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Single-partial Model of the Nonlinear Resonant Medium

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Abstract: Responses of the nonlinear resonant medium represented by set oscillators with various types of nonlinearity are investigated. Solutions of the nonlinear equations of oscillator in the form of final Volterra series in the time and frequency domains, corresponding to anharmonicity are received. Integral transformation of input signals responses' character is displayed. Both the duality of mediums under consideration as well as classical nonlinear circuits and the opportunity of realization of signals real time processing in those mediums attention is paid to.

Keyword: oscillator, resonance medium, Volterra series, nonlinearity, dualism, signal processing.

1 Introduction

Due to the time-frequency dualism nonlinear resonant (NRM) medium makes possible to calculate integral transformations of the convolution type in frequency space with the same connectivity as multiplication in time space.
In this case nonlinear effects will lead not to frequency mixing resulting in generation of oscillations with combinational frequencies, but to time mixing, i.e. to generation of signals (pulses) at combinational instants of time [1, 3]. This time-frequency dualism phenomenon is illustrated by fig. 1.

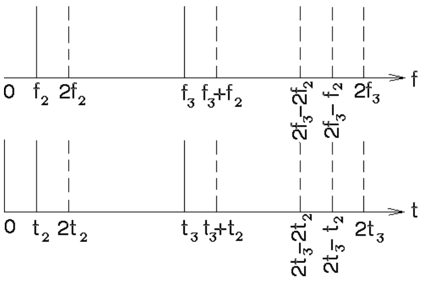

Fig. 1. Responses of nonlinear systems to multisignal excitation: above – responses of the nonlinear circuit to a series of harmonic excitations, below – responses of the nonlinear resonant medium to the excitation in the form of delta functions.

The time positions of responses are as rigidly connected to the time position of excitation pulses in nonlinear frequency space as combinational frequencies arising in a nonlinear circuit are connected to the excitation frequencies.

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Let us define the resonant medium as a set of high-Q oscillators, resonating in a frequency band. Such representation depicts the medium's local heterogeneity. The term "oscillator" here covers such concepts as separate micro particles or medium collective excitations – quasi-particles – under quantum-mechanical consideration, and as molecules or even the macroscopical particles carrying all properties of the substance - at the classical approach. In such a model the nonlinear properties of the medium can be provided both by the interaction of external excitation with a separate oscillator, and by the interaction between separate excited oscillators and thus reduced to the following types

- Anharmonicity;
- Nonlinear excitation;
- Nonlinear attenuation;
- Nonlinear interaction between oscillators.

In the latter case it is required to resolve a problem of many particles for the description of the model while solving one-partial problem is sufficient for the first three kinds of nonlinearity. The medium's response to the external excitation will be calculated by summing the responses of separate oscillators regarding with respect to their frequency distribution density $g(\omega)$. It is appropriate to mention here that the resonant medium represented by a set of oscillators is a real frequency space and it is convenient to describe it in terms of frequency representation.

The response of such nonlinear resonant medium - echo - signal – is a result of in-phase summation of oscillations of the excited oscillators, therefore the term «phased echo» is frequently used for this signal definition.

Specific physical and mathematical models distinguished by both the wide variety, and significant complexity are used in various type echo researches. In the applied perspective the theory of a spin echo [2] is most elaborated, still in this field the analysis is limited to small-signal approximation. The statistical analysis of the known physical and mathematical models of echo phenomenon in various media, not limited by the small-signal approximation framework, represents significant mathematical difficulties. The volume of such calculations even more increases due to the wide variety of specific physical mechanisms of echo – signals formation.

The purpose of the given article is to elaborate a unified description of the echo phenomenon regardless of the specific physical mechanism of its formation, suitable for the analysis of EP operation constituting a part of various radio engineering systems affected by signals and interference of any intensity. The mechanisms of nonlinearity mentioned above have the peculiarities related to responses' amplitude behavior and responses' phase - exciting pulses' phase dependence. The dependence of responses' shape on the shape of excitation pulses is the same for all types of nonlinearity. Therefore one kind of nonlinearity, that is anharmonicity, is considered in the given article.

2 NRM model with anharmonic oscillators.

Let us present the equation of the i-th anharmonic oscillator as follows

$$D_i\, y_i\, (t) + F_i\, [y_i\, (t)] = x\, (t), \qquad (1)$$

where $x\ (t)$ - external excitation, $D_i = \frac{d^2}{dt^2} + 2\sigma_i \frac{d}{dt} + \omega_{0i}^2$ - linear operator, σ_i and ω_{0i} - loss characteristic and resonant frequency of linear approximation, correspondingly, $y_i(t)$ – response of the i-th oscillator, $F\left[y(t)\right] = \sum_{k=2}^{p} a_k y^k(t)$ is a polynomial of the p-th degree, a_k - the constants including power constants and geometrical values. Later, due to the equity of all oscillators the index i will be omitted. To solve (1) let us pass to the integral relation (2)

$$y(t) = \int_{t_0}^{T} h(\tau)x(t-\tau)d\tau - \int_{t_0}^{T} h(\tau)F\left[y(t-\tau)\right]d\tau \qquad (2)$$

where $h(\tau) = \int_{0}^{\infty} \left[F\{y\}\right]^{-1} e^{j\omega t} \frac{d\omega}{2\pi}$ - (3)

pulse function of the linear part of (1), $[F\ \{\cdot\}]^{-1}$ – inverse Fourier transform. Substituting specific operator D in (3) we will have

$$h(\tau) = \begin{cases} \frac{1}{\omega_0} e^{-\sigma\tau} \sin \omega_e \tau, & \tau > 0, \\ 0, & \tau \le 0, \end{cases} \qquad (4)$$

where $\omega_e = \sqrt{\omega_0^2 - \sigma^2} \approx \omega_0$.

