Chaotic Modeling and Simulation (CMSIM) 1: 63–70, 2018

A note on Hamiltonian dynamics in a geometric framework

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Abstract. The flow associated with an autonomous Hamiltonian system can be reformulated as a geodesic flow in a Riemannian manifold endowed with the Jacobi-Maupertuis metric. In this note we discuss the possibility of existence of chaos in Hamiltonian systems with two degrees of freedom in the context of a geometric Riemannian framework given by Pettini, Valdettaro and Cerruti-Sola [6,4].
Keywords: Hamiltonian system, Gaussian curvature, stability, chaos.

1 Introduction

In the Riemannian geometric approach of classical mechanics the trajectories can be characterized as geodesics on the configuration space M with respect to the Jacobi-Maupertuis metric which is the conformal modification of the kinematic metric by the factor $h-V$, where V and h are the potential function and the total energy, respectively.

It is far from being complete the understanding of the mechanisms responsible for the existence of regular or chaotic motions in a system. Chaos is also related to the Riemannian curvature as Anosov shows [1] through his known result which asserts that negative curvature in a connected compact manifold leads to chaos. The implication for positive curvature it is not clear. In this way, to look into the connection between dynamics and curvature is of utmost importance. This note is inspired through the papers by Pettini, Valdettaro and Cerruti-Sola [6,4], where the authors claimed that the origin of chaos in the nonlinear Hamiltonian dynamics, viewed as a geodesic flow on a curved mechanical manifold would come from the fluctuations of the curvature and also from the hyperbolicity of the flow.

This paper is organized as follows: in the first part we introduce the concepts needed to have a framework where to analyze the claim given in [6,4] by Pettini, Valdettaro and Cerruti-Sola. In the last part we present two examples were we

Received: 30 December 2017 / Accepted: 23 January 2018

ISSN 2241-0503

exhibit that fluctuations of the curvature and instability of geodesics is not the only mechanism that give rise to non regular motions.

2 Preliminaries

A classical mechanical system is given by a triplet (M, T, V), being M a smooth manifold that corresponds to the configuration space where motion takes place, T is the *kinetic energy* and V is the *potential energy*. Such systems can be framed in a Riemannian geometry setting.

Assume that M is endowed with a Riemannian metric, denoted by g, and expressed as $\sum g_{ij} dq^i dq^j$, where q^k, $k = 1, \ldots, n$, are local coordinates for M. These local q^i induce fiber coordinates v^i on TM by expanding any arbitrary vector $v \in T_qM$ as $v = \sum v^i(\partial/\partial q^i)$, that is, in terms of the local coordinate vector fields $(\partial/\partial q^1, \ldots, \partial/\partial q^n)$ for TM, see [7].

It is well-known that Hamiltonian systems described by a hamiltonian function $H = T + V$ are conservative and the value of the energy h is a conserved quantity along the trajectories, and also its configuration space M has a differentiable manifold structure. In this kind of systems, for a fixed value of energy h, the accessible part of the configuration space is not the whole space M, but only the subspace of admissible motions, also called the *Hill region*, which is defined by

$$\mathcal{H} = \{q \in M : h - V(q) \geq 0\},$$

and corresponds to the projection of the phase space into the configuration space. The boundary $\partial\mathcal{H}$ of the Hill region is the set of points with zero–velocity.

The *kinetic energy* is defined by

$$T = \frac{1}{2} g(v, v) = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij} v^i v^j.$$

Thus the metric on M is given by g, whose components are g_{ij}.

