Asymptotic Behaviour of a Viscoelastic Thermosyphon Model

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Abstract. Thermoyphons, in the engineering literature, is a device composed of a closed loop containing a fluid who motion is driven by several actions such as gravity and natural convection. Their dynamics are governing for a coupled differential nonlinear systems. In several previous work we show chaos in the fluid, even with a viscoelastic fluid. We study the asymptotic behavior depending on the relevant parameters for this kind of viscoelastic fluid in some particular thermosyphon models.

Keywords: Thermosyphon, Asymptotic behaviour, Inertial Manifold, Viscoelastic fluid.

1 Introduction

In engineering literature a thermosyphon is a device composed of a closed loop *pipe* containing a fluid whose motion is driven by the effect of several actions such as gravity and natural convection.The flow inside the loop is driven by an energetic balance between thermal energy and mechanical energy.

Here, we consider a thermosyphon model in which the confined fluid is viscoelastic. This has some *a-priori* interesting peculiarities that could affect the dynamics with respect to the case of a Newtonian fluid. On the one hand, the dynamics has memory so its behavior depends on the whole past history and, on the second hand, at small perturbations the fluid behaves like an elastic solid and a characteristic resonance frequency could, eventually, be relevant (consider for instance the behavior of jelly or toothpaste).

The simplest approach to viscoelasticity comes from the so-called Maxwell model Morrison [14]. In this model, both Newton's law of viscosity and Hooke's law of elasticity are generalized and complemented through an evolution equation for the stress tensor, σ.

Viscoelastic behavior is common in polymeric and biological suspensions and, consequently, our results may provide useful information on the dynamics of this sort of systems inside a thermosyphon.

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In a thermosyphon the equations of motion can be greatly simplified because of the quasi-one-dimensional geometry of the loop. Thus, we assume that the section of the loop is constant and small compared with the dimensions of the physical device, so that the arc length co-ordinate along the loop (x) gives the position in the circuit. The velocity of the fluid is assumed to be independent of the position in the circuit, i.e. it is assumed to be a scalar quantity depending only on time. This approximations comes from the fact that the fluid is assumed to be incompressible. On the contrary temperature is assumed to depend both on time and position along the loop.

The derivation of the thermosyphon equations of motion is similar to that in Ref. Keller [12] and are obtained in Yasappan and Jiménez-Casas et al. [10]. Finally, after adimensionalizing the variables (to reduce the number of free parameters) we get the followingo ODE/PDE system (see Yasappan and Jiménez-Casas et al. [10] and Bravo-Gutierrez and Castro et al. [1]):

$$\begin{cases} \varepsilon \dfrac{d^2 v}{dt^2} + \dfrac{dv}{dt} + G(v)v = \oint Tf, \\ \dfrac{\partial T}{\partial t} + v\dfrac{\partial T}{\partial x} = l(v)(T_a - T) \end{cases} \tag{1}$$

with $v(0) = v_0$, $\frac{dv}{dt}(0) = w_0$ and $T(0,x) = T_0(x)$.

Here $v(t)$ is the velocity, $T(t,x)$ is the distributions of the temperature of the viscoelastic fluid into the loop, $G(v)$, is the friction law at the inner wall of the loop, the function f is the geometry of the loop and the distribution of gravitational forces. In this cases $l(v)(T_a - T)$ is the Newton's linear cooling law as in Jiménez-Casas and Rodríguez-Bernal [5–7], Yasappan, Jiménez-Casas et al. [10], Morrison [14] , Rodríguez-Bernal and Van Vleck [18] or Welander [20], where l represents the heat transfer law across the loop wall, and is a positive queantity depending on the velocity, and T_a is the (given) ambient temperature distribution. ε in Eq. (1) is the viscoelastic parameter, wich is the dimensionless version of the viscoelastic time. Roughly speaking, it gives the time scale in which the transition from elastic to fluid-like occurs in the fluid.