The solution of (2) will be found by the iterative method that results in the representation of $y\ (t)$ in the form of Volterra finite series in case of weak nonlinearity (a_k <<1, k = 2, 3, …, p):

$$y(t) = h_p + \sum_{p=1}^{n} \int_{E^p} h_p(\tau_1,...,\tau_p) \prod_{i=1}^{p} x(t-\tau_i)d\tau_i\ , \qquad (5)$$

where E^p - *p-dimensional* Euclidean space, in which Volterra kernels $h_p\ (\tau_1,\ \tau_2,\ \ldots,\ \tau_p)$, representing pulse functions of nonlinear transformation of the p-th order are determined. So, for example,

$$h_2(\tau_1,\tau_2) = \begin{cases} \int_{-\infty}^{\infty} h(\tau)h(\tau_1-\tau)h(\tau_2-\tau)d\tau, & \tau_1,\ \tau_2 \ge 0, \\ 0 \ \ \text{for all other values } \tau. \end{cases}$$

Outside the framework of the simple model of the isotropic medium without space nonlocal coupling considered above, $y\ (t)$ and $x\ (t\text{-}t_i)$ in (4) should be vectors, and $h_p(\tau_1,\ldots,\tau_p)$ – tensor of the p + 1-th rank, having such values as charges, masses, geometrical and power constants, as well as values characterizing dissipation of energy as constants. This more general case which doesn't lead to any change of the final conclusions is not considered here. However, transition to this case is quite obvious.

Find the explicit form of the third order Volterra kernel

$$h_3(\tau_1,\tau_2,\tau_3)=\int_{E^1} h(\tau)h(\tau_1-\tau)h(\tau_2-\tau)h(\tau_3-\tau)\,d\tau$$

The area of integration corresponding to a causal kernel, is shown as shaded in fig. 2, according to which

$$h_3(\tau_1,\tau_2,\tau_3)=\int_0^{\min\{\tau_1,\tau_2,\tau_3\}} h(\tau)h(\tau_1-\tau)h(\tau_2-\tau)h(\tau_3-\tau)\,d\tau$$

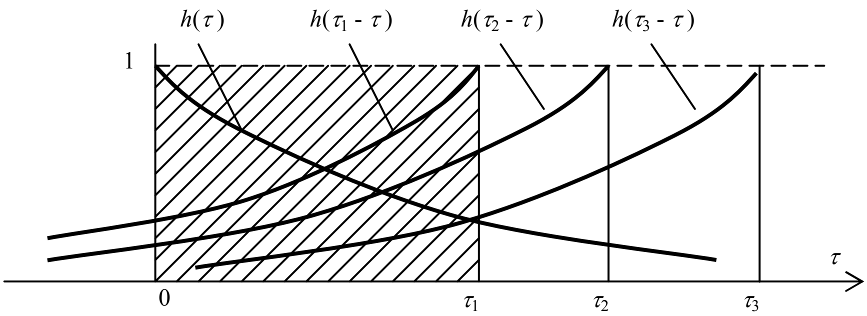

Fig. 2. The area of integration corresponding to a causal kernel.

Let $a = min\ \{\tau_1,\tau_2,\tau_3\}$. Taking (1.2.4) into account we have

$$h_3(\tau_1,\tau_2,\tau_3)=-\frac{1}{16\sigma\omega_e^4}e^{-\sigma(\tau_1+\tau_2+\tau_3)}\left(e^{2\sigma a}-1\right)\times$$
$$\left[\cos\omega_e(\tau_1-\tau_2-\tau_3)+\cos\omega_e(\tau_1+\tau_2-\tau_3)+\cos\omega_e(\tau_1-\tau_2+\tau_3)\right];$$

Under condition of $\omega_e >> \sigma$ the terms having the factor $1/16\omega_e^5$ are rejected here.

