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A Simpler Variational Principle for Stationary non-Barotropic Ideal Magnetohydrodynamics

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Abstract. Variational principles for magnetohydrodynamics (MHD) were introduced by previous authors both in Lagrangian and Eulerian form. In this paper we introduce simpler Eulerian variational principles from which all the relevant equations of non-barotropic stationary magnetohydrodynamics can be derived for certain field topologies. The variational principle is given in terms of three independent functions for stationary non-barotropic flows in which magnetic field lines lie on entropy surfaces. This is a smaller number of variables than the eight variables which appear in the standard equations of non-barotropic magnetohydrodynamics which are the magnetic field $\boldsymbol{B}$ the velocity field $\boldsymbol{v}$, the entropy s and the density ρ. The reduction of variables constraints the possible chaotic motion available to such a system.
Keywords: Magnetohydrodynamics, Variational Principles, Reduction of Variables.

1 Introduction

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [1] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [2] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field $\boldsymbol{B}$ the velocity field $\boldsymbol{v}$ and the pressure P. Kats [3] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Moreover, Kats has shown that the variables he suggested can be utilized to describe the motion of arbitrary discontinuity surfaces [4,5]. Sakurai [6] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [1]. A method of solving the equations for those two variables was introduced by Yang, Sturrock & Antiochos [8]. Yahalom & Lynden-Bell [9] combined the Lagrangian of Sturrock [1] with the Lagrangian

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of Sakurai [6] to obtain an **Eulerian** Lagrangian principle for barotropic magnetohydrodynamics which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [12] (see also [13]). Yahalom [10] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [11] are topological local conserved quantities.

Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non barotropic magnetohydrodynamics can be found in the work of Bekenstein & Oron [14] in terms of 15 functions and V.A. Kats [3] in terms of 20 functions. The author of this paper suspect that this number can be somewhat reduced. Moreover, A. V. Kats in a remarkable paper [15] (section IV,E) has shown that there is a large symmetry group (gauge freedom) associated with the choice of those functions, this implies that the number of degrees of freedom can be reduced. Yahalom [16] have shown that only five functions will suffice to describe non barotropic magnetohydrodynamics in the case that we enforce a Sakurai [6] representation for the magnetic field. Morrison [7] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [7]). The work of Yahalom [16] was concerned with general non-stationary flows. A separate work [17] was concerned with stationary flows and introduced a 8 variable stationary variational principle, here we shall attempt to improve on this and obtain a 3 variable stationary variational principle for non-barotropic MHD. This will be done for the restricted case in which the magnetic field lines lie on entropy surfaces.

We anticipate applications of this study both to linear and non-linear stability analysis of known non barotropic magnetohydrodynamic configurations [24,26] and for designing efficient numerical schemes for integrating the equations of fluid dynamics and magnetohydrodynamics [32,33,35]. Another possible application is connected to obtaining new analytic solutions in terms of the variational variables [36].

The plan of this paper is as follows: First we introduce the standard notations and equations of non-barotropic magnetohydrodynamics for the stationary and non-stationary cases. Then we introduce the concepts of load and metage. The variational principle follows.

2 Standard formulation of non-barotropic magnetohydrodynamics

The standard set of equations solved for non-barotropic magnetohydrodynamics are given below:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0, \tag{3}$$

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}) = -\boldsymbol{\nabla} p(\rho, s) + \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}. \tag{4}$$

$$\frac{ds}{dt} = 0. \tag{5}$$

The following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and $\boldsymbol{\nabla}$ has its standard meaning in vector calculus. $\boldsymbol{B}$ is the magnetic field vector, $\boldsymbol{v}$ is the velocity field vector, ρ is the fluid density and s is the specific entropy. Finally $p(\rho, s)$ is the pressure which depends on the density and entropy (the non-barotropic case).