We assume that $G(v)$ is positive and bounded away from zero. This function has been usually taken to be $G(v) = G_0$, a positive constant for the linear friction case [12], or $G(v) = |v|$ for the quadratic law [4,13], or even a rather general function given by $G(v) = g(Re)|v|$,where Re is a Reynolds-like number that is assumed to be large [17,19] and proportional to $|v|$. The functions G, f, l and h incorporate relevant physical constants of the model, such as the cross sectional area, D, the length of the loop, L, the Prandtl, Rayleigh, or Reynolds numbers, etc see [19]. Usually G, l are given continuous functions, such that $G(v) \geq G_0 > 0$, and $l(v) \geq l_0 > 0$, for G_0 and l_0 positive constants.

Finally, for physical consistency, it is important to note that all functions considered must be 1-periodic with respect to the spatial variable, and $\oint = \int_0^1 dx$ denotes integration along the closed path of the circuit. Note that $\oint f = 0$.

The contribution in this paper (Section 3) is to prove that, under suitable conditions, any solution either converges to the rest state or the oscillations of velocity around $v = 0$ must be large enough. This result, generalizes the one proposed in Rodríguez-Bernal and Van Vleck [18] for a thermosyphon model with a one-component **viscoelastic** fluid in the case of friction linear law.

2 Previous results about well posedness and global attractor

First, we note that in this section we consider the case in which all periodic functions in Eq. (1) have zero average, i.e. we work in $\mathcal{Y} = I\!R^2 \times \dot{H}^1_{per}(0,1)$, where

$$\dot{L}^2_{per}(0,1) = \{u \in L^2_{loc}(I\!R),\, u(x+1) = u(x) a.e., \oint u = 0\},$$

$$\dot{H}^m_{per}(0,1) = H^m_{loc}(I\!R) \cap \dot{L}^2_{per}(0,1).$$

In effect, we observe that, if we integrate the equation for the temperature along the loop, we have that $\frac{d}{dt}(\oint T) = l(v) \oint (T_a - T)$. Therefore,$\oint T \to \oint T_a$, exponentially as $t \to \infty$, for every $\oint T_0$.

Moreover, if we consider $\theta = T - \oint T$, then from the second equation of system (1), we obtain the θ verifies the equation:

$$\frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial x} = l(v))(\theta_a - \theta), \quad \theta(0) = \theta_0 = T_0 - \oint T_0 \tag{2}$$

and $\theta_a = T_a - \oint T_a$.

Therefore, if we consider now $\theta = T - \oint T$ then from the second equation of system Eq. (1), we obtain that θ verifies the equations (2).

Finally, since $\oint f = 0$, we have that $\oint T f = \oint \theta f$, and the equation for v reads

$$\varepsilon \frac{d^2 v}{dt^2} + \frac{dv}{dt} + G(v)v = \oint \theta . f, \quad v(0) = v_0, \frac{dv}{dt}(0) = w_0. \tag{3}$$

Thus, from Eqs. (2) and (3) we have (v, θ) verifies system Eq.(1) with θ_a, θ_0 replacing T_a, T_0 respectively and now $\oint \theta = \oint \theta_a = \oint \theta_0 = \oint f = 0$.

Thus, we consider the system Eq. (1) with $\oint T_0 = \oint T_a = 0$ and $\oint T(t) = 0$ for every $t \geq 0$.

Therefore, we can apply the result about sectorial operator of Henry [3] to prove the existence of solutions of system (1). Moreover, we consider some aditionally hypothesis (H) to add for the friction G using in the tecnique Lemma 5 in Yasappan and Jiménez-Casas [10], which are satisfied for all friction functions G consider in the previous works,i.e., the thermsyphon models where G is constant or linear or quadratic law, and also for $G(s) \equiv A|s|^n$, as $s \to \infty$. Then, we have the next result.