Thus:

if min{τ1,τ2,τ3} = τ1 then

$$h_3(\tau_1,\tau_2,\tau_3)=\frac{1}{16\sigma\omega_e^4}\left[e^{-\sigma(\tau_1+\tau_2+\tau_3)}-e^{-\sigma(-\tau_1+\tau_2+\tau_3)}\right]\times \quad (6)$$
$$\left[\cos\omega_e(\tau_1-\tau_2-\tau_3)+\cos\omega_e(\tau_1+\tau_2-\tau_3)+\cos\omega_e(\tau_1-\tau_2+\tau_3)\right];$$

2) if $min\{\tau_1, \tau_2, \tau_3\} = \tau_3$ then

$$h_3(\tau_1,\tau_2,\tau_3)=\frac{1}{16\sigma\omega_e^4}\left[e^{-\sigma(\tau_1+\tau_2+\tau_3)}-e^{-\sigma(\tau_1+\tau_2-\tau_3)}\right]\times$$
$$\left[\cos\omega_e(\tau_1-\tau_2-\tau_3)+\cos\omega_e(\tau_1+\tau_2-\tau_3)+\cos\omega_e(\tau_1-\tau_2+\tau_3)\right];$$

3) if $min\{\tau_1, \tau_2, \tau_3\} = \tau_2$ then

$$h_3(\tau_1,\tau_2,\tau_3)=\frac{1}{16\sigma\omega_e^4}\left[e^{-\sigma(\tau_1+\tau_2+\tau_3)}-e^{-\sigma(\tau_1-\tau_2+\tau_3)}\right]\times$$
$$\left[\cos\omega_e(\tau_1-\tau_2-\tau_3)+\cos\omega_e(\tau_1+\tau_2-\tau_3)+\cos\omega_e(\tau_1-\tau_2+\tau_3)\right];$$

Another way to write the third order kernel is following below:

$$h_3(t,\tau_1,\tau_2,\tau_3)=\int\limits_{\max\{\tau_1,\tau_2,\tau_3\}}^{t} h(t-\tau)h(t-\tau_1)h(t-\tau_2)h(t-\tau_3)d\tau=$$
$$\frac{1}{16\sigma\omega_e^4}\left[e^{-\sigma(3t-\tau_1-\tau_2-\tau_3)}-e^{-\sigma(t+\tau_1-\tau_2-\tau_3)}\right]\theta,$$

if $\tau_1 = \max\{\tau_1, \tau_2, \tau_3\}$;

$$h_3(t,\tau_1,\tau_2,\tau_3)=\frac{1}{16\sigma\omega_e^4}\left[e^{-\sigma(3t-\tau_1-\tau_2-\tau_3)}-e^{-\sigma(t-\tau_1+\tau_2-\tau_3)}\right]\theta,$$

if $\tau_2 = \max\{\tau_1, \tau_2, \tau_3\}$;

$$h_3(t,\tau_1,\tau_2,\tau_3)=\frac{1}{16\sigma\omega_e^4}\left[e^{-\sigma(3t-\tau_1-\tau_2-\tau_3)}-e^{-\sigma(t-\tau_1-\tau_2+\tau_3)}\right]\theta, \quad (7)$$

if $\tau_3 = \max\{\tau_1, \tau_2, \tau_3\}$;

Here $\theta = \cos\omega_e(t+\tau_1-\tau_2-\tau_3)+\cos\omega_e(t-\tau_1+\tau_2-\tau_3)+\cos\omega_e(t-\tau_1-\tau_2+\tau_3)$,

Find the oscillator respond to an excitation in the form of three δ-functions (fig. 3, at $t_1 = 0$):

$$x(t) = c_1\delta(t) + c_2\delta(t-t_2) + c_3\delta(t-t_3)$$

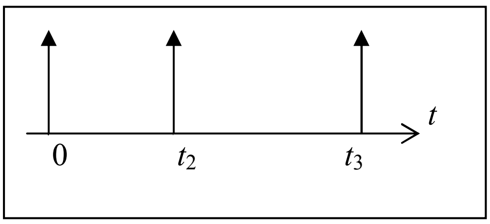

Fig. 3. Three-pulse excitation.

In the third approximation of the solution we have:

$$y_3(t)=\int_{E^1} h(t,\tau)x(\tau)d\tau - a_3\int_{E^3} h_3(t,\tau_1,\tau_2,\tau_3)\prod_{r=1}^{3} x(\tau_r)d\tau_r =$$

$$c_1h(t)+c_2h(t-t_2)+c_3h(t-t_3)-$$

$$c_1c_2c_3\{a_3\left[h_3(t,t_3,0,t_2)+h_3(t,t_3,t_2,0)\right]_{\tau_1=\max\{\tau_1,\tau_2,\tau_3\}}+$$

$$a_3\left[h_3(t,0,t_3,t_2)+h_3(t,t_3,t_2,0)\right]_{\tau_2=\max\{\tau_1,\tau_2,\tau_3\}}+$$

$$a_3\left[h_3(t,0,t_2,t_3)+h_3(t,t_2,0,t_3)\right]_{\tau_3=\max\{\tau_1,\tau_2,\tau_3\}}\}-$$

$$a_3c_1c_2^2[h_3(t,0,t_2,t_2)_{\tau_2,\tau_3=\max\{\tau_1,\tau_2,\tau_3\}}+h_3(t,t_2,0,t_2)_{\tau_1,\tau_3=\max\{\tau_1,\tau_2,\tau_3\}}+$$

$$h_3(t,t_2,t_2,0)_{\tau_1,\tau_2=\max\{\tau_1,\tau_2,\tau_3\}}]+\ldots$$

Here the first three terms correspond to the response of a linear circuit with the pulse characteristic *h (t)* and are not of interest from the echo – phenomena point of view. The expression in the first curly brackets defines the three-pulse response $y_3^{(123)}$, arising at the instant of time $t = t_2 + t_3$ (the top index corresponds to the numbers of stimulating pulses):

$$y_3^{(123)}=\frac{6a_3c_1c_2c_3}{16\sigma\omega_e^4}\left[e^{-\sigma(3t-t_2-t_3)}-e^{-\sigma(t-t_2+t_3)}\right]\cos\omega_e(t-t_2-t_3)$$

