New Criteria for Generalized Synchronization Preserving the Chaos Type

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Abstract. We provide new conditions for the presence of generalized synchronization in unidirectionally coupled systems. One of the main results in the paper is the preservation of the chaos type of the drive system. The analysis is based on the Devaney definition of chaos. Appropriate simulations which illustrate the generalized synchronization are depicted.
Keywords: Generalized synchronization, Devaney chaos, Chaotic set of functions.

1 Introduction

The most general ideas about the synchronization of different chaotic systems with an unrestricted form of coupling can be found in paper [1]. Rulkov et al. [2] realized this proposal by introducing the concept of generalized synchronization (GS) for unidirectionally coupled systems. The concept of GS [2]-[5] characterizes the dynamics of a response system that is driven by the output of a chaotic driving one.

In the present paper, the drive system will be considered in the following form

$$x' = F(x), \tag{1}$$

where $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function, and the response is assumed to have the form

$$y' = Ay + g(x, y), \tag{2}$$

where $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function in all its arguments and the constant $n \times n$ real valued matrix A has real parts of eigenvalues all negative. We assume that system (1) admits a chaotic attractor.

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GS is said to occur if there exist sets I_x, I_y of initial conditions and a transformation $\varphi : \mathbb{R}^m \to \mathbb{R}^n$, defined on the chaotic attractor of the drive system, such that for all $x(0) \in I_x$, $y(0) \in I_y$ the relation $\lim_{t\to\infty} \|y(t) - \varphi(x(t))\| = 0$ holds. In this case, a motion which starts on $I_x \times I_y$ collapses onto a manifold $M \subset I_x \times I_y$ of synchronized motions. The transformation φ is not required to exist for the transient trajectories [2,3].

According to the results of [3], GS occurs if and only if for all $x_0 \in I_x$, $y_{10}, y_{20} \in I_y$, the following criterion holds:

(A) $\lim_{t\to\infty} \|y(t, x_0, y_{10}) - y(t, x_0, y_{20})\| = 0,$

where $y(t, x_0, y_{10}), y(t, x_0, y_{20})$ denote the solutions of (2) corresponding to the initial data $y(0, x_0, y_{10}) = y_{10}$, $y(0, x_0, y_{20}) = y_{20}$ with the same $x(t)$, $x(0) = x_0$.

A consequence of GS is the ability to predict the behavior of $y(t)$, based on the knowledge of $x(t)$ and φ only. If φ is invertible $x(t)$ is also predictable from $y(t)$. The usage of statistical estimations of predictability [2], analysis of conditional Lyapunov exponents [3] and the auxiliary system approach [4] are the main approaches to the observation of GS.

Let us introduce the ingredients of Devaney chaos [6] for continuous time dynamics. Denote by

$$\mathscr{B} = \{\psi(t) \mid \psi : \mathbb{R} \to K \text{ is continuous}\}$$

a collection of functions, where $K \subset \mathbb{R}^q$ is a bounded region.

We say that $\mathscr{B}$ is sensitive if there exist positive numbers ϵ and Δ such that for every $\psi(t) \in \mathscr{B}$ and for arbitrary $\delta > 0$ there exist $\overline{\psi}(t) \in \mathscr{B}$, $t_0 \in \mathbb{R}$ and an interval $J \subset [t_0, \infty)$, with length not less than Δ, such that $\left\|\psi(t_0) - \overline{\psi}(t_0)\right\| < \delta$ and $\left\|\psi(t) - \overline{\psi}(t)\right\| > \epsilon$, for all $t \in J$.

On the other hand, the collection $\mathscr{B}$ is said to possess a dense function $\psi^*(t) \in \mathscr{B}$ if for every $\psi(t) \in \mathscr{B}$, arbitrary small $\epsilon > 0$ and arbitrary large $E > 0$, there exist a number $\xi > 0$ and an interval $J \subset \mathbb{R}$, with length E, such that $\|\psi(t) - \psi^*(t + \xi)\| < \epsilon$, for all $t \in J$. We say that $\mathscr{B}$ is transitive if it possesses a dense function.

