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On the local chaos in reaction-diffusion equations

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Abstract. There is shown that a non-dissipative reaction-diffusion partial differential equation with local nonlinearity can have an infinite number of different nonstable solutions traveling along the space axis with varying speeds, traveling impulses as well as an infinite number of different states of spatio-temporal (diffusion) chaos. These solutions are generated by cascades of bifurcations governed by the corresponding steady states. The behavior of these solutions is analyzed in detail and, as an example, it is explained how space-time chaos can arise. Questions of same type are also studied in the case of a nonlocal nonlinearity.
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1 Introduction

We are interested in stydying the chaotic behavior of solutions to reaction-diffusion type equations with nonlocal and local nonlinearities; namely, we will deal with the following problems

$$\frac{\partial u}{\partial t} - \varDelta u + g\left(t, x, u\right) = 0, \quad (t, x) \in (0, T) \times \varOmega, \tag{1}$$

$$u\left(0, x\right) = u_0\left(x\right) \in W^{1,2}\left(\varOmega\right), \quad x \in \varOmega, \quad T > 0 \tag{2}$$

$$u\Big|_{\,[0,T)\times\partial\varOmega} = 0, \quad \varOmega \subseteq R^n,\ n \geq 1, \quad \partial\varOmega \in Lip \tag{3}$$

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where $\Omega \subseteq R^n$ is an open domain, the boundary $\partial\Omega$ satisfies the Lipschitz condition, $g : L^{p_1}((0,T)\times\Omega) \longrightarrow L^{p_2}((0,T)\times\Omega)$ is a nonlinear operator and $p_1, p_2 > 1$ are fixed numbers. We assume that $g(t,x,u)$ is represented in one of the following forms:

$$(\alpha) : g(t,x,u) := a\,\|u\|_2^{\rho}\, u + h(t,x)\,, \quad or\; (\beta) : g(t,x,u) := a(t,x)\,|u|^{\rho}\, u + h(t,x)\,, \tag{4}$$

where $\|\cdot\|_2$ denotes the norm in $L^2(\Omega)$, $h(t,x)$ and $u_0(x)$ are given functions and $\rho > 0$, $a > 0$ are given constants. The problems posed above are investigated for the both cases separately: in the case of a nonlocal nonlinearity (i.e. $(4(\alpha))$), and in the case of a local nonlinearity (i.e. $(4(\beta))$).

We shall demonstrate that the partial differential equation (1) in the case of the nonlocal nonlinearity (i.e. the case $(4(\alpha))$) can possess an infinite number of different non-stable solutions. In the local case $(4(\beta))$, which differs markedly from the nonlocal case $(4(\alpha))$, the problem allows an infinite number of different both non-stable solutions, traveling along the space axis with arbitrary speeds, and traveling impulses, as well as an infinite number of different spatio-temporal (diffusion) chaotic states. These solutions are generated by cascades of static bifurcations of the evolution equation, which were studied, in particular, in [20]. As Ya. Sinai asserts in [19] "... the future of the chaos theory will be connected with new phenomena in nonlinear PDEs and other infinite-dimensional dynamical systems, where we can encounter absolutely unexpected phenomena".

The dynamics becomes much more complicated in the case of dynamical systems generated by partial differential equations (PDEs) largely due to the formation of spatially chaotic patterns. More generally, such systems may display interactions between spatially and temporally chaotic modes. One of the most challenging problems in this field is that of turbulence which displays statistical behavior in temporal and spatial directions, whose correlations decay with distance in space and time, see e.g. [9,14,25]. It should be pointed out that there have been many investigations on this and related topics (see, for example, [1,2,6,7,5,11,15–17,19,25,26] and the references therein).

In what follows we study the Cauchy problem for an equation of the non-dissipative reaction-diffusion equation type with an infinite-dimensional solution space; in particular, the corresponding steady-state problem has also infinite number of the different solutions. We show that the trajectories of solutions in the phase space depend on choosing the starting point from a sphere of the initial values. To be more precise, this choice determines how the solution behave beginning at this initial value and depending on the related Lyapunov exponent. The choice of starting point allows to determine the point at which the solution of the problem will end up. If the limiting set is not one-dimensional, more complications can arise, including even the existence of absorbing manifolds. Moreover, if such absorbing manifolds exist, their associated dynamics tends to be chaotic. We study this type of dynamic behavior and explain how space-time chaos can arise.