At the instant of time $t = t_2 + t_3$ all oscillators oscillate in the same phase and produce the pulse with the amplitude

$$y_3^{(123)}(t_2+t_3)=\frac{3}{8}a_3c_1c_2c_3e^{-2\sigma(t_3-t_2)}e^{-2\sigma t_2}\left(1-e^{-2\sigma t_2}\right)$$

The last terms in square brackets describe a two-pulse echo $y_3^{(12)}$ from the first two pulses:

$$y_3^{(12)}(t)=\frac{3a_3c_1c_2^2}{8\sigma\omega_e^4}\left[e^{-\sigma t}-e^{-\sigma(3t-2t_2)}\right]\cos\omega_e(t-2t_2)$$

There are two more echoes of two-pulse type from the second and third pulses $y_3^{(23)}(t)$ and from the first and third pulses $y_3^{(13)}(t)$

A distinctive feature of the echo - signals provided by anharmonicity is growth of the echo amplitude as the delay t_{12} between the first and the second pulses of excitation first rises up to maximum and then almost exponentially recesses [4], fig. 4.

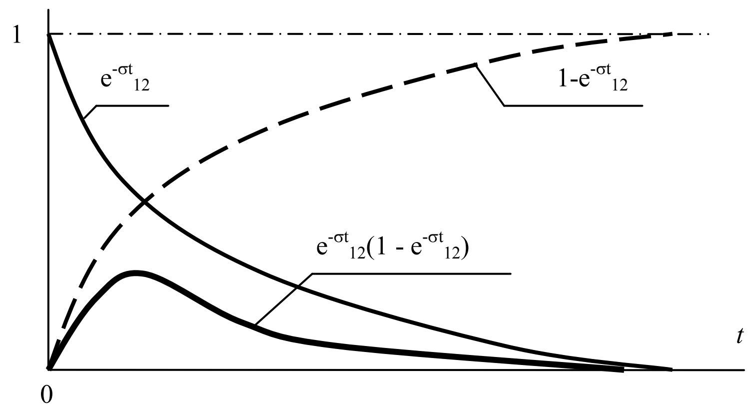

Fig. 4. Amplitude dependences of responses

Research the response of the anharmonic oscillator system to the excitation of three finite signals of the optional form within the framework of the third approximation.

The distinctive features associated with the record of the transformation kernel for various combinations of min $\{\tau_1,\tau_2,\tau_3\}$ and max $\{\tau_1,\tau_2,\tau_3\}$ will be shown in this calculation , as well as rules of calculating Volterra functionals stated.
The solution of the basic equation to a third approximation contains the first and third order functionals:

$$y_3(t) = \int_{E^1} h(t,\tau)x(\tau)d\tau - a_3 \int_{E^3} h_3(t,\tau_1,\tau_2,\tau_3)\prod_{r=1}^{3} x(\tau_r)d\tau_r\ , \quad (8)$$

among which only last one contains echo – responses .

As the all oscillators system response is of interest it should be specified that the solution (8) corresponds to the oscillator with the resonant frequency ω_e. Omitting a linear functional, we have:

$$y_3(t,\omega_e) = -a_3 \int_E h_3(t,\tau_1,\tau_2,\tau_3)\prod_{r=1}^{3} x(\tau_r)\,d\tau_r\ , \quad (9)$$

The total system response in view of a frequency distribution density is as follows : $y_3(t) = \int_{-\infty}^{\infty} g(\omega)\, y_3(t,\omega_e)\, d\omega_e$

First find the three-pulse echo expression provided by the product of all three input pulses in (9). For this purpose we write of all terms resulting from the opening the brackets in the product

$$\prod_{r=1}^{3} x(\tau_r) = \left[x_1(\tau_1) + x_2(\tau_1 - \tau) + x_3(\tau_1 - T)\right]\left[x_1(\tau_2) + x_2(\tau_2 - \tau) + x_3(\tau_2 - T)\right]\times$$
$$\left[x_1(\tau_3) + x_2(\tau_3 - \tau) + x_3(\tau_3 - T)\right],$$

those which are of interest for us

$$\begin{aligned}\prod_{r=1}^{3} x(\tau_r) &= \ldots + x_1(\tau_1)x_2(\tau_1 - \tau)x_3(\tau_1 - T) + x_1(\tau_1)x_2(\tau_3 - \tau)x_3(\tau_2 - T) + \\ & x_1(\tau_2)x_2(\tau_1 - \tau) + x_3(\tau_3 - T) + x_1(\tau_2)x_2(\tau_3 - \tau)x_3(\tau_1 - T) + \\ & x_1(\tau_3)x_2(\tau_1 - \tau)x_3(\tau_2 - T) + x_1(\tau_3)x_2(\tau_2 - \tau)x_3(\tau_1 - T) + \ldots\end{aligned} \tag{10}$$