Proposition 1. *We suppose that $H(r) = rG(r)$ is locally Lipschitz, $f, l \in \dot{L}^2_{per}(0,1)$ $l(v) \geq l_0 > 0$ and $T_a \in \dot{H}^1_{per}(0,1)$. Then, given $(w_0, v_0, T_0) \in \mathcal{Y} = I\!R^2 \times \dot{H}^1_{per}(0,1)$, there exists a unique solution of (1) satisfying*

$$(w, v, T) \in C([0,\infty], \mathcal{Y}) \text{ and } (\dot{w}, w, \frac{\partial T}{\partial t}) \in C([0,\infty), I\!R^2 \times \dot{L}^2_{per}(0,1)),$$

where $w = \dot{v} = \frac{dv}{dt}$ *and* $\dot{w} = \frac{d^2v}{dt^2}$. *In particular, (1) defines a nonlinear semi-group,* $S(t)$ *in* $\mathcal{Y}$, *with* $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$.

Moreover, from (H) (see [10]) Eq. (1) *has a global compact and connected attractor,* $\mathcal{A}$, *in* $\mathcal{Y}$. *Also if* $T_a \in \times \dot{H}^m_{per}(0,1)$ *with* $m \geq 1$, *the global attractor* $\mathcal{A} \subset I\!R^2 \times \dot{H}^{m+2}_{per}(0,1)$ *and is compact in this space.*

Proof: This result has been proved in Theorem 3, Theorem 5 and Corollary 11 from Yasappan and Jiménez-Casas at al.[10].

3 Asymptotic behaviour of finite length solutions

In this section we consider the linear friction law [12] where $G(v) = G_0$, a positive constant. First, we will see a result about the existence and behavior of constant solutions with respect to the spatial variable, i.e. depending only on time.

Proposition 2. *Under the hypotheses of Proposition 1 if we suppose that* T_a *is a constant function, i.e.* $T_a \in I\!R$, *then the solutions* $(w(t), v(t), T(t))$ *converge to* $(0, 0, T_a)$ *exponentially when the times goes to infinity.*

Proof: If $\frac{\partial T}{\partial x} = 0$ then taking into account that $\oint f = 0$ we have that $\oint Tf = 0$. Therefore from the system the constan solutions in x for T are given by $T(t) = T_a + (T_0 - T_a)e^{-\int_0^t l(v)dr}$, and hence $T(t) \to T_a$ exponentially, when $t \to \infty$.

I) First, we will note that $v(t) \to 0$ as $t \to \infty$ since from (1) the equation for $v(t)$ is now a linear homogeneous equation given by:

$$\varepsilon \frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0 v = 0 \text{ with } \varepsilon, G_0 > 0. \tag{4}$$

In effect, if we denoted by $v_H(t)$ any solution of this linear homogeneous equation then $v_H(t) \to 0$ as $t \to \infty$ since there exits a base of solutions given by exponential functions which converge to zero.

II) Second, we will also prove that if $v(t) \to 0$ then $w(t) \to 0$ as $t \to \infty$ exponentially and we conclude.

In effect, if $v(t) \to 0$, for every $\delta > 0$ there exists t_0 such that $|G_0 v| \leq \delta$ and $\varepsilon \frac{d|w|}{dt} + |w| \leq \delta$ for every $t \geq t_0$, this is

$$|w(t)| \leq |w(t_0)|e^{-\frac{1}{\varepsilon}(t-t_0)} + \delta[1 - e^{-\frac{1}{\varepsilon}(t-t_0)}] \leq \delta$$

i.e $w(t) \to 0$ as $t \to \infty$ exponentially.

□

In previous works, like Yasappan and Jiménez-Casas et al. [10], Jiménez-Casas and Castro at al.[8,11], the asymptotic behaviour of the system for the viscoelastic fluid as Eq.(1) for large enough time is studied.

In this sense the existence of a inertial manifold associated to the functions f (loop-geometry) and T_a (given ambient temperature) have proved. The abstract operators theory (Henry[3], Foias et al. [2] and Rodríguez-Bernal[16,15]) has been used for this purpose.

In this section we will prove in Proposition 3, for the linear friction case [12] the results which rise an important consequence: for large time the velocity reaches the equilibrium - null velocity -, or takes a value to make its integral diverge, which means that either it remains with a constant value without changing its sign or it will alternate an infinite number of times so the oscillations around zero become large enough to make the integral diverge.