2 Existence in the autonomous case

2.1 The nonhomogeneous case

We begin by studying the problem in the case $(4(\alpha))$ when $g(t,x,u) := a\|u\|_2^{\rho} u + h(x)$, i.e. we consider the problem

$$\frac{\partial u}{\partial t} - \Delta u - a\|u\|_H^{\rho} u = h(x), \quad (t,x) \in (0,T)\times\Omega, \tag{5}$$

$$u(0,x) = u_0(x) \in W_0^{1,2}(\Omega) := H_0^1(\Omega), \quad u\Big|_{[0,T)\times\partial\Omega} = 0. \tag{6}$$

From (5) we compute that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 - a\|u(t)\|_2^{\rho+2} = \langle h,u\rangle, \quad \|u(0)\|_2^2 = \|u_0\|_2^2 \tag{7}$$

which entails the inequalities

$$\tfrac{d}{dt}\|u(t)\|_2^2 \le -\|\nabla u(t)\|_2^2 + 2a\|u(t)\|_2^{\rho+2} + \|h\|_{H^{-1}}^2 \le$$
$$\le -\lambda_1 ||u||_2^2) + 2a\|u(t)\|_2^{\rho+2} + \|h\|_{H^{-1}}^2,$$

where $\|u(t)\|_{H_0^1} := \|\nabla u(t)\|_2$ and $\lambda_1 > 0$ is the first eigenvalue of the Laplace operator $-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$.

Now we consider the solvability of this problem, which will be analyzed making use of the general results from [21]. We take $u_0 \in B_{r_0}^{H_0^1}(0)$, where $r_0 < \lambda_1$, and study the operator A generated by the problem: it acts, by definition, from $X := W^{1,2}\left(0,T;H^{-1}(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right) \cap \{u(t,x) \mid u(0,x) = u_0\}$ to $L^2\left(0,T;H^{-1}(\Omega)\right)$. Next, we study the image of this operator A on the ball $B_r^X(0)$ for $r \in (0,r_0)$; more precisely, we define a subset M of the space $L^2\left(0,T;H^{-1}(\Omega)\right)$ and a number $r \in (0,r_0)$, such that $A\left(B_r^X(0)\right) \subseteq M \subset L^2\left(0,T;H^{-1}(\Omega)\right)$. In other words, we shall show that the problem is solvable in $B_{r_0}^X(0)$ for any $(h,u_0) \in M \times B_{r_0}^{H_0^1}(0)$. A detailed investigation requires some preliminary estimates.

So, let $u_0 \in B_{r_0}^{H_0^1}(0)$ for some number $r_0 < \lambda_1$; then we obtain

$$\frac{d}{dt}\|u(t)\|_2^2 \le -\lambda_1\|u(t)\|_2^2 + 2a\|u(t)\|_2^{\rho+2} + \|h\|_{H^{-1}}^2.$$

Consequently, it is enough to study the following initial problem

$$d(y+c_1)/dt + \lambda_1(y+c_1) \le 2a(y+c_1)^{\rho_1+1}, \quad y(0) = \|u_0\|_{H_0^1}^2, \tag{8}$$

where $y(t) := \|u(t)\|_2^2$, $\rho_1 = \frac{\rho}{2}$. Assume that the constant c_1 is chosen in such a way that the inequality

$$2a(y+c_1)^{\rho_1+1} - \lambda_1 c_1 \ge \|h\|_{H^{-1}}^2 + 2ay^{\rho_1+1}$$

holds.