Furthermore, $\mathscr{B}$ admits a dense collection $\mathscr{G} \subset \mathscr{B}$ of periodic functions if for every function $\psi(t) \in \mathscr{B}$, arbitrary small $\epsilon > 0$ and arbitrary large $E > 0$, there exist $\widetilde{\psi}(t) \in \mathscr{G}$ and an interval $J \subset \mathbb{R}$, with length E, such that $\left\|\psi(t) - \widetilde{\psi}(t)\right\| < \epsilon$, for all $t \in J$.

The collection $\mathscr{B}$ is called a Devaney chaotic set if: (i) $\mathscr{B}$ is sensitive; (ii) $\mathscr{B}$ is transitive; (iii) $\mathscr{B}$ admits a dense collection of periodic functions.

We present two main results in the paper. The first one is the the occurrence of GS in system (1)+(2), and the second one is the preservation of the chaos type of the drive system. The GS is verified in the next section by means of the criterion (A). The third section is devoted for the presence of Devaney chaos in the response system. Moreover, an example that supports our theoretical discussions is presented in the last section.

2 Preliminaries

Throughout the paper, the uniform norm $\|\Gamma\| = \sup_{\|v\|=1} \|\Gamma v\|$ for matrices will be used.

Since the matrix A, which is aforementioned in system (2), is supposed to admit eigenvalues all with negative real parts, there exist positive real numbers N and ω such that $\left\|e^{At}\right\| \leq Ne^{-\omega t}$, $t \geq 0$. These numbers will be used in the last condition below.

The following assumptions on systems (1) and (2) are needed throughout the paper:

(A1) There exists a number $H_0 > 0$ such that $\sup_{x \in \mathbb{R}^m} \|F(x)\| \leq H_0$;

(A2) There exists a number $L_0 > 0$ such that $\|F(x_1) - F(x_2)\| \leq L_0 \|x_1 - x_2\|$, for all $x_1, x_2 \in \mathbb{R}^m$;

(A3) There exists a number $M_0 > 0$ such that $\sup_{x \in \mathbb{R}^m, y \in \mathbb{R}^n} \|g(x, y)\| \leq M_0$;

(A4) There exist numbers $L_1 > 0$ and $L_2 > 0$ such that

$$L_1 \|x_1 - x_2\| \leq \|g(x_1, y) - g(x_2, y)\| \leq L_2 \|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathbb{R}^m$, $y \in \mathbb{R}^n$;

(A5) There exists a number $L_3 > 0$ such that

$$\|g(x, y_1) - g(x, y_2)\| \leq L_3 \|y_1 - y_2\|,$$

for all $x \in \mathbb{R}^m$, $y_1, y_2 \in \mathbb{R}^n$;

(A6) $NL_3 - \omega < 0$.

Using the technique presented in the book [7], for a given solution $x(t)$ of system (1), one can verify the existence of a unique bounded on $\mathbb{R}$ solution $\phi_{x(t)}(t)$ of the system $y' = Ay + g(x(t), y)$, which satisfies the following integral equation

$$\phi_{x(t)}(t) = \int_{-\infty}^{t} e^{A(t-s)} g(x(s), \phi_{x(t)}(s)) ds. \tag{3}$$

Our main assumption is the existence of a nonempty set $\mathscr{A}_x$ of all solutions of system (1), uniformly bounded on $\mathbb{R}$. That is, there exists a positive real number H such that $\sup_{t \in \mathbb{R}} \|x(t)\| \leq H$, for all $x(t) \in \mathscr{A}_x$.

Let us introduce the following set of functions

$$\mathscr{A}_y = \left\{ \phi_{x(t)}(t) \mid x(t) \in \mathscr{A}_x \right\}.$$

We note that for all $y(t) \in \mathscr{A}_y$ one has $\sup_{t \in \mathbb{R}} \|y(t)\| \leq M$, where $M = NM_0/\omega$. Moreover, if $x(t) \in \mathscr{A}_x$ is periodic then $\phi_{x(t)}(t) \in \mathscr{A}_y$ is periodic with the same period, and vice versa.

