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The Navier-Stokes Equations and Turbulence or Chaos

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Abstract: *The motion of fluids which are incompressible could be described by the Navier-Stokes differential equations. Although they are relatively simple-looking, the three-dimensional Navier-Stokes equations misbehave very badly. Even with nice, smooth, reasonably harmless initial conditions, the solutions could wind up being extremely unstable. The field of fluid mechanics would be dramatically altered through a mathematical understanding of the outrageous behaviour of these equations. An explanation why the three-dimensional Navier-Stokes equations are not solvable, i.e., the equations cannot be used to model turbulence or chaos (which is a three-dimensional phenomenon), would be provided.*

Keywords: Navier-Stokes equations; turbulence; forecast; calculus; continuity; financial market; physical world; discontinuities; irregularities; solutions; experimentalist; lotteries.

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1 Failure of the Navier-Stokes Equations as a Model for Turbulence or Chaos

Sir George Stokes obtained the general equations of motion for a viscous fluid in 1845. The fundamental equation (in vectorial form) governing the flow of a viscous fluid is as follows:-

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$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{p}\nabla Pe - \nabla\varphi + \frac{\eta}{p}\nabla 2v\ ,$$

where v represents the velocity of the fluid as a function of position, Pe the pressure, φ the gravitational potential, p the density and η the viscosity.

The scientist normally utilises the Navier-Stokes equations as a model to make a forecast of the outcome of a flow. However, for the case of turbulence or chaos making this forecast would be very difficult, if it could be done at all. If turbulence or chaos could be predicted, forecasted or modeled by the Navier-Stokes equations it is by definition not turbulence or chaos, as turbulence or chaos implies lack of predictability, lack of pattern or order and puzzlement. As turbulent flows are three-dimensional, nonlinear and highly unsteady, it would be practically impossible for the Navier-Stokes equations to model them.

Calculus, of which the Navier-Stokes equations are an example, relies on continuity, which is also a common human assumption. If one sees a person running at one moment here and a half-hour later there, one normally assumes the person has run a continuous line covering all the ground in between. It does not occur to him that the person might have stopped to rest or had even hitched a ride, i.e., there had been discontinuities in the path covered by the runner. Calculus, which was co-invented by Isaac Newton and Gottfried von Leibniz and was the greatest innovation of seventeenth-century mathematics, was designed to study continuous change; Leibniz believed deeply in what he described a "principle of continuity". Economists also normally assume continuity when using calculus to model the economy. Continuity is also a fundamental assumption of conventional finance, e.g., the financial mathematics of Batchelier, Black-Scholes, Sharpe and Markowitz all assume continuous change from one price to the next, without which their formulae would not work. However, this assumption is wrong and thus is their mathematics.

In the financial market, prices of course jump, skip and leap, sharply moving up and down, i.e., there are lots of discontinuity in the movement of financial prices. In comparison, in classical physics, in a perfect gas, e.g., as molecules collide and exchange heat, their billions of individually infinitesimal transactions together produce a true "average" temperature, around which smooth gradients move up or down the scale. However, in a financial market, the news which influences an investor could be minor or major. The investor's buying power could be insignificant or market-moving. His decision could be based on an instantaneous change of decision, from bull to bear and back again. This results in a much wilder distribution of price changes – not just movements of prices but also price dislocations, which are especially noticeable in our information age with the instantaneous broadcasts by trading-room screen, television and internet. Market influencing news such as a terrorist attack or political change in a country flashes across the world to millions of investors in seconds. The investors could act on the news, not bit by bit in a progressive wave, as conventional theorists normally assume, but all at once and instantaneously. The effect could be exhilarating or heart-stopping, depending on whether one gains or loses.

A sudden price drop could cause panic in investors. The mutual fund industry sometimes takes extraordinary measures to "manage" emotions. For example, in 2000 a Milwaukee mutual fund company, Heartland Advisors Inc., hit turbulence when the market value of some of its bond investments nose-dived to $80 per $100 face value, from as high as $98, which did not show up immediately in its daily price reports and instead, according to the Securities and Exchange Commission (SEC), the fund's data supplier recorded a long, slow and gentle decline at 50 cents a day over a period of weeks. When word finally got out, Heartland investors panicked and stampeded to the exits. The data supplier was later sued by SEC but the case was settled without the data supplier admitting or denying the charges.