3.1 Previous results and notations

In this section we assume also that $G^*(r) = rG(r)$ is locally Lipschitz satisfying (H) (see [10]), and $f, T_a \in \dot{L}^2_{per}$ are given by following Fourier expansions

$$T_a(x) = \sum_{k \in \mathbb{Z}^*} b_k e^{2\pi kix}; \qquad f(x) = \sum_{k \in \mathbb{Z}^*} c_k e^{2\pi kix}; \tag{5}$$

where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, while $T_0 \in \dot{H}^2_{per}$ is given by $T_0(x) = \sum_{k \in \mathbb{Z}^*} a_{k0} e^{2\pi kix}$.

Finally assume that $T(t,x) \in \dot{H}^2_{per}$ is given by

$$T(t,x) = \sum_{k \in \mathbb{Z}^*} a_k(t) e^{2\pi kix} \tag{6}$$

where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, We note that $\bar{a_k} = -a_k$ since all functions consider are real and also $a_0 = 0$ since they have zero average.

Now we observe the dynamics of each Fourier mode and from Eq. (1), we get the following system for the new unknowns, v and the coefficients $a_k(t)$.

$$\begin{cases} \varepsilon \frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \sum_{k \in \mathbb{Z}^*} a_k(t) c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + l(v(t))\right] a_k(t) = l(v(t)) b_k \end{cases} \tag{7}$$

- Assume that the given ambient temperature $T_a \in \dot{H}^m_{per}$, are given by

  $$T_a(x) = \sum_{k \in K} b_k e^{2\pi kix},$$

  and $b_k \neq 0$ for every $k \in K \subset \mathbb{Z}$ with $0 \neq K$, since $\oint T_a = 0$. We denote by V_m the clousure of the subspace of $\dot{H}^m_{per}$ generated by $\{e^{2\pi kix}, k \in K\}$. Then we have from Theorem 13 in Yasappan and Jiménez-Casas et al.[10] the set $\mathcal{M} = \mathbb{R}^2 \times V_m$ is an **inertial manifold** for the flow of $S(t)(w_0, v_0, T_0) = (w(t), v(t), T(t))$ in the space $\mathcal{Y} = \mathbb{R}^2 \times \dot{H}^m_{per}(0,1)$. By this, the dynamics of the flow is given by the flow in $\mathcal{M}$ associated to the given ambient temperature T_a. This is

  $$\begin{cases} \varepsilon \frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \sum_{k \in K} a_k(t) c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + l(v(t))\right] a_k(t) = l(v(t)) b_k, \ k \in K \end{cases} \tag{8}$$

- Moreover, we assume that the function associated to the geometry of the loop f, are given by
$$f(x) = \sum_{k\in J} c_k e^{2\pi kix}$$
and $c_k \neq 0$ for every $k \in J \subset \mathbb{Z}$ with $0 \neq K$, since $\oint f = 0$.
We note also that on the inertial manifold
$\oint Tf = \sum_{k\in K\cap J} a_k(t)c_{-k}$. Thus, the dynamics of the system depends only on the coefficients in $K \cap J$.
- Herafter, we consider de functions T_a and f are given by following Fourier expansions
$$T_a(x) = \sum_{k\in K} b_k e^{2\pi kix}; \qquad f(x) = \sum_{k\in J} c_k e^{2\pi kix}; \tag{9}$$
where
$K = \{k \in \mathbb{Z}^*/b_k \neq 0\}, J = \{k \in \mathbb{Z}^*/c_k \neq 0\}$ with $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, First, from the equations Eq. (7) we can observe the velocity of the fluid is independent of the coefficients for temperature $a_k(t)$ for every $k \in \mathbb{Z}^* - (K \cap J)$. That is, the **relevant coefficients** for the velocity are only $a_k(t)$ with k belonging to the set $K \cap J$. This important result about the asymptotic behaviour has been proved in Propositions 14 and 15 from Yasappan and Jiménez-Casas at al.[10].