Now, lets recall Jacobi's reformulation of mechanics. It asserts that the solutions to Newton's equations with energy h are, after a time reparametrization, exactly the geodesic curves on the manifold M relative to the metric

$$\hat{g} = 2(h - V)g.$$

This metric is referred to as the Jacobi-Maupertuis metric, and its arc-length element is given by

$$ds_{\hat{g}}^2 = g_{ij} dq^i dq^j = 2(h - V) \frac{dq^i}{dt} \frac{dq^j}{dt} dt^2 = 4(h - V)^2 dt^2.$$

The geodesic equations in terms of the local coordinate frame $(q^1, \ldots, q^n)$ are

$$\frac{d^2 q^k}{ds^2} + \Gamma_{ij}^k \frac{dq^i}{ds} \frac{dq^j}{ds} = 0, \qquad (1)$$

where Γ_{ij}^k are the Christoffel symbols, which can be written as

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^n g^{kl}\left(\frac{\partial g_{jl}}{\partial q^i} + \frac{\partial g_{li}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^l}\right) = \sum_{l=1}^n g^{kl}\frac{1}{2}\left(\frac{\partial g_{jl}}{\partial q^i} + \frac{\partial g_{li}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^l}\right),$$

in shorthand,

$$\Gamma_{ij}^k = g^{kl}\Gamma_{ijl},$$

where Einstein's summation notation is assumed and

$$\Gamma_{ijl} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial q^i} + \frac{\partial g_{li}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^l}\right).$$

It is well known that given X, Y, Z and W vector fields on M, the *curvature operator* and the *Riemann curvature tensor* are defined, respectively, by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and

$$R(X,Y,Z,W) = g(R(X,Y)Z,W),$$

where ∇ is the Levi-Civita connection and $[\cdot,\cdot]$ is the Lie bracket. By taking $X = X_i = \partial/\partial q^i, Y = X_j = \partial/\partial q^j$ and $Z = X_k = \partial/\partial q^k$, since $[X_i, X_j] = 0$, we have that $R(X_i,X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k$. In shorthand, we define $R_{ij}^k = R(X_i,X_j)X_k$. From this, as usual, the tensor metric is defined in terms of the elements of the basis of the tangent space as $R_{ijkl} = g(R(X_i,X_j)X_k,X_l)$, which can be written in the form

$$R_{ijkl} = \sum_{s=1}^n g_{sl}\left(\frac{\partial \Gamma_{jk}^s}{\partial q^i} - \frac{\partial \Gamma_{ik}^s}{\partial q^j} + \sum_{r=1}^n\left(\Gamma_{jk}^r\Gamma_{ir}^s - \Gamma_{ik}^r\Gamma_{jr}^s\right)\right), \qquad (2)$$

that is, in terms of the elements g_{ij} and the Christoffel symbols of the second kind and their partial derivatives.

The curvature tensor satisfies the following properties:

$$\begin{aligned} R_{ijkl} &= -R_{jikl} && \text{first skew symmetry}\\ R_{ijkl} &= -R_{ijlk} && \text{second skew symmetry}\\ R_{ijkl} &= R_{klij} && \text{block symmetry}\\ R_{ijkl} + R_{iklj} + R_{iljk} &= 0 && \text{Bianchi's identity.}\end{aligned} \qquad (3)$$

Let us consider the special case when our Riemannian manifold is a two-dimensional surface, where is well known that the sectional curvature coincides with the (intrinsic) Gaussian curvature K. Furthermore, in two dimensions, the scalar curvature (or the Ricci scalar) $\mathcal{R}$ is twice the Gaussian curvature, and completely characterizes the curvature of a surface. It follows from the above symmetry relations that

$$R_{1111} = R_{2222} = R_{1122} = R_{1112} = R_{1121} = R_{1222} = R_{2122} = 0,$$

while R_{1212} being the only independent component of the Riemann tensor which possibly does not vanish.

There is a close and beautiful connection between the sectional curvature and the Riemannian curvature tensor which is given by the following formula:

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}. \tag{4}$$

Next, we obtain a calculable expression for R_{1212}.

Proposition 1. *For a two-dimensional Riemannian manifold endowed with the Jacobi-Maupertuis metric, it holds that*

$$R_{1212} = \frac{\Delta V}{2} + \frac{|\nabla V|^2}{2W},$$

where $W = h - V(q)$, $|\cdot|$, Δ and ∇ refer to the norm, Laplacian and gradient for the Euclidean metric ds^2, respectively.