The justification for those equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example [1]). The above applies to a collision-dominated plasma in local thermodynamic equilibrium. Such conditions are seldom satisfied by physical plasmas, certainly not in astrophysics or in fusion-relevant magnetic confinement experiments. Never the less it is believed that the fastest macroscopic instabilities in those systems obey the above equations [11], while instabilities associated with viscous or finite conductivity terms are slower. It should be noted that due to a theorem by Bateman [37] every physical system can be described by a variational principle (including viscous plasma) the trick is to find an elegant variational principle usually depending on a small amount of variational variables. The current work will discuss only ideal magnetohydrodynamics while viscous magnetohydrodynamics will be left for future endeavors.

Equation (1) describes the fact that the magnetic field lines are moving with the fluid elements (”frozen” magnetic field lines), equation (2) describes the fact that the magnetic field is solenoidal, equation (3) describes the conservation of mass and equation (4) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\boldsymbol{J} = \frac{\boldsymbol{\nabla} \times \boldsymbol{B}}{4\pi}, \tag{6}$$

is the electric current density which is not connected to any mass flow. Equation (5) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic magnetohydrodynamics and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight $(\boldsymbol{v}, \boldsymbol{B}, \rho, s)$ and the number of equations (1,3,4,5) is also eight. Notice that equation (2) is a condition on the initial $\boldsymbol{B}$ field and is satisfied automatically for any other time due to equation (1). For the stationary case in which the physical fields do not depend on time we obtain the following set of stationary equations:

$$\boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = 0, \tag{7}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{8}$$

$$\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0, \tag{9}$$

$$\rho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p(\rho, s) + \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}. \tag{10}$$

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0. \tag{11}$$

3 Load and Metage

The following section follows closely a similar section in [9]. Consider a thin tube surrounding a magnetic field line as described in figure 1, the magnetic

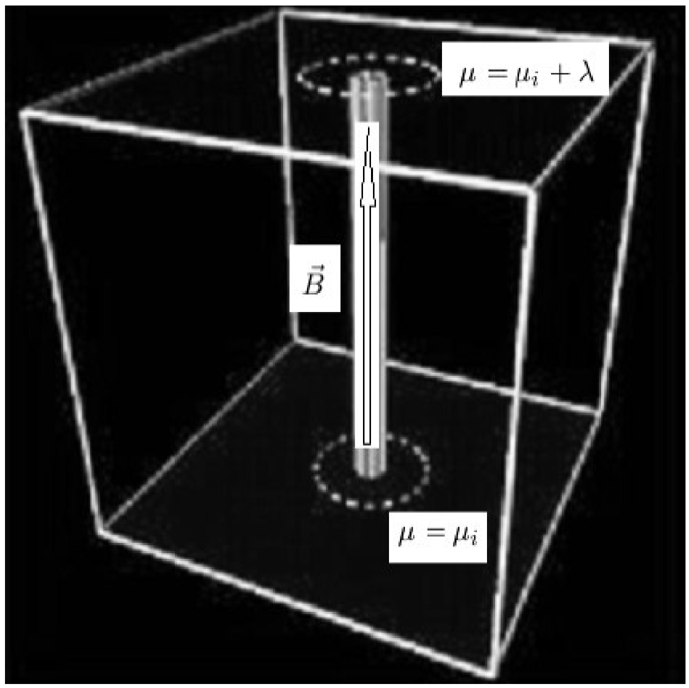

Fig. 1. A thin tube surrounding a magnetic field line

flux contained within the tube is:

$$\Delta\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{S} \tag{12}$$

and the mass contained with the tube is:

$$\Delta M = \int \rho d\boldsymbol{l} \cdot d\boldsymbol{S}, \tag{13}$$

in which dl is a length element along the tube. Since the magnetic field lines move with the flow by virtue of equation (1) and equation (3) both the quantities $\Delta\Phi$ and ΔM are conserved and since the tube is thin we may define the conserved magnetic load:

$$\lambda = \frac{\Delta M}{\Delta \Phi} = \oint \frac{\rho}{B} dl, \tag{14}$$

in which the above integral is performed along the field line. Obviously the parts of the line which go out of the flow to regions in which $\rho = 0$ have a null contribution to the integral. Notice that λ is a **single valued** function that can be measured in principle. Since λ is conserved it satisfies the equation:

$$\frac{d\lambda}{dt} = 0. \tag{15}$$

By construction surfaces of constant magnetic load move with the flow and contain magnetic field lines. Hence the gradient to such surfaces must be orthogonal to the field line:

$$\boldsymbol{\nabla}\lambda \cdot \boldsymbol{B} = 0. \tag{16}$$

Now consider an arbitrary comoving point on the magnetic field line and denote it by i, and consider an additional comoving point on the magnetic field line and denote it by r. The integral:

$$\mu(r) = \int_i^r \frac{\rho}{B} dl + \mu(i), \tag{17}$$

is also a conserved quantity which we may denote following Lynden-Bell & Katz [18] as the magnetic metage. $\mu(i)$ is an arbitrary number which can be chosen differently for each magnetic line. By construction:

$$\frac{d\mu}{dt} = 0. \tag{18}$$

Also it is easy to see that by differentiating along the magnetic field line we obtain:

$$\boldsymbol{\nabla}\mu \cdot \boldsymbol{B} = \rho. \tag{19}$$

Notice that μ will be generally a **non single valued** function.

At this point we have two comoving coordinates of flow, namely λ, μ obviously in a three dimensional flow we also have a third coordinate. However, before defining the third coordinate we will find it useful to work not directly with λ but with a function of λ. Now consider the magnetic flux within a surface of constant load $\Phi(\lambda)$ as described in figure 2 (the figure was given by Lynden-Bell & Katz [18]). The magnetic flux is a conserved quantity and depends only on the load λ of the surrounding surface. Now we define the quantity:

$$\chi = \frac{\Phi(\lambda)}{2\pi}. \tag{20}$$

Obviously χ satisfies the equations:

$$\frac{d\chi}{dt} = 0, \qquad \boldsymbol{B} \cdot \boldsymbol{\nabla}\chi = 0. \tag{21}$$

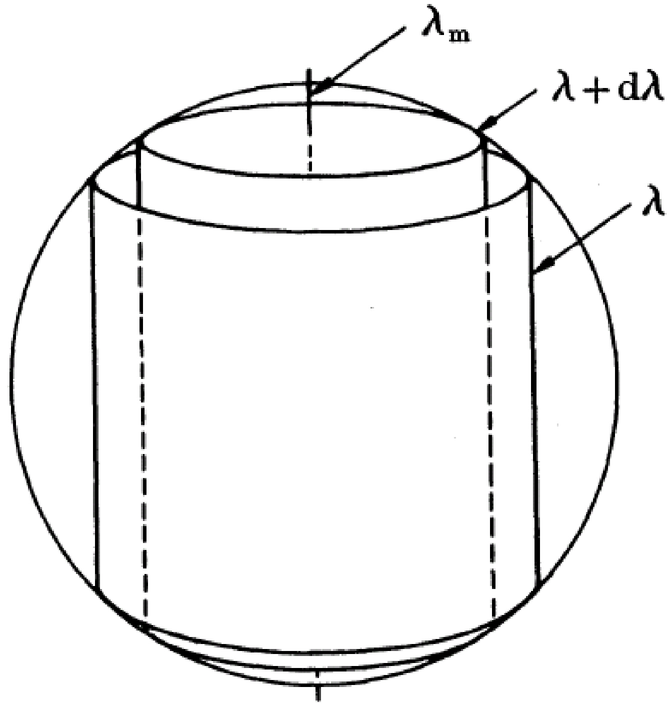

Fig. 2. Surfaces of constant load

Let us now define an additional comoving coordinate η^* since $\boldsymbol{\nabla}\mu$ is not orthogonal to the $\boldsymbol{B}$ lines we can choose $\boldsymbol{\nabla}\eta^*$ to be orthogonal to the $\boldsymbol{B}$ lines and not be in the direction of the $\boldsymbol{\nabla}\chi$ lines, that is we choose η^* not to depend only on χ. Since both $\boldsymbol{\nabla}\eta^*$ and $\boldsymbol{\nabla}\chi$ are orthogonal to $\boldsymbol{B}$, $\boldsymbol{B}$ must take the form:

$$\boldsymbol{B} = A\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\eta^*. \tag{22}$$