We also note that $0 \notin K \cap J$ and since $K = -K$ and $J = -J$ then the set $K \cap J$ has an even number of elements, wihch we denote by $2n_0$. Therefore the number of the positive elements of $K \cap J$, $(K \cap J)_+$ is n_0. Moreover the equations for a_{-k} are conjugates of the equations for a_k and therefore we have $\sum_{k\in K\cap J} a_{k(t)}c_{-k} = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k})$.

Thus,
$$\oint Tf = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}). \tag{10}$$

The aim is to prove the Proposition 3 which generalize the result of thermosyphon model for Newtoniann fluids of Rodríguez-Bernal and Van Vleck [18] in the case of a linear friction law. To do so we examine which are these steady-state solutions, also called *equilibrium points.*

We have to make the difference between equilibrium points (constants respect to the time) null velocity, called *rest equilibrium*, and equilibrium points with non-vanishing constant velocity.

Equilibrium conditions.

i) The system Eq. (7) presents the *rest equilibrium* $w = v = 0$, $a_k = b_k$ $\forall k \in K \cap J$ under the assumption of the following orthogonality condition:
$$I_0 = Re(\sum_{k\in(K\cap J)_+} b_k c_{-k}) = 0. \tag{11}$$

ii) Any other equilibrium position will have a non-vanishing velocity and the equilibrium is given by:

$$\begin{cases} G(v)v = 2Re(\sum_{k\in(K\cap J)_+} a_k c_{-k}) \\ a_k = \frac{l(v)b_k}{l(v)+2\pi kiv} \end{cases} \tag{12}$$

3.2 Asymptotic behaviour

Lemma 1. *We consider the linear equation given by*

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0 v = I(t), \tag{13}$$

then there exist $v_p(t)$ particular solution of (13) such that

$$limsup_{t\to\infty}|v_p(t)| \leq \frac{1}{G_0} limsup_{t\to\infty}|I(t)| \tag{14}$$

and

$$liminf_{t\to\infty}|v_p(t)| \geq \frac{1}{G_0} liminf_{t\to\infty}|I(t)| \tag{15}$$

Proof: We consider $\alpha_i = e^{\phi_i t}, i = 1, 2$ a base of solutions of linear homogeneous equations

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0 v = 0$$

with $\phi_1 + \phi_2 = \frac{-1}{\varepsilon}$ and $\phi_1.\phi_2 = \frac{G_0}{\varepsilon}$, then we have that

$$v_p(t) = (\alpha_1, \alpha_2)\int_{t_0}^{t} W^{-1}((\alpha_1(s), \alpha_2(s))(0, \frac{I(s)}{\varepsilon})^{\perp} ds \tag{16}$$

is a particular solution of (13), with W denoting the Wrouskiann. Then we have that

$$v_p(t) = \frac{e^{\phi_2 t}}{\varepsilon(\phi_2 - \phi_1)}\int_{t_0}^{t} I(s)e^{(\phi_1+\frac{1}{\varepsilon})s} - \frac{e^{\phi_1 t}}{\varepsilon(\phi_2 - \phi_1)}\int_{t_0}^{t} I(s)e^{(\phi_2+\frac{1}{\varepsilon})s}.$$

Finally, using L'Hopital's Lemma from [18] we have for example:

$$limsup_{t\to\infty}\frac{\int_{t_0}^{t} I(s)e^{(\phi_1+\frac{1}{\varepsilon})s}}{e^{-\phi_2 t}} \leq$$

$$limsup_{t\to\infty}\frac{I(t)e^{(\phi_1+\frac{1}{\varepsilon})t}}{-\phi_2 e^{-\phi_2 t}} = \frac{-1}{\phi_2} limsup_{t\to\infty} I(t),$$

since $\phi_1 + \phi_2 = \frac{-1}{\varepsilon}$, and we conclude.

□

Lemma 2. *If we assume that a solution of Eq.* (7) *satisfies* $\int_0^\infty |v(s)|ds < \infty$, *then for every* $\eta > 0$ *there exists* t_0 *such that*

$$\int_{t_0}^{t} l(r)e^{-\int_r^t l}(e^{-\int_r^t 2\pi ikv} - 1)dr \leq \eta, \tag{17}$$

where $l(r) = l(v(r))$*, and* $t \geq t_0$.