Next, we will reveal that if the set $\mathscr{A}_x$ is an attractor with basin $\mathscr{U}_x$, that is, for each $x(t) \in \mathscr{U}_x$ there exists $\overline{x}(t) \in \mathscr{A}_x$ such that $\|x(t) - \overline{x}(t)\| \to 0$ as $t \to \infty$, then the set $\mathscr{A}_y$ is also an attractor in the same sense. In the following lemma we specify the basin of attraction of $\mathscr{A}_y$.

Suppose that the set $\mathscr{U}_y$ consists of solutions of the system $y' = Ay + g(x(t), y)$, where $x(t)$ belongs to $\mathscr{U}_x$.

Lemma 1. *$\mathscr{U}_y$ is a basin of $\mathscr{A}_y$.*

Proof. Fix an arbitrary $\epsilon > 0$ and let $y(t) \in \mathscr{U}_y$. There exists $\overline{x}(t) \in \mathscr{A}_x$ such that $\|x(t) - \overline{x}(t)\| \to 0$ as $t \to \infty$. Set $\alpha = \frac{\omega - NL_3}{\omega - NL_3 + NL_2}$ and $\overline{y}(t) = \phi_{\overline{x}(t)}(t)$. One can find $R_0 = R_0(\epsilon) > 0$ such that if $t \geq R_0$ then $\|x(t) - \overline{x}(t)\| < \alpha\epsilon$ and $N\|y(R_0) - \overline{y}(R_0)\| e^{(NL_3-\omega)t} < \alpha\epsilon$. Using the equation

$$\begin{aligned} y(t) - \overline{y}(t) &= e^{A(t-R_0)}(y(R_0) - \overline{y}(R_0)) \\ &+ \int_{R_0}^{t} e^{A(t-s)} \left[g(x(s), y(s)) - g(x(s), \overline{y}(s))\right] ds \\ &+ \int_{R_0}^{t} e^{A(t-s)} \left[g(x(s), \overline{y}(s)) - g(\overline{x}(s), \overline{y}(s))\right] ds, \end{aligned}$$

we obtain for $t \geq R_0$ that

$$\begin{aligned} & e^{\omega t}\|y(t) - \overline{y}(t)\| \leq Ne^{\omega R_0}\|y(R_0) - \overline{y}(R_0)\| + \frac{NL_2\alpha\epsilon}{\omega}\left(e^{\omega t} - e^{\omega R_0}\right) \\ & + NL_3\int_{R_0}^{t} e^{\omega s}\|y(s) - \overline{y}(s)\| ds. \end{aligned}$$

Applying Gronwall's inequality we attain that

$$\begin{aligned} & e^{\omega t}\|y(t) - \overline{y}(t)\| \leq \frac{NL_2\alpha\epsilon}{\omega}e^{\omega t} + N\|y(R_0) - \overline{y}(R_0)\| e^{\omega R_0}e^{NL_3(t-R_0)} \\ & -\frac{NL_2\alpha\epsilon}{\omega}e^{\omega R_0}e^{NL_3(t-R_0)} + \frac{N^2L_2L_3\alpha\epsilon}{\omega(\omega - NL_3)}e^{\omega t}\left(1 - e^{(NL_3-\omega)(t-R_0)}\right). \end{aligned}$$

Thus, we have

$$\|y(t) - \overline{y}(t)\| < N\|y(R_0) - \overline{y}(R_0)\| e^{(NL_3-\omega)(t-R_0)} + \frac{NL_2\alpha\epsilon}{\omega - NL_3}, \ t \geq R_0.$$

For $t \geq 2R_0$, one can show that $\|y(t) - \overline{y}(t)\| < \left(1 + \frac{NL_2}{\omega - NL_3}\right)\alpha\epsilon = \epsilon$. Consequently, $\|y(t) - \overline{y}(t)\| \to 0$ as $t \to \infty$. □

One can verify using Lemma 1 that for a fixed $x(t) \in \mathscr{U}_x$, any two solutions $y(t), \overline{y}(t)$ of the system $y' = Ay + g(x(t), y)$ satisfy the criterion (A). Therefore, we have the following theorem.