However, using equation (2) we have:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla}A \cdot (\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\eta^*) = 0. \tag{23}$$

Which implies that A is a function of χ, η^*. Now we can define a new comoving function η such that:

$$\eta = \int_0^{\eta^*} A(\chi, \eta^{'*}) d\eta^{'*}, \qquad \frac{d\eta}{dt} = 0. \tag{24}$$

In terms of this function we obtain the Sakurai (Euler potentials) presentation:

$$\boldsymbol{B} = \boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\eta. \tag{25}$$

Hence we have shown how χ, η can be constructed for a known $\boldsymbol{B}, \rho$. Notice however, that η is defined in a non unique way since one can redefine η for example by performing the following transformation: $\eta \to \eta + f(\chi)$ in which

$f(\chi)$ is an arbitrary function. The comoving coordinates χ, η serve as labels of the magnetic field lines. Moreover the magnetic flux can be calculated as:

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{S} = \int d\chi d\eta. \tag{26}$$

In the case that the surface integral is performed inside a load contour we obtain:

$$\Phi(\lambda) = \int_\lambda d\chi d\eta = \chi \int_\lambda d\eta = \begin{cases} \chi[\eta] \\ \chi(\eta_{max} - \eta_{min}) \end{cases} \tag{27}$$

There are two cases involved; in one case the load surfaces are topological cylinders, in this case η is not single valued and hence we obtain the upper value for $\Phi(\lambda)$. In a second case the load surfaces are topological spheres, in this case η is single valued and has minimal η_{min} and maximal η_{max} values. Hence the lower value of $\Phi(\lambda)$ is obtained. For example in some cases η is identical to twice the latitude angle θ. In those cases $\eta_{min} = 0$ (value at the "north pole") and $\eta_{max} = 2\pi$ (value at the "south pole").

Comparing the above equation with equation (20) we derive that η can be either **single valued** or **not single valued** and that its discontinuity across its cut in the non single valued case is $[\eta] = 2\pi$.

So far the discussion did not differentiate the cases of stationary and non-stationary flows. It should be noted that even for stationary flows one can have a non-stationary η coordinates as the magnetic field depends only on the gradient of η (see equation (25)), in particular if η is stationary than $\eta + g(t)$ which is clearly not stationary will produce according to equation (25) a stationary magnetic field. In what follows we find it advantageous to use the following form of η:

$$\eta = \bar{\eta} - t, \tag{28}$$

in which $\bar{\eta}$ is stationary.

4 A Simpler variational principle of stationary non-barotropic magnetohydrodynamics

In a previous paper [17] we have shown that stationary non-barotropic magnetohydrodynamics can be described in terms of eight first order differential equations and by an action principle from which those equations can be derived. Below we will show that one can do better for the case in which the magnetic field lines lie on an entropy surface, in this case three functions will suffice to describe stationary non-barotropic magnetohydrodynamics.

Consider equation (21), for a stationary flow it takes the form:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} \chi = 0. \tag{29}$$

Hence $\boldsymbol{v}$ can take the form:

$$\boldsymbol{v} = \frac{\boldsymbol{\nabla} \chi \times \boldsymbol{K}}{\rho}. \tag{30}$$

However, the velocity field must satisfy the stationary mass conservation equation (3):

$$\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0. \tag{31}$$

We see that a sufficient condition (although not necessary) for $\boldsymbol{v}$ to solve equation (31) is that $\boldsymbol{K}$ takes the form $\boldsymbol{K} = \boldsymbol{\nabla} N$, where N is an arbitrary function. Thus, $\boldsymbol{v}$ may take the form:

$$\boldsymbol{v} = \frac{\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla} N}{\rho}. \tag{32}$$

Let us now calculate $\boldsymbol{v} \times \boldsymbol{B}$ in which $\boldsymbol{B}$ is given by Sakurai's presentation equation (25):

$$\begin{aligned}\boldsymbol{v} \times \boldsymbol{B} &= (\frac{\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla} N}{\rho}) \times (\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\eta) \\ &= \frac{1}{\rho}\boldsymbol{\nabla}\chi(\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla} N) \cdot \boldsymbol{\nabla}\eta.\end{aligned} \tag{33}$$