Proof: If $\int_0^\infty |v(s)|ds < \infty$, then for all δ there exits $t_0 > 0$ such that for every $t_0 \leq r \leq t$ we have $|\int_r^t v| \leq \delta$. Then, for any $\eta > 0$ we can take t_0 large enough such that

$$|e^{-\int_r^t 2\pi ikv} - 1| \leq \eta \text{ for all } t_0 \leq r \leq t. \tag{18}$$

Therefore,writing $l(r) = l(v(r))$ and taking into account that l is istrictly bounded away from zero, we get

$$\int_{t_0}^{t} l(r)e^{-\int_r^t l}(e^{-\int_r^t 2\pi ikv} - 1)dr \leq \eta(1 - e^{-\int_r^t l}) \leq \eta$$

□

Proposition 3. *We consider the linear friction case, i.e.* $G(v) = G_0$ *with* G_0 *a positive constant.*

i) if $K \cap J = \emptyset$*, then the global attractor for system Eq.* (1) *in* $I\!R^2 \times \dot{H}^1_{per}(0,1)$ *is reduced to a point* $\{(0,0,T_a)\}$.

ii) We assume that $I_0 = Re(\sum_{k\in(K\cap J)_+ b_k c_{-k}}) = 0$*, with* $K \cap J$ *finite set, and that a solution of Eq.* (7) *satisfies* $\int_0^\infty |v(s)|ds < \infty$. *Then the system reaches the rest stationary solution, that:*

$$\begin{cases} v(t) \to 0, \text{ and } w(t) \to 0, \text{ as } t \to \infty \\ a_k(t) \to b_k, \text{ as } t \to \infty \end{cases}$$

This is, he also have in this situation the global attractor for system Eq. (1) *in* $I\!R^2 \times \dot{H}^1_{per(0,1)}$ *is reduced to a point* $\{(0,0,T_a)\}$.

iii) Conversely, if $I_0 = Re(\sum_{k\in(K\cap J)_+} b_k c_{-k}) \neq 0$ *then for every solution* $\int_0^\infty |v(s)|ds = \infty$*, and* $v(t)$ *does not converge to zero.*

Proof: First, we study the behaviour for large time of the coefficients $a_k(t)$.

The distance between the coefficients that represents the solution of the system, $a_k(t)$ to the values of those coefficients in the equilibrium, b_k is computed.

For t_0 enough large, noting $l(v(r)) = l(r)$, we known that for every $t > t_0$ we have

$$a_k(t) = a_k(t_0)e^{-\int_{t_0}^t 2\pi ikv + l} + b_k \int_{t_0}^{t} l(r)e^{-\int_r^t 2\pi ikv + l}dr \tag{19}$$

and

$$a_k(t) - (1 - e^{-\int_0^t 2\pi kiv})b_k = a_k(t_0)e^{-\int_{t_0}^t 2\pi ikv+l} +$$
$$+ b_k \int_{t_0}^t l(r)e^{-\int_r^t l}(e^{-\int_r^t 2\pi ikv} - 1)dr.$$

Taking limits when $t \to \infty$, we get

$a_k(t) - (1 - e^{-\int_0^t 2\pi kiv})b_k \to 0$, since $a_k(t_0)e^{-\int_0^t 2\pi ikv+l} \to 0$ and from Eq. (17) we have that $b_k \int_{t_0}^t l(r)e^{-\int_r^t l}(e^{-\int_r^t 2\pi kiv} - 1) \to 0$. Now taking into account that $(1 - e^{-\int_0^t 2\pi ikv})b_k$ converges to b_k for large time we conclude that:

$$\begin{cases} a_k(t) \to b_k \\ I(t) = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}) \to I_0 \end{cases} \tag{20}$$

with $I_0 = 2Re(\sum_{k\in(K\cap J)_+} b_k c_{-k})$.

We note that $T(t,x) = \sum_k a_k(t)e^{2\pi kix} \to T_a = \sum_k b_k e^{2\pi kix}$.