Proof. In a two-dimensional Riemannian manifold holding the Jacobi-Maupertuis metric, we have that $g_{11} = g_{22} = h - V(q)$, $g_{12} = g_{21} = 0$ and $g^{11} = g^{22} = (h - V(q))^{-1}$, $g^{12} = g^{21} = 0$, where $g^{ij} = (g^{-1})_{ij}$. It is not hard to see that

$$\Gamma_{22}^1 = \frac{1}{2W}\frac{\partial V}{\partial q^1} \quad \text{and} \quad \Gamma_{21}^1 = -\frac{1}{2W}\frac{\partial V}{\partial q^2}.$$

So,

$$\frac{\partial \Gamma_{22}^1}{\partial q^1} = \frac{1}{2}\left(\frac{1}{W^2}\left(\frac{\partial V}{\partial q^1}\right)^2 + \frac{1}{W}\frac{\partial^2 V}{\partial {q^1}^2}\right)$$

and

$$\frac{\partial \Gamma_{21}^1}{\partial q^2} = -\frac{1}{2}\left(\frac{1}{W^2}\left(\frac{\partial V}{\partial q^2}\right)^2 + \frac{1}{W}\frac{\partial^2 V}{\partial {q^2}^2}\right).$$

Direct computations give $\Gamma_{22}^m\Gamma_{m1}^1 - \Gamma_{21}^m\Gamma_{m2}^1 = 0$, for $m = 1, 2$. Thus,

$$\begin{aligned} R_{212}^1 &= \frac{1}{2}\left(\frac{1}{W^2}\left(\frac{\partial V}{\partial q^1}\right)^2 + \frac{1}{W}\frac{\partial^2 V}{\partial {q^1}^2}\right) + \frac{1}{2}\left(\frac{1}{W^2}\left(\frac{\partial V}{\partial q^2}\right)^2 + \frac{1}{W}\frac{\partial^2 V}{\partial {q^2}^2}\right) \\ &= \frac{1}{2}\left(\frac{\Delta V}{W} + \frac{|\nabla V|^2}{W^2}\right). \end{aligned}$$

Finally, we obtain

$$R_{1212} = g_{11}R_{212}^1 + g_{12}R_{212}^2 = g_{11}R_{212}^1 = \frac{\Delta V}{2} + \frac{|\nabla V|^2}{2W}.$$

3 Stability of the geodesic flow and chaos

In [6,4], the authors use the Riemannian geometric framework to relate the stability of a geodesic flow to the curvature of manifold M for mechanical geodesic flows geometry, and make use of this relation to determine the existence of chaos.

In order to tackle the study the Hamiltonian chaos by means of tools of Riemannian geometry, we should study the stability of the geodesic flow by means of the Jacobi-Levi-Civita (JLC) equation, which describes the evolution of a vector field $\mathbf{J}$ along a geodesic $\gamma(s)$, known as Jacobi field, through which the separation between nearby geodesics can be measured, see [2]. The JLC equation reads

$$\frac{\nabla^2 \mathbf{J}}{ds^2} + R(\gamma'(s), \mathbf{J})\gamma'(s) = 0, \tag{5}$$

where s is the arc-length parameter, ∇/ds is the covariant derivative along geodesic $\gamma(s)$ and R is the curvature operator.

In the case of a two-dimensional manifold, the JLC equation can be decomposed into a system of two simpler equations.

Proposition 2. *For a two-dimensional Riemannian manifold, the Jacobi-Levi-Civita equation (5) is decomposed as*

$$\frac{d^2 f_1}{ds^2} + K(s) f_1 = 0, \tag{6}$$

$$\frac{d^2 f_2}{ds^2} = 0. \tag{7}$$

where f_1 and f_2 are, respectively, the perpendicular and parallel components of the Jacobi vector field $\mathbf{J}$, and $K(s)$ is the Gaussian curvature.