Discontinuity in the financial market could also bring profits, i.e., turbulence in the financial market is not always bad. The New York Stock Exchange has had, for more than a century, a system of "specialists" who are traders on the exchange floor who each specialises in the shares of a few companies, maintaining an order book, and, when the buys and the sells do not match, step in with their own money to complete trades. According to the rule, their job is to "ensure market continuity". They have however lately come into disrepute in the post-bubble scandals which have engulfed most of Wall Street. In the SEC study of a 1997 financial market collapse, it was discovered that specialists in the most tumultuous 24 minutes were powerful net buyers, with the volume of their purchases exceeding their sales by a ratio of 2.06. These were good bets for prices did recover.

It is evident that investing in the financial market is extremely risky due to discontinuities, turbulence and unpredictability in the market, wherein instead of the smooth flow of prices expected by the normal investor sudden and sharp changes in prices, i.e., discontinuities, often occur. The same could be said about turbulence in the physical world, for instance turbulence in fluids, only that in the case of the physical world it is probably worse. Take the case of weather forecasting, for example. It is not possible to forecast the weather more than a few days in advance even with very powerful computers. With just a few days of forecasting the weather still gives nasty surprises. This is due to the equations which model the weather being nonlinear, i.e., they involve the variables multiplied together, not just the variables themselves.

The theory behind the mathematics of weather forecasting was developed by Claude Navier in 1821 and George Gabriel Stokes in 1845. The Navier-Stokes equations are of very great interest to scientists, who are keen to unlock their secrets. When the Navier-Stokes equations are applied to the problem of fluid flow, they reveal much about the steady movements of the upper atmosphere. But the equations fail when applied near the earth's surface where air flow creates turbulence.

Though a lot is known about linear systems of equations, the Navier-Stokes equations contain nonlinear terms which render them intractable. The only practical way of solving the Navier-Stokes equations, which depend on initial conditions, is to do so numerically by utilising powerful computers.

Differential equations, such as the Navier-Stokes equations, could only make forecasts on phenomena characterised by smooth, regular, continuous flows, which turbulence is definitely not – turbulence, on the other hand, is characterised by great irregularities, discontinuities, disruptions and sharp jumps. With smooth, regular, continuous flows, which are each graphically represented by a smooth, continuous curve with gentle gradients, it would be possible to extrapolate and interpolate, i.e., forecasts are possible. This is not the case with turbulence, which does not display any discernable, set pattern or regularity at all. Hence, the Navier-Stokes equations fail when there is turbulence.

2 Conclusion

The solutions for turbulent flows therefore have to be left to the experimentalist and are not attempted by solving the Navier-Stokes equations.

If a scientist could produce an equation for forecasting turbulence, he could also probably apply the same equation with some modifications in forecasting the winning numbers of a lottery and he would be very wealthy and would very probably keep the equation a secret, for the uncertainties, randomness and unpredictability of turbulence and lotteries appear similar. Such a happy winner of lotteries has not happened so far, which says something about the uncertainties, randomness and unpredictability of turbulence as well.

In fact, the implicit trust in differential equations such as the famous Black-Scholes formula with its bell-curve assumption in making financial forecasts has led a number of financial organisations into trouble, e.g., the case of Long-Term Capital Management LP (LTCM), a hedge fund set up in 1993 by two Nobel laureates, Robert Merton and Myron Scholes, and some heavy-weight Wall Street bond traders, which had at one point 25 Ph.D.'s on its payroll and was possibly the best academic finance department in the world, which had made colossal losses caused by market turbulence and volatility leading to bankruptcy and had to be bailed out by several banks reluctantly through a $3.625 billion takeover at the behest of the Federal Reserve Board, which was concerned about a wave of bankruptcies if LTCM went bust.

All this goes to show how unreliable mathematics, in this case calculus, could be when there are wild swings, volatilities, discontinuities and irregularities as when there is turbulence, be it in the financial world or the physical world.

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