Whence, one finds that

$$z' - \lambda_1\rho_1 z \geq -a\rho, \;\; z = (y + c_1)^{-\rho_1}, \;\; z(0) = \left(\|u_0\|_2^2 + c_1\right)^{-\rho_1},$$

which gives

$$(y + c_1)^{-\rho_1} \geq \left(\|u_0\|_2^2 + c_1\right)^{-\rho_1} e^{\lambda_1\rho_1 t} + \frac{2a}{\lambda_1} - \frac{2a}{\lambda_1} e^{\lambda_1\rho_1 t} \Longrightarrow$$

$$\|u(t)\|_2^2 + c_1 \leq e^{-\lambda_1 t}\left(\|u_0\|_2^2 + c_1\right)\left[1 - \frac{2a}{\lambda_1}\left(\|u_0\|_2^2 + c_1\right)^{\rho_1}\left(1 - e^{-\lambda_1\rho_1 t}\right)\right]^{\rho_1^{-1}}. \tag{9}$$

Therefore, we see from (9) that the functions u_0 and h should be selected from balls of respective spaces so as to satisfy the inequality

$$1 - \frac{2a}{\lambda_1}\left(\|u_0\|_2^2 + c_1\right)^{\rho_1} > 0 \Longrightarrow \|u_0\|_2^2 + c_1 < \left|\frac{\lambda_1}{2a}\right|^{\frac{2}{\rho}}. \tag{10}$$

Moreover, it follows readily from (9) that $c_1 < 0$.

So, we obtained that for the solvability of the problem posed the initial data and the exterior source should be [20,21] small enough. Consequently, one can formulate the following proposition.

Proposition 1. *Let the initial data u_0 and h satisfy the inequality (10) with parameters λ_1, a, ρ and c_1 defined above hold. Then the problem (5) - (6) is solvable in the ball $B_{r_0}^X(0)$ for any $h \in M \subset H^{-1}$ where $M := \left\{h \in H^{-1} \mid \langle h, u\rangle \leq \langle A(u), u\rangle, \; u \in S_{r_0}^X(0)\right\}$ and $r_0 < \min\left\{\left(\frac{\lambda_1}{a}\right)^{\frac{2}{\rho}}; \lambda_1^{\frac{1}{\rho}}\right\}$ some number.*

Proof. We will make use of the formal solution to the problem (5) - (6):

$$u(t) = \exp\{t\Delta\} u_0 + a0t\int \exp\{(t-\tau)\Delta\}\|u(\tau)\|_H^{\rho} u(\tau)\, d\tau +$$

$$0{+}t\int \exp\{(t-\tau)\Delta\}\, d\tau \circ h\ .$$

For the evolution operator $\exp\{t\Delta\}$ one has the estimate: $\|\exp\{t\Delta\}\| \leq \exp\{-\lambda_1 t\}$ (see, for example, [13]). Thus, we obtain:

$$\|u(t)\|_H \leq \exp\{-\lambda_1 t\}\|u_0\|_H + a0t\int \exp\{-\lambda_1(t-\tau)\}\|u(\tau)\|_H^{\rho+1} d\tau +$$

$$0{+}t\int \exp\{-\lambda_1(t-\tau)\}\, d\tau \cdot \|h\|_H = e^{-\lambda_1 t}\|u_0\|_H + \frac{1 - e^{-\lambda_1 t}}{\lambda_1}\|h\|_H +$$

$$+a0t\int \exp\{-\lambda_1(t-\tau)\}\|u(\tau)\|_2^{\rho+1} d\tau,$$

where $\lambda_1 > 0$ is, as before, the first Laplace operator eigenvalue. Whence, one observes that there exists a number $r_1 = r_1(\lambda_1, \|h\|, a, \rho) < \lambda_1$ such that $\|u_0\|_2^{\rho+1} < r_1$. Then $\|u(t)\|_2$ decreases as $t \uparrow \infty$ (at least, for $t \in (0, t_1)$ where $t_1 > 0$ is some positive value), and consequently for any $t > 0$.

Now, taking account the above property of the solution in the equation

$$\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq 2a\|u(t)\|_2^{\rho+2} + \|h\|_{H^{-1}}^2,$$

it follows from (7) and (8) we obtain a priori estimates, which suffice to complete the proof of this proposition. Moreover, we have the following estimates

$$-\langle \Delta u, u\rangle - a\left\langle \|u(t)\|_2^{\rho} u, u\right\rangle = \|\nabla u(t)\|_2^2 - a\|u(t)\|_2^{\rho+2} \geq \|\nabla u(t)\|_2^2 -$$

$$\frac{a}{\lambda_1}\|u(t)\|_2^{\rho}\ \|\nabla u(t)\|_2^2 = \|\nabla u(t)\|_2^2\left(1 - \frac{a}{\lambda_1}\|u(t)\|_2^{\rho}\right) \geq \delta\|\nabla u(t)\|_2^2$$