Proof. We start by taking a basis on the tangent space, recall that we are dealing with the two-dimensional case. So, we take $\{\mathbf{e}_1, \mathbf{e}_2\}$ as such a basis, where $\mathbf{e}_1 = (\mathrm{e}_1^1, \mathrm{e}_1^2) = \left(-\frac{dq^2}{ds}, \frac{dq^1}{ds}\right)$ is parallel to the Jacobi field, whereas $\mathbf{e}_2 = (\mathrm{e}_2^1, \mathrm{e}_2^2) = \left(\frac{dq^1}{ds}, \frac{dq^2}{ds}\right)$ is orthogonal to this field. Hence the Jacobi field takes the form $\mathbf{J}(s) = f_1(s)\,\mathbf{e}_1(s) + f_2(s)\,\mathbf{e}_2(s)$. In this reference frame we get

$$\frac{\nabla^2 \mathbf{J}}{ds^2} = \frac{d^2 f_1}{ds^2}\,\mathbf{e}_1(s) + \frac{d^2 f_2}{ds^2}\,\mathbf{e}_2(s). \tag{8}$$

It follows that JLC equation (5) can be decomposed into

$$\begin{aligned} &\frac{d^2 f_1}{ds^2} + Q_{11} f_1 + Q_{12} f_2 = 0, \\ &\frac{d^2 f_2}{ds^2} + Q_{22} f_1 + Q_{21} f_2 = 0, \end{aligned} \tag{9}$$

where

$$
\begin{aligned}
Q_{11} &= R_{1212}v^1\mathrm{e}_1^2v^1\mathrm{e}_1^2 + R_{2112}v^2\mathrm{e}_1^1v^1\mathrm{e}_1^2 + R_{2121}v^2\mathrm{e}_1^1v^2\mathrm{e}_1^1 + R_{1221}v^1\mathrm{e}_1^2v^2\mathrm{e}_1^1\ ,\\
Q_{12} &= R_{1212}v^1\mathrm{e}_1^2v^1\mathrm{e}_2^2 + R_{2112}v^2\mathrm{e}_1^1v^1\mathrm{e}_2^2 + R_{2121}v^2\mathrm{e}_1^1v^2\mathrm{e}_2^1 + R_{1221}v^1\mathrm{e}_1^2v^2\mathrm{e}_2^1 = 0,\\
Q_{21} &= R_{1212}v^1\mathrm{e}_2^2v^1\mathrm{e}_1^2 + R_{2112}v^2\mathrm{e}_2^1v^1\mathrm{e}_1^2 + R_{2121}v^2\mathrm{e}_2^1v^2\mathrm{e}_1^1 + R_{1221}v^1\mathrm{e}_2^2v^2\mathrm{e}_1^1 = 0,\\
Q_{22} &= R_{1212}v^1\mathrm{e}_2^2v^1\mathrm{e}_2^2 + R_{2112}v^2\mathrm{e}_2^1v^1\mathrm{e}_2^2 + R_{2121}v^2\mathrm{e}_2^1v^2\mathrm{e}_2^1 + R_{1221}v^1\mathrm{e}_2^2v^2\mathrm{e}_2^1 = 0,
\end{aligned}
$$

and $v^i = \dfrac{dq^i}{ds}$.

Since $R_{ijkl} = g_{is}R^s_{jkl}$ and by using the symmetry properties (3), a direct calculation yields

$$Q_{11} = R_{1212}\,\frac{1}{W^2},$$

which corresponds to the Gaussian curvature $K(s)$ for the Jacobi-Maupertuis metric, see [3]. Consequently, the system (6)-(7) is obtained.

From equation (7) it follows that the parallel component of geodesic separation does not accelerate, thus only equation (6) conveys information about the behavior of the nearby geodesics. Since $K(s)$ is not always constant, the scalar equation (6) is cast in the form of a generalized Hill equation [12], and such equation gives the exact geometric description of the stability properties of the geodesics. It seems that this was what inspired Pettini, Valdettaro and Cerruti-Sola [6,4] to claim that if the solutions of the equation (6) are exponentially growing, thus the geodesic flow is unstable, if $K(s)$ is everywhere or almost everywhere negative, or if $K(s)$ is suitably varying so as to yield parametric instability. They were concluding that for two-degrees of freedom systems with physically meaningful potentials this is actually the relevant mechanism responsible for chaos.