Since the flow is stationary N can be at most a function of the three comoving coordinates $\chi, \mu, \bar{\eta}$ defined in section 3, hence:

$$\boldsymbol{\nabla} N = \frac{\partial N}{\partial \chi}\boldsymbol{\nabla}\chi + \frac{\partial N}{\partial \mu}\boldsymbol{\nabla}\mu + \frac{\partial N}{\partial \bar{\eta}}\boldsymbol{\nabla}\bar{\eta}. \tag{34}$$

Inserting equation (34) into equation (33) will yield:

$$\boldsymbol{v} \times \boldsymbol{B} = \frac{1}{\rho}\boldsymbol{\nabla}\chi\frac{\partial N}{\partial \mu}(\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\mu) \cdot \boldsymbol{\nabla}\bar{\eta}. \tag{35}$$

Rearranging terms and using Sakurai's presentation equation (25) we can simplify the above equation and obtain:

$$\boldsymbol{v} \times \boldsymbol{B} = -\frac{1}{\rho}\boldsymbol{\nabla}\chi\frac{\partial N}{\partial \mu}(\boldsymbol{\nabla}\mu \cdot \boldsymbol{B}). \tag{36}$$

However, using equation (19) this will simplify to the form:

$$\boldsymbol{v} \times \boldsymbol{B} = -\boldsymbol{\nabla}\chi\frac{\partial N}{\partial \mu}. \tag{37}$$

Inserting equation (37) into equation (7) will lead to the equation:

$$\boldsymbol{\nabla}(\frac{\partial N}{\partial \mu}) \times \boldsymbol{\nabla}\chi = 0. \tag{38}$$

However, since N is at most a function of $\chi, \mu, \bar{\eta}$ it follows that $\frac{\partial N}{\partial \mu}$ is some function of χ:

$$\frac{\partial N}{\partial \mu} = -F(\chi). \tag{39}$$

This can be easily integrated to yield:

$$N = -\mu F(\chi) + G(\chi, \bar{\eta}). \tag{40}$$

Inserting this back into equation (32) will yield:

$$\boldsymbol{v} = \frac{\boldsymbol{\nabla}\chi \times (-F(\chi)\boldsymbol{\nabla}\mu + \frac{\partial G}{\partial \bar{\eta}}\boldsymbol{\nabla}\bar{\eta})}{\rho}. \tag{41}$$

Let us now replace the set of variables $\chi, \bar{\eta}$ with a new set $\chi', \bar{\eta}'$ such that:

$$\chi' = \int F(\chi) d\chi, \qquad \bar{\eta}' = \frac{\bar{\eta}}{F(\chi)}. \tag{42}$$

This will not have any effect on the Sakurai representation given in equation (25) since:

$$\boldsymbol{B} = \boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\eta = \boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\bar{\eta} = \boldsymbol{\nabla}\chi' \times \boldsymbol{\nabla}\bar{\eta}'. \tag{43}$$

However, the velocity will have a simpler representation and will take the form:

$$\boldsymbol{v} = \frac{\boldsymbol{\nabla}\chi' \times \boldsymbol{\nabla}(-\mu + G'(\chi', \bar{\eta}'))}{\rho}, \tag{44}$$

in which $G' = \frac{G}{F}$. At this point one should remember that μ was defined in equation (17) up to an arbitrary constant which can vary between magnetic field lines. Since the lines are labelled by their $\chi', \bar{\eta}'$ values it follows that we can add an arbitrary function of $\chi', \bar{\eta}'$ to μ without effecting its properties. Hence we can define a new μ' such that:

$$\mu' = \mu - G'(\chi', \bar{\eta}'). \tag{45}$$

Notice that μ' can be multi-valued. Inserting equation (45) into equation (44) will lead to a simplified equation for $\boldsymbol{v}$:

$$\boldsymbol{v} = \frac{\boldsymbol{\nabla}\mu' \times \boldsymbol{\nabla}\chi'}{\rho}. \tag{46}$$