as in this case $\|u_0\|_2^{\rho} < \frac{\lambda_1}{a}$, and consequently $\|u(t)\|_2^{\rho} < \frac{\lambda_1}{a}$ for $t > 0$. Hence, it is enough to use the existence theorem from [21] on the ball $B_{r_0}^X(0)$, where $r_0 \leq \left(\frac{\lambda_1}{a}\right)^{\frac{2}{\rho}}$ and $u \in S_{r_0}^X(0)$. Indeed, this implies following inequalities on $W^{1,2}\left(0, T; H_0^1\right)$

$$\left\langle \frac{\partial u}{\partial t} - \Delta u - a\|u\|^{\rho} u, \frac{\partial u}{\partial t} + u\right\rangle = \left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 -$$

$$\frac{a}{\rho+2}\frac{d}{dt}\|u\|^{\rho+2} + \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 - a\|u\|^{\rho+2} \Longrightarrow$$

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla u\|^2 - a\|u\|^{\rho+2} + \frac{d}{dt}\left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{a}{\rho+2}\|u\|^{\rho+2}\right] \geq$$

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla u\|^2\left(1 - \frac{a}{\lambda_1}\|u\|^{\rho}\right) + \frac{d}{dt}\left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{a}{\rho+2}\|u\|^{\rho+2}\right], \tag{11}$$

which when integrated with respect to t yield

$$0T\int\left[\left\|\frac{\partial u}{\partial t}\right\|^2 + \delta_1\|\nabla u\|^2\right]dt + \frac{1}{2}\|u\|^2(T) + \frac{1}{2}\|\nabla u\|^2(T) - \frac{a}{\rho+2}\|u\|^{\rho+2}(T) -$$

$$\frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|\nabla u_0\|^2 + \frac{a}{\rho+2}\|u_0\|^{\rho+2} \geq 0T\int\left[\left\|\frac{\partial u}{\partial t}\right\|^2 + \delta_1\|\nabla u\|^2\right]dt + \frac{1}{2}\|u\|^2(T) +$$

$$\|\nabla u\|^2(T)\left[\frac{1}{2} - \frac{a}{\lambda_1(\rho+2)}\|u\|^{\rho}(T)\right] - \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|\nabla u_0\|^2 + \frac{a}{\rho+2}\|u_0\|^{\rho+2} \geq$$

$$0T\int\left[\left\|\frac{\partial u}{\partial t}\right\|^2 + \delta_1\|\nabla u\|^2\right]dt + \frac{1}{2}\|u\|^2(T) + \delta_2\|\nabla u\|^2(T) - C\left(\|u_0\|_{H^1}, a, \rho\right).$$

Now we consider the following inequality $(u-v=w)$

$$\left\langle \frac{\partial u}{\partial t} - \Delta u - a\left\|u\right\|^{\rho} u - \frac{\partial v}{\partial t} + \Delta v + a\left\|v\right\|^{\rho} v, u-v \right\rangle =$$

$$\left\langle \frac{\partial w}{\partial t} - \Delta w - a\left(\left\|u\right\|^{\rho} u - \left\|v\right\|^{\rho} v\right), w \right\rangle = \frac{1}{2}\left\|w\right\|^{2}(t) + \left\|\nabla w\right\|^{2}(t) -$$

$$a\left\langle \left\|u\right\|^{\rho} u - \left\|v\right\|^{\rho} v, w\right\rangle \geq \frac{1}{2}\left\|w\right\|^{2}(t) + \left\|\nabla w\right\|^{2}(t) - a(\rho+1)\left\|\widehat{u}\right\|^{\rho}\left\|w\right\|^{2}(t) \quad (12)$$

where $\widehat{u} = l(u,v)$ is a bilinear mapping. Thus, we conclude that all conditions of the general theorem from [21] are fulfilled on the ball $B_{r_0}^{X}(0)$, thereby completing th proof of the proposition by virtue of the inequalities (11) and (12).

