $T_q(x)$  (a  $p$ -adic analogue of the  $\beta$ -transformation). Then the topological entropy of the  $p$ -adic  $\beta$ -transformation is given by  $h(T_q) = \log^+ |q|_p$  ([5], Theorem 4.1). If  $|q|_p \geq 1$  then the map  $T_q$  is ergodic with respect to Haar measure for  $|q|_p > 1$  and is not ergodic for  $|q|_p = 1$  ([5], Theorem 4.2). Let  $Per_n(T_q)$  denotes the subgroup of  $\mathbf{Z}_p$  consisting of elements of period  $n$  under  $T_q$ . Let  $U$  be the set of unit roots of  $\mathbf{Q}_p$  and  $q \in \mathbf{Q}_p \setminus U$ . Then

$$\log |Per_n(T_q)| = n \log^+ |q|_p.$$

([5], Theorem 4.3). The authors use the topological entropy and measure theoretical arguments based on volume growth rate and arithmetic of  $\mathbf{Z}_p$ .

Let  $Q$  be a rational point of an elliptic curve over  $\mathbf{Q}$  and let  $\hat{h}(Q)$  be the global canonical height on rational points of the elliptic curve. Then with the definitions and assumptions of the paper [5] and  $q = a/b = x(Q)$ , (i) the entropy of  $T_Q$  is given by  $h(T_Q) = 2\hat{h}(Q)$ , and (ii) the asymptotic growth rate of the periodic points is given by the division polynomial  $\nu_n(x)$ :  $\log |Per_n(T_Q)| \sim \log |b^n \nu_n(q)|$  as  $n \rightarrow \infty$ . ([5], Theorem 5.2). In the case authors use also the elliptic analogue of Baker's theorem, which described in paper [6] and in paper [7].

### 3 Dynamical systems on probability spaces

Let  $(X, B, \mu, T)$  be a dynamical system on standard probability space with  $T : X \rightarrow X$  is measurable, almost surely one to one, preserves  $\mu$ , for which it is an ergodic transformation. Random dynamical systems relate a partial case of bundle dynamical systems by I. Cornfeld, S. Fomin, and Ya. Sinai [8]. Measurable partition of the space  $X$  transforms the initial random dynamical system into a symbolic dynamical system. We will present novel symbolic dynamical systems and their applications.

### 4 Rigid and weakly mixing ergodic transformations

In papers [9] and [10] authors present recent results on genericity of rigid and multiply recurrent infinite measure preserving and nonsingular transformations and on measurable sensitivity. In the paper [11] authors investigate properties of uniformly rigid transformations and analyze the compatibility of uniform rigidity and measurable weak mixing along with some of their asymptotic convergence properties. All spaces of the paper under review are considered simultaneously as topological spaces and as measure spaces. Presented results concern either the measurable dynamics on the spaces or the interplay between the measurable and topological dynamics. The notion of uniform rigidity was introduced as a topological version of rigidity by S. Glasner and D. Maon [12]. Authors of the paper [11] consider functional analytic properties of uniform rigidity that is similar to the properties of rigidity. Theorem 1 ([11]). Every

totally ergodic finite measure-preserving transformation on a Lebesgue space has a representation that is not uniformly rigid, except in the case where the space consists of a single atom.

The proof of the theorem connects with results of authors of the theorem that uniform rigidity and weak mixing are mutually exclusive notions on a Cantor set, and follows from the Jewett-Krieger Theorem by K. Peterson [13].