We notice that for the first term in (10) $\tau_3 = \max\{\tau_1,\tau_2,\tau_3\}$, and $\tau_1 = \min\{\tau_1,\tau_2,\tau_3\}$. Consequently, the kernel (7) should be used with the first term. In this kernel we should take that one of the three cosines in whose argument $\min\{\tau_1,\tau_2,\tau_3\}$ has a sign "plus", i.e. $\cos\omega(t+\tau_1-\tau_2-\tau_3)$. Thus, the transformation kernel for the first term (10) $x_1(\tau_1)x_2(\tau_2-\tau)x_3(\tau_3-\tau)$ is

$$\begin{aligned} h_3(t,\tau_1,\tau_2,\tau_3)\Big|_{\substack{\tau_3=\max(\tau_1,\tau_2,\tau_3)\\ \tau_1=\min(\tau_1,\tau_2,\tau_3)}} &= \frac{1}{16\sigma\omega_e^4}\times \\ \left[e^{-\sigma(3t-\tau_1-\tau_2-\tau_3)} - e^{-\sigma(t-\tau_1-\tau_2+\tau_3)}\right] & \cos(t+\tau_1-\tau_2-\tau_3)\end{aligned} \tag{11}$$

The other terms' kernels should be found in the same way. For example, in the fourth term $\tau_1 = \max\{\tau_1,\tau_2,\tau_3\}$, and $\tau_2 = \min\{\tau_1,\tau_2,\tau_3\}$. This term's kernel is

$$\begin{aligned} h_3(t,\tau_1,\tau_2,\tau_3)\Big|_{\substack{\tau_1=\max(\tau_1,\tau_2,\tau_3)\\ \tau_2=\min(\tau_1,\tau_2,\tau_3)}} &= \frac{1}{16\sigma\omega_e^4}\times \\ \left[e^{-\sigma(3t-\tau_1-\tau_2-\tau_3)} - e^{-\sigma(t+\tau_1-\tau_2-\tau_3)}\right] & \cos(t-\tau_1+\tau_2-\tau_3)\end{aligned} \tag{12}$$

As for the physical model under consideration Volterra kernels are symmetric the appropriate replacement of arguments in the terms (10) and in the corresponding kernels will reduce both all the terms and kernels to one. Indeed, the consequent change $\tau_1 \leftrightarrow \tau_3$, $\tau_2 \leftrightarrow \tau_1$ reduces the kernel (12) to the kernel (10). Such arguments replacement simultaneously reduces the fourth term to the first one.

Thus, the property of kernels symmetry appears to be rather useful and should be applied when possible in order to minimize rather bulky calculations of Volterra functionals.

All the terms in expression (10) are reduced to the same type, therefore (9) transforms in:

$$y_3(t,\omega) = -6a_3 \int_{E^3} h_3(t,\tau_1,\tau_2,\tau_3)\Big|_{\substack{\tau_3=\max(\tau_1,\tau_2,\tau_3)\\ \tau_1=\min(\tau_1,\tau_2,\tau_3)}} \times$$
$$x_1(\tau_1)x_2(\tau_2-\tau)x_3(\tau_3-T)d\tau_1 d\tau_2 d\tau_3$$

In this case it appears simpler to calculate all oscillators system response than that of one oscillator due to possibility of using spectral densities of the input signals:

$$y_3(t) = \mathrm{Re}\left\{\begin{aligned}&-e^{-\sigma t}\int_{-\infty}^{\infty} cg(\omega_{\acute{y}})e^{j\omega t}\int_0^{t_1} e^{(\sigma+j\omega)\tau_1}x_1(\tau_1)d\tau_1 \times\\ &\int_{\tau}^{\tau'} e^{(\sigma-j\omega)\tau_2}x_2(\tau_2)d\tau_2 e^{-2\sigma\mathring{a}} \times\\ &\left[\int_T^{T'} e^{(\sigma-j\omega)\tau_3}x_3(\tau_3)d\tau_3 - \int_T^{T'} e^{(\sigma+j\omega)\tau_3}x_3(\tau_3)d\tau_3\right]d\omega_e\end{aligned}\right\}, \quad (13)$$

$$c = 6a_3/16\sigma\omega_e^4.$$

Internal integrals (13) represent spectral density of the input signals multiplied by the exponents exp ($\pm\sigma\tau_i$). Introducing definitions

$$S_{\pm\sigma i}(\omega_e) = \int_z^{z'} x_i(t)e^{\pm\sigma t}e^{-j\omega_e t}dt = \tilde{S}_{\pm\sigma i}(\omega_e)e^{\pm\sigma z}e^{-j\omega \mathrm{t}},$$