2.2 The homogeneous case

Next, we consider the homogeneous case of (5) - (6) and study the behavior of its solutions. First, we need note that in this case (i.e. $h(t,x)=0$, $a=1$) that (8) and (10) yield the following inequalities:

$$\left\|u(t)\right\|_2^{-\rho} \geq \frac{2a}{\lambda_1} + 2\left[\left\|u_0\right\|_2^{-\rho} - \frac{a}{\lambda_1}\right] e^{\lambda_1 \frac{\rho}{2} t},$$

or, at $a=1$,

$$\left\|u(t)\right\|_2 \leq \left[\frac{2}{\lambda_1} + 2\left(\left\|u_0\right\|_2^{-\rho} - \frac{1}{\lambda_1}\right) e^{\lambda_1 \frac{\rho}{2} t}\right]^{-\rho}.$$

Therefore, $\frac{2}{\lambda_1} + 2\left(\left\|u_0\right\|_2^{-\rho} - \frac{1}{\lambda_1}\right) e^{\lambda_1 \frac{\rho}{2} t} > 0$ for any $t>0$, for which the the inequality $\left\|u_0\right\|_2^{\rho} \leq \lambda_1$ is sufficient.

We now set $h(t,x)=0$, $a=1$ and investigate the problem (5) - (6) for the initial data satisfying the condition $u_0 \in B_{r_0}^{H_0^1(\Omega)}(0) \subset H_0^1(\Omega)$ if $r_0^{\rho} < \lambda_1$. In this case we have

$$\frac{d}{dt}\left\|u(t)\right\|_2^2 + 2\left\|\nabla u(t)\right\|_2^2 - 2r^{\rho}(t)\left\|u(t)\right\|_2^2 = 0,$$

which gives rise to the differential inequality

$$\frac{d}{dt}\left\|u(t)\right\|_2^2 \leq -2\left(\lambda_1 - r^{\rho}(t)\right)\left\|u(t)\right\|_2^2,$$

equivalent to

$$\left\|u(t)\right\|_2^2 \leq \exp\left\{-2\left(t0\int\left(\lambda_1 - r^{\rho}(\tau)\right)d\tau\right)\right\}\left\|u(0)\right\|_2^2.$$

Under the conditions imposed above there exists, owing to the continuity of the function $u(t)$, an interval $(0,t')$, $(t'>0)$ such that $\lambda_1 - r^{\rho}(t) > 0$ for $t \in (0,t')$. Thus, one easily computes that

$$\left\|u(t)\right\|_2^2 < \exp\left\{-2\left(\lambda_1 - r_0^{\rho}\right)t\right\}\left\|u_0\right\|_2^2 < \left\|u_0\right\|_2^2 < r_0^2 \quad (13)$$

for any $t>0$. As in the previous case, it is easy to prove the following result.

Proposition 2. *Let $h(t,x)=0$ and $a=1$. Then for any $u_0 \in B_{r_0}^{H_0^1(\Omega)}(0) \subset H_0^1(\Omega)$ the problem (5) - (6) is solvable for any $t>0$ if $r_0^{\rho}<\lambda_1$. Moreover the mapping (semi-flow) $F(t): u_0 \longrightarrow u(t)$ is such that L^2 strongly $F(t)\left(B_{r_0}^{H_0^1(\Omega)}(0)\right) \longrightarrow 0$ as $t \uparrow \infty$.*

3 Longtime behavior of solutions

Let $h \in L^2\left((0,\infty);H^{-1}(\Omega)\right)$ and $u_0 \in H_0^1(\Omega)$ with the norm $\|\circ\|_{H_0^1(\Omega)}^2 := \|\nabla\circ\|_2^2$. We assume that the Laplace operator $-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ has only a point spectrum, i.e. $\sigma(-\Delta) := \sigma_P(-\Delta) \subset (0,\infty)$. Denote the eigenvalues of the Laplace operator $-\Delta$ by λ_j, $j=1,2,...$ ($\sigma_P(-\Delta) := \{\lambda_j \mid j \in N\}$). This, of course, requires that the domain Ω is sufficiently regular in a geometric sense.

Now we consider the problem (5) - (7) in the case $g(t,x,u) := \|u\|_2^{\rho} u + h(t,x)$, and investigate the inverse mapping of the operator $f(t): B_{r_0}^{H_0^1(\Omega)}(0) \subset H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$, where $f(t)u := \partial u/\partial t - \Delta u - \|u\|_2^{\rho} u$ and $r_0>0$ is some positive number.