In the following the primes on $\chi, \mu, \bar{\eta}$ will be ignored. The above equation is analogues to Vladimirov and Moffatt's [2] equation 7.11 for incompressible flows, in which our μ and χ play the part of their A and Ψ. It is obvious that $\boldsymbol{v}$ satisfies the following set of equations:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla}\mu = 0, \qquad \boldsymbol{v} \cdot \boldsymbol{\nabla}\chi = 0, \qquad \boldsymbol{v} \cdot \boldsymbol{\nabla}\bar{\eta} = 1, \tag{47}$$

to derive the right hand equation we have used both equation (18) and equation (25). Hence μ, χ are both comoving and stationary. As for $\bar{\eta}$ it satisfies the same equation as $\bar{\eta}$ defined in equation (28). It can be easily seen that if:

$$basis = (\boldsymbol{\nabla}\chi, \boldsymbol{\nabla}\bar{\eta}, \boldsymbol{\nabla}\mu), \tag{48}$$

is a local vector basis at any point in space than their exists a dual basis:

$$dual\ basis = \frac{1}{\rho}(\boldsymbol{\nabla}\bar{\eta} \times \boldsymbol{\nabla}\mu, \boldsymbol{\nabla}\mu \times \boldsymbol{\nabla}\chi, \boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\bar{\eta}) = (\frac{\boldsymbol{\nabla}\bar{\eta} \times \boldsymbol{\nabla}\mu}{\rho}, \boldsymbol{v}, \frac{\boldsymbol{B}}{\rho}). \tag{49}$$

Such that:

$$basis_i \cdot dual\ basis_j = \delta_{ij}, \qquad i,j \in [1,2,3], \tag{50}$$

in which δ_{ij} is Kronecker's delta. Hence while the surfaces $\chi, \mu, \bar{\eta}$ generate a local vector basis for space, the physical fields of interest $\boldsymbol{v}, \boldsymbol{B}$ are part of the dual basis. By vector multiplying $\boldsymbol{v}$ and $\boldsymbol{B}$ and using equations (46,25) we obtain:

$$\boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{\nabla}\chi, \tag{51}$$

this means that both $\boldsymbol{v}$ and $\boldsymbol{B}$ lie on χ surfaces and provide a vector basis for this two dimensional surface. The above equation can be compared with Vladimirov and Moffatt [2] equation 5.6 for incompressible flows in which their J is analogue to our χ.

5 The action principle

In the previous subsection we have shown that if the velocity field $\boldsymbol{v}$ is given by equation (46) and the magnetic field $\boldsymbol{B}$ is given by the Sakurai representation equation (25) than equations (7,8,9) are satisfied automatically for stationary flows. To complete the set of equations we will show how the Euler equations (4) can be derived from the action:

$$\begin{aligned} A &\equiv \int \mathcal{L} d^3x dt, \\ \mathcal{L} &\equiv \rho(\frac{1}{2}\boldsymbol{v}^2 - \varepsilon(\rho, s)) - \frac{\boldsymbol{B}^2}{8\pi}, \end{aligned} \tag{52}$$

in which both $\boldsymbol{v}$ and $\boldsymbol{B}$ are given by equation (46) and equation (25) respectively and the density ρ is given by equation (18):

$$\rho = \boldsymbol{\nabla}\mu \cdot \boldsymbol{B} = \boldsymbol{\nabla}\mu \cdot (\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\eta) = \frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}. \tag{53}$$