Theorem 1. *GS occurs in the coupled system (1)+(2).*

3 The chaotic dynamics

We will prove that if the drive system (1) is Devaney chaotic then the response system (2) is also chaotic in the same sense. The three ingredients of Devaney chaos will be considered individually. We start with sensitivity in the next lemma.

Lemma 2. *Sensitivity of the set $\mathscr{A}_x$ implies the same feature for the set $\mathscr{A}_y$.*

Proof. Fix an arbitrary $\delta > 0$ and $y(t) \in \mathscr{A}_y$. There exists $x(t) \in \mathscr{A}_x$ such that $y(t) = \phi_{x(t)}(t)$. Choose a sufficiently small number $\overline{\epsilon} = \overline{\epsilon}(\delta) > 0$ such that $\left(1 + \frac{NL_2}{\omega - NL_3}\right)\overline{\epsilon} < \delta$, and take $R = R(\overline{\epsilon}) < 0$ sufficiently large in absolute value such that $\frac{2M_0N}{\omega} e^{(\omega - NL_3)R} < \overline{\epsilon}$. Set $\delta_1 = \delta_1(\overline{\epsilon}, R) = \overline{\epsilon} e^{L_0 R}$. Since $\mathscr{A}_x$ is sensitive, there exist $\epsilon_0 > 0$, $\Delta > 0$ such that $\|x(t_0) - \overline{x}(t_0)\| < \delta_1$ and $\|x(t) - \overline{x}(t)\| > \epsilon_0$, $t \in J$, for some $\overline{x}(t) \in \mathscr{A}_x$, $t_0 \in \mathbb{R}$ and for some interval $J \subset [t_0, \infty)$ whose length is not less than Δ.

By means of continuous dependence on initial conditions, one can verify that $\|x(t) - \overline{x}(t)\| < \overline{\epsilon}$, $t \in [t_0 + R, t_0]$. Denote $\overline{y}(t) = \phi_{\overline{x}(t)}(t)$. Using the relation (3) for both $y(t)$ and $\overline{y}(t)$, we obtain for $t \in [t_0 + R, t_0]$ that

$$e^{\omega t}\|y(t) - \overline{y}(t)\| \le NL_3 \int_{t_0+R}^{t} e^{\omega s}\|y(s) - \overline{y}(s)\| ds$$
$$+ \frac{NL_2\overline{\epsilon}}{\omega}(e^{\omega t} - e^{\omega(t_0+R)}) + \frac{2M_0N}{\omega} e^{\omega(t_0+R)}.$$

Applying Gronwall's Lemma to the last inequality we attain that

$$\|y(t) - \overline{y}(t)\| \le \frac{NL_2\overline{\epsilon}}{\omega - NL_3} + \frac{2M_0N}{\omega} e^{(NL_3 - \omega)(t - t_0 - R)},\ t \in [t_0 + R, t_0].$$

Consequently, we have $\|y(t_0) - \overline{y}(t_0)\| \le \frac{NL_2\overline{\epsilon}}{\omega - NL_3} + \frac{2M_0N}{\omega} e^{(\omega - NL_3)R} < \delta$.

Next, we will show the existence of a positive numbers ϵ_1, $\overline{\Delta}$ and an interval $J^1 \subset J$ with length $\overline{\Delta}$ such that the inequality $\|y(t) - \overline{y}(t)\| > \epsilon_1$ holds for all $t \in J^1$.

Suppose that $g(x, y) = (g_1(x, y), g_2(x, y), \ldots, g_n(x, y))$, where each g_j, $1 \le j \le n$, is a real valued function.