Let us remember that, in 1967, Anosov [1] proved that the geodesic flow is chaotic in a connected compact manifold of constant negative curvature. This leads to instability of the geodesic curves in the sense that nearby orbits diverge exponentially. Since then, chaos could often be understood as a manifestation of negative curvature.

Next, we use the Kepler problem and Hénon-Heiles system described by Hamiltonian systems with two-degree of freedom, to illustrate that what is claimed in [6,4] is not always verified.

From now on, in order to avoid confusion in notation we use x and y instead of q^1 and q^2, respectively.

3.1 The Newtonian Kepler problem

Let us consider two bodies moving under the influence of their Newtonian mutual attraction. We denote their position vectors with $\mathbf{r}_1$ and $\mathbf{r}_2$, and refer to the origin at $(0,0)$. Thus, the relative position vector is given by $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (x,y)$ and $r = |\mathbf{r}|$ being its magnitude. By choosing the reduced mass and the gravitational constant to be equal to one, its potential energy is $V(x,y) = -(x^2+y^2)^{-1/2}$. It is well known that the Kepler problem is an integrable system, that is, its equation of motion can be solved and solutions correspond to conic sections – ellipses, parabolas and hyperbolas, see [2].

Substitution of the derivatives of respective potential in (4), the Gaussian curvature yields

$$K = -\frac{h}{2r^3(h-V)^3}.$$

This shows that the sign of curvature is determined by the sign of the energy. Thus, the curvature is negative when $h > 0$ and the geodesics are hyperbolic orbits, such that the distance between points on different hyperbolas is a convex function of time, and so they are unstable orbits, see [11]. Since the configuration space is unbounded, we are not able to apply the Lobatchevsky-Hadamard theorem [10], which states that the geodesic flow on a connected and compact Riemannian manifold of negative curvature all trajectories diverge exponentially for all time. However, the authors in [6,4] suggest that negativity of curvature by itself and its fluctuations could arise to chaos. But, negativity of curvature alone does not imply chaos, since the curvature is negative for positive energy in the Kepler problem and it is a completely integrable system.

3.2 The Hénon-Heiles system

M. Hénon and C. Heiles [5] provided with a planar model which describes the motion of a star in the axisymmetric potential of the galaxy described by a Hamiltonian function $H = T + V$ given by two one-dimensional harmonic oscillators coupled by a cubic term with potential function

$$V(x,y) = \frac{1}{2}(x^2+y^2) + x^2y - \frac{1}{3}y^3, \tag{10}$$

where x and y are the radius and the altitude of the star orbit, respectively. We reduce our analysis to a fixed level of energy $\mathcal{E}_h = \{H = h\}$, which is a three–dimensional manifold.

The topology of the energy manifolds $\mathcal{E}_h$ is determined entirely by the level sets of the potential energy $V(x,y)$. For $h = 0$, $\mathcal{E}_0$ has a compact component and $\mathcal{E}_h = \mathbb{S}^3$ for $h \in (0, 1/6]$; while for $h > 1/6$, the surfaces $\mathcal{E}_h$ are not compact. As a matter of fact, for $0 < h < 1/6$, the Hill region has four connected components, see [8].

In this system the curvature is given by

$$K = \frac{2}{(h-V)^2}\left[2 + \frac{1}{h-V}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right\}\right],$$

which is positive everywhere.

It has been reported by Toda in [9], that the local divergence of trajectories for Hénon-Heiles system for $h > 1/12$ experiences exponential instability in part of the phase space. So, the Hénon-Heiles system gives us an example where the configuration manifold M has non negative scalar curvature, but shows chaotic motions for energy values bigger than $h = 1/12$. Thus, chaos in this system cannot come from the negative curvature in the manifold. It is seen that the chaotic phenomena is achieved for the fluctuating values of the curvature along the geodesics. Therefore, even on strictly positively curved manifolds chaos can appear.

4 Acknowledgements

The third author was supported by the Universidad de Cartagena, Grant number 066-2013.

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