## 5 Superrigidity for groups

The concept of superrigidity was introduced by G. D. Mostow [14] and by G. A. Margulis [15] in the context of studying the structure of lattices in rank one and higher rank Lie groups respectively. The notion of property (T) for locally compact groups was defined by D. Kazhdan [16] and the notion of relative property (T) for inclusion of countable groups  $\Gamma_0 \subset \Gamma$  was defined by G. Margulis [17]. Now consider the orbit equivalence (OE) superrigidity. One of the first result of this type of superrigidity was obtained by A. Furman [18], who combined the cocycle superrigidity by R. Zimmer [19] with ideas from geometric group theory to show that the actions  $SL_n(\mathbf{Z})$  on  $\mathbf{T}^n$  ( $n \geq 3$ ) are OE superrigid. The deformable actions of rigid groups are OE superrigid by S. Popa [20]. The main result of the paper by A. Ioana [21] is the Theorem A on orbit equivalence (OE) superrigidity. As a consequence of Theorem A the author of the paper [21] can construct uncountable many non-OE profinite actions for the arithmetic groups  $SL_n(\mathbf{Z})$  ( $n \geq 3$ ), as well as for their finite subgroups, and for the groups  $SL_m(\mathbf{Z}) \times \mathbf{Z}^m$  ( $m \geq 2$ ). The author deduces Theorem A as a consequence of the Theorem B on cocycle superrigidity.

Let the action of  $\Gamma$  on  $X$  be a free ergodic measure-preserving profinite action (i.e., an inverse limit of actions  $\Gamma$  on  $X_n$  with  $X_n$  finite) of a countable property (T) group  $\Gamma$  (more generally, of a group  $\Gamma$  which admits an infinite normal subgroup  $\Gamma_0$  such that the inclusion  $\Gamma_0 \subset \Gamma$  has relative property (T) and  $\Gamma/\Gamma_0$  is finitely generated) on a standard probability space  $X$ . The author prove that if  $\omega : \Gamma \times X \rightarrow A$  is a measurable cocycle with values in a countable group  $A$ , then  $\omega$  is cohomologous to a cocycle  $\omega'$  which factors through the map  $\Gamma \times X \rightarrow \Gamma \times X_n$ , for some  $n$ . As a corollary, he shows that any free ergodic measure-preserving action  $A$  on  $Y$  comes from a (virtual) conjugacy of actions.

## 6 Equidistribution for orbits of nonabelian semigroups on the torus

Furstenberg [22] and Berent [23] have investigated the action of abelian semigroups on the torus  $\mathbf{T}^d$  for  $d = 1$  and  $d > 1$  respectively. Their results answer problems raising by H. Furstenberg [24]. Authors of the paper [25] extend to the noncommutative case some results of Furstenberg and Berent

## 7 Zeta functions from spaces and dynamical systems

Recall that Dedekind has defined zeta function for polynomials over prime finite field. The zeta function is trivial and equal to  $\frac{1}{1-pz}$ . However, combining the zeta function with Chebyshev-Mobius inversion formula we obtain the number of monic irreducible over  $\mathbf{F}_p$  polynomials of natural degree  $m$ . Riemann and Dedekind zeta functions are first examples of motivic zeta functions. The authors of the paper [26] investigate sufficient conditions for (i) the existence of trace formulae for the Reidemeister number of a group endomorphism; (ii) the rationality of the Reidemeister zeta function and the convergence of the Nielsen zeta function; (iii) the equality of Reidemeister torsion of a group endomorphism to a special value of the Reidemeister zeta. This interesting survey [26] includes recent results on trace formulae, rationality and convergence of zeta functions and relations between special values of zeta functions and some simply homotopy invariants. The general setting of the paper [27] is braided zeta functions in  $q$ -deformed geometry. In the framework authors define a zeta function for any rigid object in a ribbon braided category. In the ribbon case they define braided Hilbert series for objects in an Abelian braided category. We will present some other types of zeta-functions.

## 8 Dynamical Systems from Arithmetic Surfaces

### 8.1 Sato-Tate case

Let  $y^2 = f(x)$ ,  $f(x) = x^3 + cx + d$  be a cubic polynomial in prime finite field  $\mathbf{F}_p$ . For the number  $\#C_p$  of points of the curve  $C : y^2 = f(x)$  in  $\mathbf{F}_p$  the well known formula

$$\#C_p = \sum_{x=0}^{p-1} \left( 1 + \left( \frac{f(x)}{p} \right) \right),$$

take place, where  $\left( \frac{f(x_0)}{p} \right)$  is the Legendre symbol with a numerator which is equal to the value of the polynomial  $f(x_0)$  in point  $x_0 \in \mathbf{F}_p$ . It is ease to see that  $\#C_p = p - a_p$ , where