Since $\mathscr{A}_x$ and $\mathscr{A}_y$ are both equicontinuous on $\mathbb{R}$, and the function $\overline{g} : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ defined as $\overline{g}(x_1, x_2, x_3) = g(x_1, x_3) - g(x_2, x_3)$ is uniformly continuous on the compact region

$$\mathscr{D} = \{(x_1, x_2, x_3) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \mid \|x_1\| \le H, \|x_2\| \le H, \|x_3\| \le M\},$$

the set $\mathscr{F}$ with elements of the form $g_j(x(t), \phi_{x(t)}(t)) - g_j(\overline{x}(t), \phi_{x(t)}(t))$, $1 \le j \le n$, where $x(t), \overline{x}(t) \in \mathscr{A}_x$, is an equicontinuous family on $\mathbb{R}$. Therefore, there exists a positive number $\tau < \Delta$, independent of $x(t), \overline{x}(t) \in \mathscr{A}_x$, $y(t), \overline{y}(t) \in \mathscr{A}_y$, such that for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \tau$ the inequality

$$\begin{aligned} &|(g_j(x(t_1), y(t_1)) - g_j(\overline{x}(t_1), y(t_1))) - (g_j(x(t_2), y(t_2)) - g_j(\overline{x}(t_2), y(t_2)))| \\ &< \frac{L_1\epsilon_0}{2n} \end{aligned} \tag{4}$$

holds, for all $1 \le j \le n$.

Condition $(A4)$ implies that for each $t \in J$, there exists an integer $j_0 = j_0(t)$, $1 \le j_0 \le n$, such that $|g_{j_0}(x(t), y(t)) - g_{j_0}(\overline{x}(t), y(t))| \ge \frac{L_1}{n}\|x(t) - \overline{x}(t)\|$.

Let s_0 be the midpoint of the interval J and $\theta = s_0 - \tau/2$. One can find an integer $j_0 = j_0(s_0)$, $1 \le j_0 \le n$, such that

$$|g_{j_0}(x(s_0), y(s_0)) - g_{j_0}(\overline{x}(s_0), y(s_0))| \ge \frac{L_1}{n}\|x(s_0) - \overline{x}(s_0)\| > \frac{L_1\epsilon_0}{n}. \tag{5}$$

According to (4), for all $t \in [\theta, \theta+\tau]$ we obtain that

$$|g_{j_0}(x(s_0), y(s_0)) - g_{j_0}(\overline{x}(s_0), y(s_0))| - |g_{j_0}(x(t), y(t)) - g_{j_0}(\overline{x}(t), y(t))| < \frac{L_1\epsilon_0}{2n}$$

and therefore by means of (5), the following inequality:

$$|g_{j_0}(x(t), y(t)) - g_{j_0}(\overline{x}(t), y(t))| > \frac{L_1\epsilon_0}{2n},\ t \in [\theta, \theta+\tau].$$

The last inequality implies that

$$\left\| \int_{\theta}^{\theta+\tau} [g(x(s), y(s)) - g(\overline{x}(s), y(s))]\, ds \right\| > \frac{\tau L_1 \epsilon_0}{2n}.$$

Therefore, we have

$$\begin{aligned} &\max_{t\in[\theta,\theta+\tau]} \|y(t) - \overline{y}(t)\| \geq \|y(\theta+\tau) - \overline{y}(\theta+\tau)\| \\ &> \frac{\tau L_1\epsilon_0}{2n} - [1+\tau(L_3 + \|A\|)] \max_{t\in[\theta,\theta+\tau]} \|y(t) - \overline{y}(t)\|, \end{aligned}$$

and hence, $\max_{t\in[\theta,\theta+\tau]} \|y(t) - \overline{y}(t)\| > \dfrac{\tau L_1\epsilon_0}{2n[2+\tau(L_3+\|A\|)]}$.