For simplicity, we assume that the eigenfunctions and adjoint eigenfunctions are total in the space $H_0^1(\Omega)$; moreover, we may assume without loss of generality that they generate an orthogonal basis in this space.

Let $\inf\{\lambda_j \in \sigma_P(-\Delta): \ \lambda_j > r_0^{\rho},\ j=1,2,...\} = \lambda_{k_0}$. Then, we can represent [12,13,18,20,24] the space $H_0^1(\Omega)$ in the form $H_0^1(\Omega) := H_{k_0} \oplus H_{-k_0}$, where the subspace $H_{k_0} \subset H_0^1(\Omega)$ is related to $\{\lambda_j\}_{j=1}^{k_0-1}$ and has dimension $\dim H_{k_0} = k_0-1$ and H_{-k_0} is a subspace of $\operatorname{codim} H_{-k_0} = k_0-1$. We can now introduce the projections Q_{k_0} and P_{k_0}; $Q_{k_0}: H_0^1(\Omega) \longrightarrow H_{-k_0} \subset H_0^1(\Omega)$ and $P_{k_0}: H_0^1(\Omega) \longrightarrow H_{k_0} \subset H_0^1(\Omega)$, giving rise to the splitting $u := Q_{k_0}u + P_{k_0}u$. (It is well known that such a decomposition allows to introduce either a spectral measure or a family of spectral projections (see, for example, [10,**?**], etc.)

Thus, it is easy to see that $-\Delta : H_{k_0} \longrightarrow H_{k_0}^-$ and $-\Delta : H_{-k_0} \longrightarrow H_{-k_0}^-$, where the subspaces $H_{k_0}^-, H_{-k_0}^-$ possess bi-orthogonal bases (see, for instance, [10,**?**,**?**]), and owing to the evident commutativity of operators P_{k_0} and Q_{k_0} with the Laplacian Δ in $H_0^1(\Omega)$, one can rewrite the problem as

$$\frac{\partial}{\partial t}P_{k_0}u - \Delta P_{k_0}u - \|u\|_2^{\rho} P_{k_0}u = P_{k_0}^* h(t,x), \tag{14}$$

$$\frac{\partial}{\partial t}Q_{k_0}u - \Delta Q_{k_0}u - \|u\|_2^{\rho} Q_{k_0}u = Q_{k_0}^* h(t,x), \tag{15}$$

$$P_{k_0}u(0,x) = P_{k_0}u_0(x) \in H_{k_0} \subset H_0^1(\Omega), \tag{16}$$

$$Q_{k_0}u(0,x) = Q_{k_0}u_0(x) \in H_{-k_0} \subset H_0^1(\Omega), \tag{17}$$

with imposed above the condition (3), where $P_{k_0}^*$ and $Q_{k_0}^*$ are the adjoint operators to P_{k_0} and Q_{k_0}, respectively.

For the investigation of the problem above we shall consider the following differential-functional expression

$$E(u) := \frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u\|_2^2(t) - \|u\|_2^{\rho}(t)\cdot\|u\|_2^2(t),$$

for $u \in H^1((0,T)\times\Omega)$.

As our aim is the investigation of the behavior of solutions of the problem under the condition $u_0 \in B_{r_0}^{H_0^1(\Omega)}(0)$, $0 < r_0^\rho < \lambda_{k_0}$ it is enough to study the homogeneous equation case. Therefore, we assume $h(t,x) = 0$. Then for the problem (14) - (15) we obtain

$$E(P_{k_0}u) = \frac{1}{2}\frac{d}{dt}\|P_{k_0}u\|_2^2(t) + \|\nabla P_{k_0}u\|_2^2(t) - \left(\|u\|_2^\rho \|P_{k_0}u\|_2^2\right)(t) \ = 0, \quad (18)$$

$$\langle P_{k_0}u, P_{k_0}u\rangle \mid_{\ t=0} \ = \|P_{k_0}u\|_2^2(0) = \|P_{k_0}u_0\|_2^2 . \quad (19)$$