$$S_{\pm\sigma i}(\omega_e) = \int_z^{z'} x_i(t)e^{\pm\sigma t}e^{j\omega_e t}dt = \tilde{S}^*_{\pm\sigma i}(\omega_e)e^{\pm\sigma z}e^{j\omega \mathrm{t}},$$

where $\tilde{S}_{\pm\sigma i}(\omega_e)$ - spectral density of the signal $x_i(t)e^{\pm\sigma t}$ whose rise-up portion coincides with the time zero, we can write (13) as follows

$$y_3(t) = \mathrm{Re}\left\{\frac{3}{8}\frac{a_3}{\sigma}\int_{-\infty}^{\infty} g_1(\omega)\tilde{S}^*_{\sigma 1}(\omega)\tilde{S}_{\sigma 2}(\omega)\times\right.$$
$$\left.\left[\tilde{S}_{-\sigma 3}(\omega)e^{-\sigma T} - \tilde{S}_{\sigma 3}(\omega)e^{-\sigma(2t-T)}\right]e^{-\sigma(t-\tau)}e^{j\omega(t-T-\tau)}d\omega\right\},$$

where $g_1(\omega) = g(\omega)/\omega^4$.

As entrance signals duration is supposed to be negligible with reference to the decay time constant of the oscillators (*ti <<1/s*, *i* = 1, 2, 3), one can consider the approximate equality $\tilde{S}_{\sigma i} \approx \tilde{S}_{-\sigma i}$ valid. Then

$$y_3(t) = \mathrm{Re}\left\{ \frac{3}{8} \frac{a_3}{\sigma} e^{-\sigma(t-\tau+\mathrm{T})} \left[1 - e^{-2\sigma(t-T)}\right] \times \right. \\ \left. \int_{-\infty}^{\infty} g_1(\omega) \widetilde{S}_{\sigma 1}^{*}(\omega) \widetilde{S}_{\sigma 2}(\omega) \widetilde{S}_{\sigma 3}(\omega) e^{j\omega(t-T-\tau)} d\omega \right\}. \qquad (14)$$

It follows from the last expression that the three-pulse echo signal form is defined by the product of spectral densities of the entrance signals, the spectrum of the first signal should be taken in a complex conjugated form. Composed of exponents factor at the integral sign determines the dependence of the three-pulse echo- signal amplitude on the entrance signals time positions. The echo - signal clusters about the moment of time $t = T + \tau$.
(14) can be written in the other form

$$y_3(t) = \frac{3}{8} \frac{a_3}{\sigma} e^{-\sigma(t-\tau+T)} \left[1 - e^{-2\sigma(t-T)}\right] g_1(t) * x_{\sigma 1}(t) \otimes x_{\sigma 2}(t-\tau) * x_{\sigma 3}(t-T),$$

where symbol * means convolution, and ⊗ - correlation.
Introducing multidimensional Fourier transforms of Volterra kernels instead of Volterra kernels themselves we can pass to the frequency representation for y(t)

$$K_n(\omega_1, \ldots, \omega_n) = \int_0^{\infty} \ldots \int_0^{\infty} h_n(\tau_1, \ldots, \tau_n) e^{-j(\omega_1 \tau_1 + \ldots + \omega_n \tau_n)} d\tau_1, \ldots d\tau_n, , \qquad (15)$$

$$y(t) = \sum_{p=1}^{n} \frac{1}{2^p} \int_0^{\infty} \ldots \int_0^{\infty} K_p(\omega_1, \ldots, \omega_p) \prod_{k=1}^{p} S(\omega_k) e^{j\omega_0 t} d\omega_k , \qquad (16)$$

where $S(\omega)$ - spectral density of excitation $x(t)$ which use is determined by the assumed pulse character of $x(t)$. The reduced form of the record used in [5] is done in (16). In this record the total number of terms in the subintegral expression is equal to $(C_p^0 + C_p^1 + \ldots + C_p^p) = 2^p$, where C_p^k - number of combinations of p elements over k, containing k "minus" signs in the arguments of a gain. The first term in (16) describes signal $x(t)$ transmission through the linear quadripole with the gain $K_1(\omega)$, the others depict nonlinear signal transformation. The gain characteristics (16) are presented in [5].
If the frequencies in the gain arguments are not equal, i.e. $\omega_1 \neq \omega_2 \neq \ldots \neq \omega_p$, the ratio (16) describes occurrence of the new spectral components with frequencies $\pm\omega_1 \pm \ldots \pm \omega_k \pm \ldots \pm \omega_p$, passing through filters with the gain $K_p(\omega_1, \ldots, \omega_k, \ldots, \omega_p)$. Assuming the Q-quality of oscillators to be high it is possible to consider $K_p(\omega_1, \ldots, \omega_k, \ldots, \omega_p) = 0$ at $\omega_1 \neq \ldots \neq \omega_k \neq \ldots \neq \omega_p$. To exclude the highest harmonics of the signal the number of "plus" and "minus" signs in argument K_p should differ by unit. Such integrated transformation kernel in (16) is called a kernel with sum - differential argument in [5]. In this case