To conclude, we study now when the velocity $v(t)$ and the acceleration $w(t)$ go to zero. From (10) we can reading the equation for v, the first equation of system Eq. (7), as

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = I(t).$$

we consider now, $G(v) = G_0 > 0$ and then we note that:

I) First, if we consider $v_p(t)$ the particular solution of the above equation given by Lemma 1 and we denoted by $v_H(t)$ the solution of linear homogeneous equation given by:

$$\varepsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + G_0 v = 0$$

such that $v(t) = v_p(t) + v_H(t)$. We now that since $v_H(t) \to 0$ as $t \to \infty$, we have that: $v(t) - v_p(t) \to 0$ as $t \to \infty$.

II) Second,using (20) for every $\delta > 0$ there exists t_0 such that $|I(t) - I_0| \leq \delta$ for ever $t \geq t_0$, and using Lemma 1, we conclude that

$$limsup_{t\to\infty}|v(t)| \leq \frac{\delta + I_0}{G_0}$$

for every $\delta > 0$.

i)-ii) In particular, when $K \cap J = \emptyset$ or $I_0 = 0$, we get $v(t) \to 0$ and working as in Part II) from Proposition 2 we also have that $w(t) \to 0$. Thus we conclude.

iii) Finally, we also note that

$$liminf_{t\to\infty}|v(t)| \geq \frac{\delta + I_0}{G_0}$$

for every $\delta > 0$ and in the case of $I_0 \neq 0$, we get $liminf_{t\to\infty}|v(t)| > 0$, which implies that $\int_0^\infty |v(s)|ds = \infty$. This result is in contradiction with the initial condition $\int_0^\infty |v(s)|ds < \infty$, what implies that it is not a valid hypothesis.

□

3.3 Concluding remarks

Recalling that functions associated to circuit geometry, f, and to prescribed ambient temperature, T_a, are given by $f(x) = \sum_{k \in J} c_k e^{2\pi kix}$ and $T_a(x) = \sum_{k \in K} b_k e^{2\pi kix}$, respectively. In Yasappan and Jiménez-Casas et. al [10], using the operator abstract theory, it is proved that if $K \cap J = \emptyset$, then the global attractor for system Eq. (1) in $I\!R^2 \times \dot{H}^1_{per}(0,1)$ is reduced to a point

In this sense the Proposition 3 offers the possibility to obtain the same asymptotic behaviour for the dynamics, i.e., the attractor is also reduced to a point taking functions f and T_a without this condition, that is with $K \cap J \neq \emptyset$, its enough that the set $(K \cap J) \neq \emptyset$, but $Re(\sum_{k \in (K \cap J)_+} b_k c_{-k}) = 0$, when we consider the linear friction case $G = G_0$.

We note, the result about the inertial manifold (Yasappan and Jiménez-Casas et. al [10]) reduces the asymptotic behaviour of the initial system Eq. (1) to the dynamics of the reduced explicit system Eq. (8) with $k \in K \cap J$.

We observe also that from the analysis above, it is possible to design the geometry of circuit, f, and/or ambient temperature, T_a, so that the resulting system has an arbitrary number of equations of the form $N = 4n_0 + 1$ where n_0 is the number of elements of $(K \cap J)_+$ and we consider the real an imaginary parts of relevant coefficients for the temperature $a_k(t)$ and solute concentration $d_k(t)$ with $k \in (K \cap J)_+$.

Note that it may be the case that K and J are infinite sets, but their intersection is finite. Also, for a circular circuit we have $f(x) \sim asen(x) + bcos(x)$, i.e. $J = \{\pm 1\}$ and then $K \cap J$ is either $\{\pm 1\}$ or the empty set.