Whence, it follows that for some $t_0 > 0$ for $t \in [0,t_0)$ we have $\|u\|_2^\rho(t) \le r_0^\rho + \theta < \lambda_{k_0}$, for some $\theta > 0$. Indeed if $\|u_0\|_2 = r_0$, then we have from (18) - (19) that

$$\frac{d}{dt}\|P_{k_0}u\|_2^2(t) + 2(\lambda_{k_0-1} - r^\rho(t))\|P_{k_0}u\|_2^2(t) \ge$$

$$\frac{d}{dt}\|P_{k_0}u\|_2^2(t) + 2(\lambda_{k_0-1} - r^\rho(0))\|P_{k_0}u\|_2^2(t) \ge 0$$

and consequently we obtain the inequality

$$\|P_{k_0}u\|_2^2(t) \ge \exp\{-2(\lambda_{k_0-1} - r_0^\rho)t\}\|P_{k_0}u_0\|_2^2 . \quad (20)$$

Thus, we see that if $\|Q_{k_0}u\|_2(t) \le \delta < \theta < r_0$ for some sufficiently small $\delta > 0$ and $t \in [0,t_0)$, then the solution of problem (18) - (19) exists and is an exponentially increasing function.

Now consider the problem (16) - (17) for which one easily obtains

$$E(Q_{k_0}u) = \frac{1}{2}\frac{d}{dt}\|Q_{k_0}u\|_2^2(t) + \|\nabla Q_{k_0}u\|_2^2(t) - \left(\|u\|_2^\rho \|Q_{k_0}u\|_2^2\right)(t) \ = 0, \quad (21)$$

$$\langle Q_{k_0}u, Q_{k_0}u\rangle \mid_{\ t=0} \ = \|Q_{k_0}u\|_2^2(0) = \|Q_{k_0}u_0\|_2^2 .$$

Therefore, the solution of problem (21) exists and is an exponentially decreasing function. Consequently for $\|Q_{k_0}u_0\|_2 + \|P_{k_0}u_0\|_2 = r_0$ if $\|Q_{k_0}u_0\|_2 < \|P_{k_0}u_0\|_2$ and $\|Q_{k_0}u_0\|_2$ is sufficiently small, then the solution $\|u\|_2(t)$ exists and is an increasing function up to some $t_1 > 0$.

To study in detail the behavior of solutions to the problem we will make use of the following assumption: the system of eigenfunctions $\{w_k\}_{k=1}^\infty \subset H_0^1(\Omega)$ comprises an orthonormal basis of this space. Then each function $u(t,x) \in L^2\left((0,T);H_0^1(\Omega)\right)$ has the representation $u(t,x) = k = 1\infty \sum u_k(t)\, w_k(x)$. Consequently, owing to (14) - (15), the problem is equivalent to studying the system of equations

$$\frac{1}{2}\frac{d}{dt}|u_k(t)|^2 + \lambda_k |u_k(t)|^2 - \left(i = 1\infty \sum |u_i(t)|^2\right)^{\frac{\rho}{2}} |u_k(t)|^2 = 0 \quad (22)$$

with the Cauchy data

$$|u_k(0)|^2 = |u_{0k}|^2, \quad k = 1,2,...,k_0.$$

Let $u(0,x) \in B_{r_0}^{H_0^1(\Omega)}(0)$ and $\|u_0\|_2 < r_0$,
then we have $\|u\|_2^2(t) := \left(i = 1\infty\sum |u_i(t)|^2\right) \leq r_0^2 + \varepsilon$ for sufficiently small $t = t(\varepsilon, r_0) > 0$. But $|u_k(t)|^2$ will increase in this case for $k = 1, 2, ..., \widetilde{k}_0 \leq k_0$ depending on the relationship between $\|u_0\|_2^\rho$ and λ_k (therefore, between r_0 and λ_k).

Consider the behavior of $|u_k(t)|$ for all $k = 1, 2, ...$ Define $\|u_0\|_2 := r_0$ for some $r_0 > 0$. Let us list all of possible cases: 1) $r_0^\rho < \lambda_1$, 2) $\exists \lambda_{k_0} : \lambda_{k_0-1} < r_0^\rho < \lambda_{k_0}$ and 3) $\exists \lambda_{k_0} : r_0^\rho = \lambda_{k_0}$. The case 1) was already investigated, therefore we will consider here only cases 2) and 3).