$$y(t) = \sum_{p=1}^{n} \frac{C_{2p-1}^{p-1}}{2^{2p-1}} \int_{-\infty}^{\infty} K_{2p-1}(\omega) S(\omega) \left|S(\omega)\right|^{2p-2} e^{j\omega t} d\omega + c.\tilde{n}. , \qquad (17)$$

where *c.c.* - is a complex conjugate part of the expression presented above. The reduced form of the record is also applied here. Thus, for example, total description (17) of the second term of the series looks like this:

$$\frac{1}{2^3}\{\int_{-\infty}^{\infty} K_3(\omega,\omega,-\omega)S(\omega)S(\omega)S^*(\omega)d\omega+$$

$$\int_{-\infty}^{\infty} K_3(\omega,-\omega,\omega)S(\omega)S^*(\omega)S(\omega)d\omega+$$

$$\int_{-\infty}^{\infty} K_3(-\omega,\omega,\omega)S^*(\omega)S(\omega)S(\omega)d\omega+\text{ c.c.}\}$$

The block diagram of the device providing transformation (17) with all oscillators taken into account is shown in fig. 5. Filters $\Phi_i(\omega)$, i = 1, 2, …, m, m=N, select spectral bands of a signal with a bandwidth $\Delta\omega_i$; output oscillations of these filters are subject to instantaneous nonlinear transformation in nonlinear blocks with instantaneous characteristics $a_{2k-1}\xi^{2k-1}$, $k=1,2,...,n$, linear filters $K_{2k-1}^{(i)}(\omega)$ select the first spectral bands of the transformed oscillations, adders make linear summation of all output oscillations.

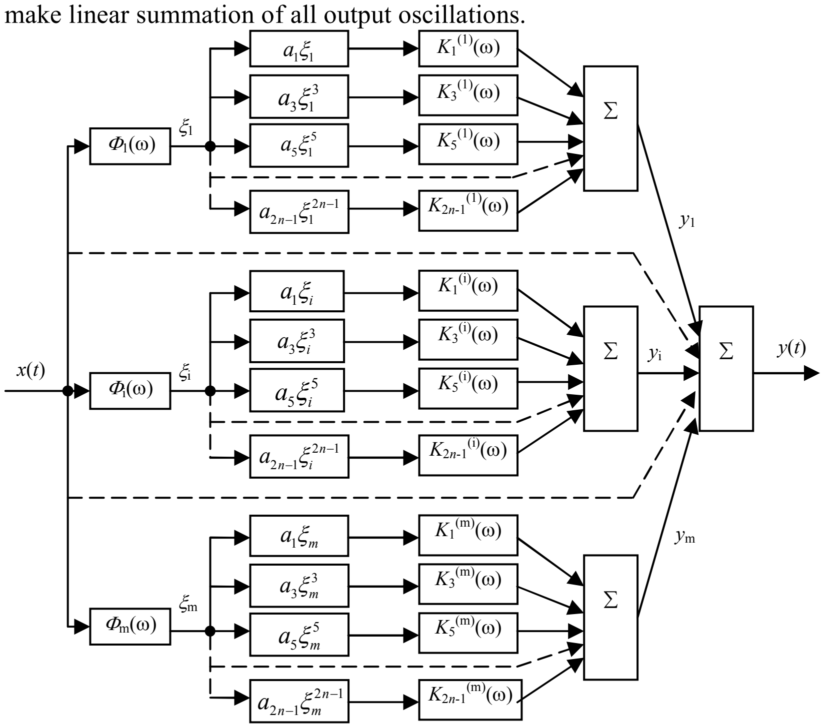

Fig. 5. The block diagram of the device making transformation (17).

The minimal order of nonlinearity in the equation (1) providing echo phenomena equals to three. This is the cubic medium and it can be represented by the radio engineering equivalent shown in Fig. 6. Putting aside thin distinctions of higher order echo time relations we can substantially simplify the block diagram transforming it as shown in Fig. 7.

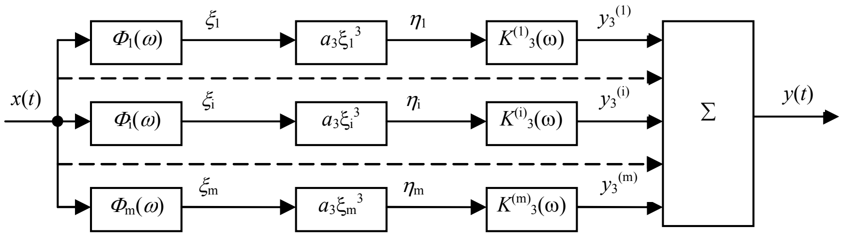

Fig. 6

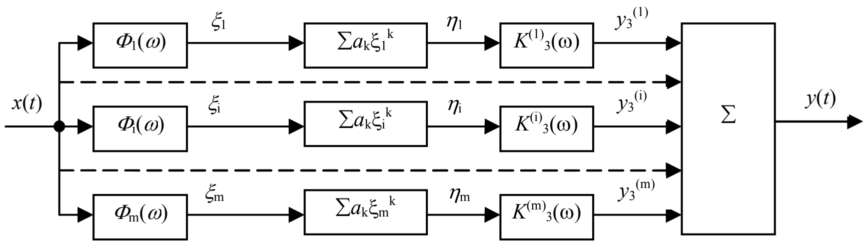

Fig. 7

We find the explicit form of the third order gain $K_3(-\omega,\omega,\omega)$ using the appropriate Volterra functional kernel (6):