Now, suppose that at the point $\eta \in [\theta, \theta+\tau]$, the function $\|y(t) - \overline{y}(t)\|$ takes its maximum. Define $\overline{\Delta} = \min\left\{\dfrac{\tau}{2}, \dfrac{\tau L_1\epsilon_0}{8n(M\|A\| + M_0)[2+\tau(L_3+\|A\|)]}\right\}$ and $\theta^1 = \begin{cases} \eta, & \text{if } \eta \leq \theta + \tau/2 \\ \eta - \overline{\Delta}, & \text{if } \eta > \theta+\tau/2 \end{cases}$. For $t \in J^1 = [\theta^1, \theta^1 + \overline{\Delta}]$, we have

$$\begin{aligned} &\|y(t) - \overline{y}(t)\| \geq \|y(\eta) - \overline{y}(\eta)\| - \left| \int_{\eta}^{t} \|A\| \, \|y(s) - \overline{y}(s)\|\, ds \right| \\ &- \left| \int_{\eta}^{t} \|g(x(s), y(s)) - g(\overline{x}(s), \overline{y}(s))\|\, ds \right| \\ &> \frac{\tau L_1\epsilon_0}{4n[2+\tau(L_3+\|A\|)]}. \end{aligned}$$

Consequently, $\|y(t) - \overline{y}(t)\| > \epsilon_1,\ t \in J^1$, where $\epsilon_1 = \frac{\tau L_1\epsilon_0}{4n[2+\tau(L_3+\|A\|)]}$ and the length of the interval J^1 does not depend on the functions $y(t), \overline{y}(t) \in \mathscr{A}_y$. $\square$

Lemma 3. *Transitivity of $\mathscr{A}_x$ implies the same feature for $\mathscr{A}_y$.*

Proof. Fix arbitrary numbers $\epsilon > 0$, $E > 0$, and $y(t) \in \mathscr{A}_y$. There exists a function $x(t) \in \mathscr{A}_x$ such that $y(t) = \phi_{x(t)}(t)$. Let $\gamma = \frac{\omega(\omega - NL_3)}{2M_0N(\omega-NL_3)+NL_2\omega}$. Since there exists a dense solution $x^*(t) \in \mathscr{A}_x$, one can find $\xi > 0$ and an interval $J \subset \mathbb{R}$ with length E such that $\|x(t) - x^*(t+\xi)\| < \gamma\epsilon$, for all $t \in J$. Without loss of generality, assume that J is a closed interval, that is, $J = [a, a+E]$ for some real number a. Denote $y^*(t) = \phi_{x^*(t)}(t)$.

Making use of the integral equation (3) for both $y(t)$ and $y^*(t)$, one can verify for $t \in J$ that

$$e^{\omega t} \|y(t) - y^*(t+\xi)\| \leq \frac{2M_0N}{\omega} e^{\omega a} + \frac{NL_2\gamma\epsilon}{\omega}\left(e^{\omega t} - e^{\omega a}\right)$$
$$+NL_3 \int_a^t e^{\omega s} \|y(s) - y^*(s+\xi)\| \, ds.$$

Application of Gronwall's Lemma to the last inequality implies that

$$\|y(t) - y^*(t+\xi)\| \leq \frac{2M_0N}{\omega} e^{(NL_3-\omega)(t-a)} + \frac{NL_2\gamma\epsilon}{\omega - NL_3}\left(1 - e^{(NL_3-\omega)(t-a)}\right).$$

Suppose that $E > \frac{2}{\omega - NL_3} \ln\left(\frac{1}{\gamma\epsilon}\right)$. If $t \in J_1 = \left[a + \frac{E}{2}, a + E\right]$, then it is true that $e^{(NL_3-\omega)(t-a)} < \gamma\epsilon$. Consequently, we have $\|y(t) - y^*(t+\xi)\| < \left[\frac{2M_0N}{\omega} + \frac{NL_2}{\omega - NL_3}\right]\gamma\epsilon = \epsilon$, for $t \in J_1$. Thus, the set $\mathscr{A}_y$ is transitive. $\square$

In a similar way to Lemma 3 one can prove the following assertion.

Lemma 4. *If $\mathscr{A}_x$ admits a dense collection of periodic functions, then the same is true for $\mathscr{A}_y$.*

The following theorem can be proved using Lemmas 2-4.