Recently, we have considered a thermosyphon model containing a viscoelastic fluid and we have shown chaos in some closed-loop thermosyphon model with one-component viscoelastic fluid not only in this model [10], also in other kind of transfer law (Jiménez-Casas and Castro [8], Yasappan and Jiménez-Casas et al. [9]), and even in some cases with a viscoelastic binary fluid (Yasappan and Jiménez-Casas et al. [11])

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References

1. Bravo-Gutierrez. M.E, Castro Mario, Hernandez-Machado. A and Poire, “Controlling Viscoelastic Flow in Microchannels with Slip”, Langmuir, ACS Publications, 27-6, 2075-2079, (2011).
2. C. Foias, G.R. Sell and R. Temam,“Inertial Manifolds for Nonlinear Evolution Equations”, J. Diff. Equ., 73, 309-353, (1985).

3. D. Henry, "Geometric Theory of Semilinear Parabolic Equations", Lectures Notes in Mathematics 840, Springer-Verlag, Berlin, New York, (1982).
4. M.A. Herrero and J.J-L. Velazquez, "Stability analysis of a closed thermosyphon", European J. Appl. Math., 1, 1-24, (1990).
5. A. Jiménez-Casas, and A. Rodríguez-Bernal, "Finite-dimensional asymptotic behavior in a thermosyphon including the Soret effect", Math. Meth. in the Appl. Sci., 22, 117-137, (1999).
6. A. Jiménez-Casas, "A coupled ODE/PDE system governing a thermosyphon model", Nonlin. Analy., 47, 687-692, (2001).
7. A. Jiménez Casas and A. Rodríguez-Bernal, "Dinámica no lineal: modelos de campo de fase y un termosifón cerrado", Editorial Académica Española, (Lap Lambert Academic Publishing GmbH and Co. KG, Germany 2012).
8. A. Jiménez-Casas and Mario Castro, "A Thermosyphon model with a viscoelastic binary fluid", Electronic Journal of Differ. Equ., ISSN: 1072-6691 (2016).
9. A. Jiménez-Casas, Mario Castro and Justine Yasappan,"Finite-dimensional behaviour in a thermosyphon with a viscoelastic fluid",Disc. and Conti. Dynamic. Syst. Supplement 2013, vol. , , 375-384, (2013).
10. Justine Yasappan, A. Jiménez-Casas and Mario Castro, "Asymptotic behavior of a viscoelastic fluid in a closed loop thermosyphon: physical derivation, asymptotic analysis and numerical experiments",Abstract and Applied Analysis, **2013**, 748683, 1-20, (2013).
11. Justine Yasappan, A. Jiménez-Casas and Mario Castro, "Stabilizing interplay between thermodiffusion and viscoelasticity in a closed-loop thermosyphon",Disc. and Conti. Dynamic. Syst. Series B., vol. 20, 9, 3267-3299, (2015).
12. J.B. Keller, "Periodic oscillations in a model of thermal convection", J. Fluid Mech., 26, 3, 599-606, (1966).
13. A. Liñan, "Analytical description of chaotic oscillations in a toroidal thermosyphon", in Fluid Physics, Lecture Notes of Summer Schools, (M.G. Velarde, C.I. Christov, eds.,) 507-523, World Scientific, River Edge, NJ, (1994).
14. F. Morrison, *Understanding rheology*, (Oxford University Press, USA, 2001).
15. A. Rodríguez-Bernal, "Inertial Manifolds for dissipative semiflows in Banach spaces", Appl. Anal., 37, 95-141, (1990).
16. A. Rodríguez-Bernal, "Attractor and Inertial Manifolds for the Dynamics of a Closed Thermosyphon", Journal of Mathematical Analysis and Applications, 193, 942-965, (1995).
17. A. Rodríguez-Bernal and E.S. Van Vleck, "Diffusion Induced Chaos in a Closed Loop Thermosyphon", SIAM J. Appl. Math., vol. 58, 4, 1072-1093, (1998).
18. A. Rodríguez-Bernal and E.S. Van Vleck, "Complex oscillations in a closed thermosyphon", in Int. J. Bif. Chaos, vol. 8, 1, 41-56, (1998).
19. J.J.L. Velázquez, "On the dynamics of a closed thermosyphon", SIAM J.Appl. Math. 54, n^o 6, 1561-1593, (1994).
20. P. Welander, "On the oscillatory instability of a differentially heated fluid loop," J. Fluid Mech. 29, No 1, 17-30, (1967).