In the above ε is the specific internal energy (internal energy per unit of mass). The reader is reminded of the following thermodynamic relations which will become useful later:

$$\begin{aligned} d\varepsilon &= T ds - P d\frac{1}{\rho} = T ds + \frac{P}{\rho^2} d\rho \\ \frac{\partial \varepsilon}{\partial s} &= T, \qquad \frac{\partial \varepsilon}{\partial \rho} = \frac{P}{\rho^2} \\ w &= \varepsilon + \frac{P}{\rho} = \varepsilon + \frac{\partial \varepsilon}{\partial \rho}\rho = \frac{\partial(\rho\varepsilon)}{\partial \rho} \\ dw &= d\varepsilon + d(\frac{P}{\rho}) = T ds + \frac{1}{\rho} dP \end{aligned} \tag{54}$$

in the above T is the temperature and w is the specific enthalpy. The Lagrangian density of equation (52) takes the more explicit form:

$$\mathcal{L}[\chi, \eta, \mu] = \rho \left(\frac{1}{2} (\frac{\boldsymbol{\nabla}\mu \times \boldsymbol{\nabla}\chi}{\rho})^2 - \varepsilon(\rho, s(\chi, \eta)) \right) - \frac{(\boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\eta)^2}{8\pi} \tag{55}$$

and can be seen explicitly to depend on only three functions. We underline that due to the assumption that the magnetic field lines lie on entropy surfaces. s must be a function of χ, η. Let us make arbitrary small variations $\delta\alpha_i = (\delta\chi, \delta\eta, \delta\mu)$ of the functions $\alpha_i = (\chi, \eta, \mu)$. Let us define a Δ variation that does not modify the α_i's, such that:

$$\Delta\alpha_i = \delta\alpha_i + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\alpha_i = 0, \tag{56}$$

in which $\boldsymbol{\xi}$ is the Lagrangian displacement, thus:

$$\delta\alpha_i = -\boldsymbol{\nabla}\alpha_i \cdot \boldsymbol{\xi}. \tag{57}$$

Which will lead to the equation:

$$\boldsymbol{\xi} \equiv -\frac{\partial \boldsymbol{r}}{\partial \alpha_i}\delta\alpha_i. \tag{58}$$

Making a variation of ρ given in equation (53) with respect to α_i will yield:

$$\delta\rho = -\boldsymbol{\nabla} \cdot (\rho\boldsymbol{\xi}). \tag{59}$$

Making a variation of s will result in:

$$\delta s = \frac{\partial s}{\partial \alpha_i}\delta\alpha_i = -\frac{\partial s}{\partial \alpha_i}\boldsymbol{\nabla}\alpha_i \cdot \boldsymbol{\xi} = -\boldsymbol{\nabla} s \cdot \boldsymbol{\xi}. \tag{60}$$

Furthermore, taking the variation of $\boldsymbol{B}$ given by Sakurai's representation (25) with respect to α_i will yield:

$$\delta\boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}). \tag{61}$$

It remains to calculate $\delta\boldsymbol{v}$ by varying equation (46) this will yield:

$$\delta\boldsymbol{v} = -\frac{\delta\rho}{\rho}\boldsymbol{v} + \frac{1}{\rho}\boldsymbol{\nabla} \times (\rho\boldsymbol{\xi} \times \boldsymbol{v}). \tag{62}$$

Varying the action will result in:

$$\begin{aligned} \delta A &= \int \delta\mathcal{L} d^3x dt, \\ \delta\mathcal{L} &= \delta\rho(\frac{1}{2}\boldsymbol{v}^2 - w(\rho, s)) - \rho T \delta s + \rho\boldsymbol{v} \cdot \delta\boldsymbol{v} - \frac{\boldsymbol{B} \cdot \delta\boldsymbol{B}}{4\pi}, \end{aligned} \tag{63}$$

Inserting equations (59,61,62) into equation (63) will yield:

$$\begin{aligned} \delta\mathcal{L} &= \boldsymbol{v} \cdot \boldsymbol{\nabla} \times (\rho\boldsymbol{\xi} \times \boldsymbol{v}) - \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B})}{4\pi} - \delta\rho(\frac{1}{2}\boldsymbol{v}^2 + w) + \rho T \boldsymbol{\nabla} s \cdot \boldsymbol{\xi} \\ &= \boldsymbol{v} \cdot \boldsymbol{\nabla} \times (\rho\boldsymbol{\xi} \times \boldsymbol{v}) - \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B})}{4\pi} + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{\xi})(\frac{1}{2}\boldsymbol{v}^2 + w) \\ &+ \rho T \boldsymbol{\nabla} s \cdot \boldsymbol{\xi}. \end{aligned} \tag{64}$$