Theorem 2. *If the set $\mathscr{A}_x$ is Devaney's chaotic, then the same is true for the set $\mathscr{A}_y$.*

In the next part, we will present an example which supports our theoretical discussions. The usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices will be used.

4 An example

We consider the Lorenz equations [8]

$$\begin{aligned} x_1' &= 10\left(-x_1 + x_2\right) \\ x_2' &= -x_2 + 28x_1 - x_1x_3 \\ x_3' &= -\frac{8}{3}x_3 + x_1x_2, \end{aligned} \tag{6}$$

as the drive system. It is known that system (6) admits sensitivity and possesses infinitely many unstable periodic solutions [8]. The equations for the response system are chosen as

$$\begin{aligned} y_1' &= -2y_1 - y_3 + 0.003y_2^2 + x_2 - \frac{1}{2}\cos x_2 \\ y_2' &= -y_1 - 2y_2 + 5x_1 + 0.01x_1^3 \\ y_3' &= y_1 - y_2 - 3y_3 + 2\tan\left(\frac{x_3 + y_2}{120}\right). \end{aligned} \tag{7}$$

System (7) is in the form of (2), where $A = \begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$. The inequality $\left\|e^{At}\right\| \leq N e^{-\omega t}$ is valid, where $N = 4.829$ and $\omega = 2$. One can verify that conditions $(A4)-(A6)$ are satisfied with constants $L_1 = \sqrt{3}/180$, $L_2 = 17\sqrt{3}$ and $L_3 = 16\sqrt{3}/75$.

According to the results of the present study, system (7) exhibits GS, saving the sensitivity feature of the drive and the existence of infinitely many unstable periodic solutions. Consider a trajectory of system (6)+(7) with $x_1(0) = 0.11$, $x_2(0) = 0.96$, $x_3(0) = 18.98$, $y_1(0) = -0.69$, $y_2(0) = -11.09$, $y_3(0) = 1.96$. Figure 1 shows the projections of this trajectory on the $y_1 - y_2 - y_3$ space, and supports the theoretical results such that the response system (7) possesses chaotic motions. According to the GS, the attractor shown in Figure 1, (a) is a nonlinear image of the chaotic attractor of system (6). Figure 1, (b), on the other hand, depicts the projection on the $x_2 - y_2$ plane, and reveals that the systems are not synchronized in the sense of identical synchronization [9].

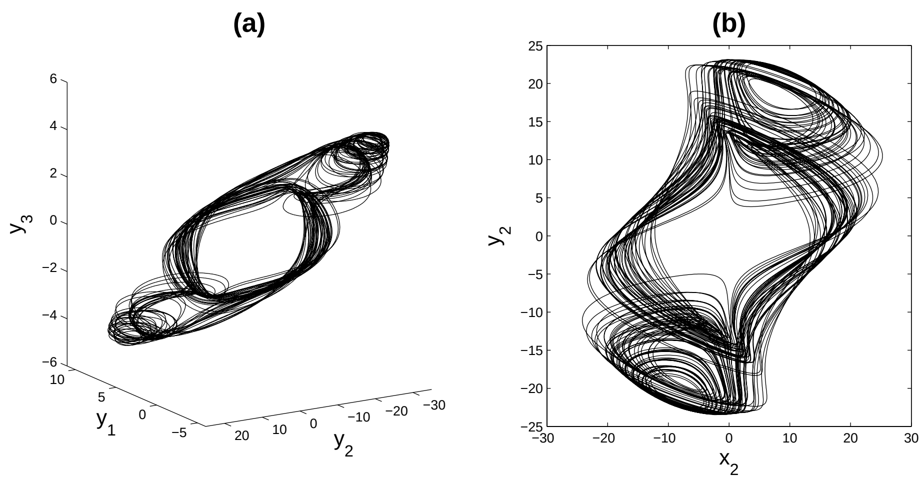

Fig. 1. The projections of the chaotic attractor generated by the coupled system (6)+(7). (a) Projection on the $y_1 - y_2 - y_3$ space; (b) Projection on the $x_2 - y_2$ plane. The pictures represent the synchronized behavior.

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