$$a_p = - \sum_{x=0}^{p-1} \left( \frac{f(x)}{p} \right)$$

If  $C$  is the elliptic curve, then the number of points  $\#C(\mathbf{F}_p)$  of the projective model of the curve in  $\mathbf{F}_p$  is represented by the formula  $\#E_p = 1 + p - a_p$ , where  $a_p = 2\sqrt{p} \cos \varphi_p$ . If  $C$  is not the elliptic curve, then the value  $a_p$  is equal 1, -1 or 0 and ease to compute. In both cases compute:  $\varphi_p = \arccos(a_p/2\sqrt{p})$  and reduce it to the interval  $[0, \pi]$ .

Let  $E$  be an elliptic curve over rational numbers  $\mathbf{Q}$  which does not admit complex multiplication. Sato and Tate [28] have given computational and theoretical evidences suggesting the distribution of angles  $\varphi_p$ .

Recently L. Clozel, M. Harris, N. Shepherd-Barron, R. Taylor and their colleagues have proved the Sato-Tate conjecture for all elliptic curves  $E$  over

**Q** (and over some its extensions) satisfying the mild condition of having multiplicative reduction at some prime.

Langlands conjectured that some symmetric power  $L$ -functions extend to an entire function and coincide with certain automorphic  $L$ -functions.

**Theorem** (Clozel, Harris, Shepherd-Barron, Taylor). Suppose  $E$  is an elliptic curve over  $Q$  with non-integral  $j$ -invariant. Then for all  $n > 0$ ,  $L(s, E, Sym^n)$  extends to a meromorphic function which is holomorphic and non-vanishing for  $Re(s) \geq 1 + n/2$ .

These conditions suffice to prove the Sato-Tate conjecture.

Theoretical considerations give

**Proposition EC.** It is possible the arithmetic modeling of the Brownian motion by quantity  $a_p$ .

### 8.2 Kloosterman sums

Let

$$T_p(c, d) = \sum_{x=1}^{p-1} e^{2\pi i(\frac{cx+d}{p})}$$

$$1 \leq c, d \leq p - 1; \quad x, c, d \in \mathbf{F}_p^*$$

be a Kloosterman sum.

By A. Weil estimate

$$T_p(c, d) = 2\sqrt{p} \cos \theta_p(c, d)$$

There are possible two distributions of angles  $\theta_p(c, d)$  on semiinterval  $[0, \pi)$  :

a)  $p$  is fixed and  $c$  and  $d$  varies over  $\mathbf{F}_p^*$ ; what is the distribution of angles  $\theta_p(c, d)$  as  $p \rightarrow \infty$  ;

b)  $c$  and  $d$  are fixed and  $p$  varies over all primes not dividing  $c$  and  $d$ .

For the case a) N. Katz [29] and A. Adolphson [30] proved that  $\theta$  are distributed on  $[0, \pi)$  with density  $\frac{2}{\pi} \sin^2 t$ .

Let

$$cd \not\equiv 0 \pmod{p}, \quad T_p(c, d) = \sum_{x=1}^{p-1} e^{2\pi i(\frac{cx+d}{p})}$$

the Kloosterman sum. By A. Weil,  $T_p(c, d) = 2\sqrt{p} \cos \theta_p(c, d)$ . Compute  $T_p$ ,  $\cos \theta_p$ ,  $\theta_p$  and reduce  $\theta_p$  to the interval  $[0, \pi]$ . Experiments demonstrate random behavior of angles of Kloosterman sums.

Theoretical considerations give

**Proposition KS.** It is possible the arithmetic modeling of the Brownian motion by Kloosterman sums.

## Conclusions

We have presented a review of new results on actions and transformations of (quantum) groups and semigroups on (quantum) spaces, have presented dynamical systems and zeta functions arising from these spaces, actions and transformations, discussed their stochastic properties.

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