$$h_3(t,\tau_1,\tau_2,\tau_3)=c\left[e^{-\sigma(\tau_1+\tau_2+\tau_3)}-e^{-\sigma(-\tau_1+\tau_2+\tau_3)}\right]\cos(\tau_1-\tau_2-\tau_3),$$

$$\tau_1=\min\{\tau_1,\tau_2,\tau_3\},\tau_2=\max\{\tau_1,\tau_2,\tau_3\}$$

Then, letting $\omega_0=\omega_e$,

$$\dot{K}_3(-\omega,\omega,\omega)=\int_0^\infty d\tau_3\int_0^{\tau_3}d\tau_2\int_0^{\tau_2}d\tau_1 h_3(\tau_1,\tau_2,\tau_3)e^{-j\omega(-\tau_1+\tau_2+\tau_3)}=$$

$$=\dot{F}(\omega-\omega_0)+\dot{F}(\omega+\omega_0)=$$

$$=\frac{1}{\left[\sigma+j(\omega-\omega_0)\right]^2}\left\{1+\frac{1}{2}\left[\frac{1}{\sigma-j(\omega-\omega_0)}+\frac{1}{\sigma+j(\omega-\omega_0)}\right]\right\}+\dot{F}(\omega+\omega_0).$$

Considering only positive frequencies domain and taking into account, that $\sigma>>1$, we write $K_3(-\omega,\omega,\omega)$ in the following form

$$K_3(-\omega,\omega,\omega) \approx \frac{1}{\left[\sigma + j(\omega-\omega_0)\right]^2}\left\{1+\frac{1}{2}\left[\frac{1}{\sigma + j(\omega-\omega_0)}+\frac{1}{\sigma - j(\omega-\omega_0)}\right]\right\}$$

Correction resulted from the summation of the conjugated factors in the square brackets, has the maximal value (1 + 1/s) at $\omega - \omega_0 = 0$ and, thus,

$$K_3(-\omega,\omega,\omega) \approx \frac{1}{\left[\sigma + j(\omega-\omega_0)\right]^2},$$

as shown in fig. 7.

Let us find the whole oscillator system response to the signal $x(t)$:

$$y(t)=\frac{1}{2\pi}\sum_{p=1}^{n} a_{2p-1}\frac{C_{2p-1}^{p-1}}{2^{2p-1}}\int_E g'(j\omega)S(j\omega)\left|S(j\omega)\right|^{2p-2} e^{j\omega}d\omega + \text{ c.c.}, \quad (18)$$

where $g'(j\omega)=g(\omega)\sum_i K_{2p-2}^{(i)}(j\omega)$ -

the total resonant characteristic of the oscillator system, function g(ω) has the meaning of the oscillator frequency distribution density. The first term of expression (18) describes linear transmission of the signal $x(t)$ through the filter with the gain $g'(j\omega)$; the second one corresponds to the nonlinear transformation of the third order and rather adequately characterizes the processes occurring in the nonlinear resonant medium:

$$y(t \mid p=2)=2\frac{3}{8}\frac{a_3}{2\pi}\operatorname{Re}\left\{\int_{-\infty}^{\infty} g'(j\omega)S(j\omega)\left|S(j\omega)\right|^2 e^{j\omega t}d\omega\right\} \quad (19)$$

The spectral density of the output signal corresponding to this transformation defines possible responses of medium.

In time domain the medium response to the excitation x (t), determined by the second approximation of the basic medium equation solution, can be presented according to (20) in the form, where $g'(t)$ - the pulse characteristic of the system with the gain $g'(j\omega)$.

$$S_y(j\omega \mid p=2)=\frac{3a_3}{8}\left[S(j\omega)\left|S(j\omega)\right|^2\right]g'(j\omega), \quad (20)$$

The last expression determines functional capabilities of the devices which can use NRM properties, i.e. producing convolution, signal correlation function and Fourier transforms in real time.

References

1. Korpel A., Chatterjee M. Nonlinear echoes, phase conjugation, time reversal, and electronic holography. Proceedings IEEE, vol. 69, pp 1539 – 1556, 1981.

2. Hahn E.L. Physical Revue, vol. 80. pp. 580 – 594. 1950.
3. Rassvetalov L. A. Radiotechnika i Electronika, vol. 31. № 1, pp 8 – 14, 1987.
4. Gould R.W. American Journal Physic, vol. 37, 6. pp. 595 – 597, 1969.
5. Kashkin V. B. Functional polynoms in problems of a statistical radio engineering. Novosibirsk, Nauka, 1981, P. 145.