Using the well known vector identity:

$$\boldsymbol{A} \cdot \boldsymbol{\nabla} \times (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{\nabla} \cdot ((\boldsymbol{C} \times \boldsymbol{A}) \times \boldsymbol{A}) + (\boldsymbol{C} \times \boldsymbol{A}) \cdot \boldsymbol{\nabla} \times \boldsymbol{A} \tag{65}$$

and the theorem of Gauss we can write now equation (63) in the form:

$$\begin{aligned}\delta A = \int dt \{ \oint d\boldsymbol{S} \cdot [\rho(\boldsymbol{\xi} \times \boldsymbol{v}) \times \boldsymbol{v} - \frac{(\boldsymbol{\xi} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} + (\frac{1}{2}\boldsymbol{v}^2 + w)\rho\boldsymbol{\xi}] \\ + \int d^3x \boldsymbol{\xi} \cdot [\rho\boldsymbol{v} \times \boldsymbol{\omega} + \boldsymbol{J} \times \boldsymbol{B} - \rho\boldsymbol{\nabla}(\frac{1}{2}\boldsymbol{v}^2 + w) + \rho T \boldsymbol{\nabla} s] \}.\end{aligned} \tag{66}$$

The time integration is of course redundant in the above expression. Also notice that we have used the current definition equation (6) and the vorticity definition $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$. Suppose now that $\delta A = 0$ for a $\boldsymbol{\xi}$ such that the boundary term in the above equation is null but that $\boldsymbol{\xi}$ is otherwise arbitrary, then it entails the equation:

$$\rho\boldsymbol{v} \times \boldsymbol{\omega} + \boldsymbol{J} \times \boldsymbol{B} - \rho\boldsymbol{\nabla}(\frac{1}{2}\boldsymbol{v}^2 + w) + \rho T \boldsymbol{\nabla} s = 0. \tag{67}$$

Using the well known vector identity:

$$\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{v}^2) = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) \tag{68}$$

and rearranging terms we recover the stationary Euler equation:

$$\rho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p + \boldsymbol{J} \times \boldsymbol{B}. \tag{69}$$

6 Conclusion

It is shown that stationary non-barotropic magnetohydrodynamics can be derived from a variational principle of three functions provided that the magnetic field lines lie on entropy surfaces. We emphasize that such a reduction in the degrees of freedom restricts the possibility of chaotic motion.

Possible applications include stability analysis of stationary magnetohydrodynamic configurations and its possible utilization for developing efficient numerical schemes for integrating the magnetohydrodynamic equations. It may be more efficient to incorporate the developed formalism in the frame work of an existing code instead of developing a new code from scratch. Possible existing codes are described in [19–23]. I anticipate applications of this study both to linear and non-linear stability analysis of known barotropic magnetohydrodynamic configurations [24–26]. I suspect that for achieving this we will need to add additional constants of motion constraints to the action as was done by [27,28] see also [29–31]. As for designing efficient numerical schemes for integrating the equations of fluid dynamics and magnetohydrodynamics one may follow the approach described in [32–35].

Another possible application of the variational method is in deducing new analytic solutions for the magnetohydrodynamic equations. Although the equations are notoriously difficult to solve being both partial differential equations and nonlinear, possible solutions can be found in terms of variational variables. An example for this approach is the self gravitating torus described in [36].

One can use continuous symmetries which appear in the variational Lagrangian to derive through Noether theorem new conservation laws. An example for such derivation which still lacks physical interpretation can be found in [38]. It may be that the Lagrangian derived in [10] has a larger symmetry group. And of course one anticipates a different symmetry structure for the non-barotropic case.

Topological invariants have always been informative, and there are such invariants in MHD flows. For example the two helicities have long been useful in research into the problem of hydrogen fusion, and in various astrophysical scenarios. In previous works [9,11,40] connections between helicities with symmetries of the barotropic fluid equations were made. The variables of the current variational principles are helpful for identifying and characterizing new topological invariants in MHD [41,42].

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