Consider either the case 2) or 3), i.e. $\exists \lambda_{k_0} : \lambda_{k_0-1} < r_0^\rho < \lambda_{k_0}$ and $\exists \lambda_{k_0} : r_0^\rho = \lambda_{k_0}$. We have the following system of equations

$$0 = \frac{1}{2}\frac{d}{dt}|u_k(t)|^2 + \lambda_k |u_k(t)|^2 - r(t)^\rho |u_k(t)|^2 =$$

$$= \frac{1}{2}\frac{d}{dt}|u_k(t)|^2 + (\lambda_k - r(t)^\rho)|u_k(t)|^2, \ |u_k(0)|^2 = |u_{0k}|^2, k = 1, 2, ..., \quad (23)$$

where $u(t,x) := k = 1\infty\sum u_k(t) w_k(x)$. It is easy to see that this system of equations is of interest only for the cases $k \geq k_0$, $k \leq k_0 - 1$ and $k = k_0$ separately. In case 2) if $k \geq k_0$ this part of the system has a solution that is unique as $t \leq t_2(k, r_0)$ for some $t_2(k, r_0) > 0$. Formally, we can determine the solution of each equation from (21) to be

$$|u_k(t)|^2 = \exp\left\{-2\left(\lambda_k t - t0\int r(\tau)^\rho d\tau\right)\right\}|u_{0k}|^2 \ . \quad (24)$$

Thus, if we consider the expression (21) for $k \leq k_0 - 1$ in the case 2), it follows from (22) that

$$|u_k(t)|^2 = \exp\left\{-2\left(\lambda_k t - t0\int r(\tau)^\rho d\tau\right)\right\}|u_0|^2 \geq \exp\left\{2\left(r(0)^\rho - \lambda_k\right)t\right\}|u_0|^2,$$

as $r(t)^\rho > \lambda_k$ for $1 \leq k \leq k_0 - 1$ and some $t > 0$. Consequently, the sequence $|u_k(t)|$ increases for each $k : 1 \leq k \leq k_0 - 1$ leading to the the increase of $r(t)$ as long as $\|P_{k_0} u_0\|_2$ is sufficiently greater than $\|Q_{k_0} u_0\|_2$.

For case 3) for some $k = k_0$ one has $r_0^\rho = \lambda_{k_0}$ for which

$$|u_{k_0}(t)|^2 = \exp\left\{-2t0\int (\lambda_{k_0} - r(\tau)^\rho) d\tau\right\}|u_{0k_0}|^2$$

by virtue of (22). Here the function $r_1(t) := \lambda_{k_0} - r(t)^\rho$ equals zero at $t = 0$, but in general its variation is not known. Thus, it is impossible to obtain a monotonicity result for a solution to this equation, since the behavior of $r(t)$ is not known. As we shall explain further in the sequel, the behavior of $r(t)$ depends on the geometrical properties of the initial data u_0 of sphere $S_r^{H_0^1}(0)$, $0 < r \leq r_0$.

From the previously mentioned relationships it is clear that in order to investigate the behavior of the parameter $r(t)$ one should study both $\|P_{k_0} u\|_2$

and $\|Q_{k_0}u\|_2$. It is easy to see that if the condition 2) is assumed, then $\|P_{k_0}u\|_2$ increases, and $\|Q_{k_0}u\|_2$ decreases as $t>0$ at least nearby zero in virtue of (18) and (22). Using the orthogonal splitting $u=P_{k_0}u+Q_{k_0}u$, we also find that

$$\|u\|_2^2=\|P_{k_0}u\|_2^2+\|Q_{k_0}u\|_2^2 . \tag{25}$$

Thus, the behavior of the functional $\|u\|_2^2(t)$ depends on the relationship between the values $\|P_{k_0}u_0\|_2$ and $\|Q_{k_0}u_0\|_2$. Let $\|u_0\|_2:=r_0$, $\lambda_{k_0-1}<r_0^\rho<\lambda_{k_0}$ and consider (25) , i.e.

$$\|u\|_2^2(t)=\|P_{k_0}u\|_2^2(t)+\|Q_{k_0}u\|_2^2(t)=k\le k_0\sum|u_k(t)|^2+k>k_0\sum|u_k(t)|^2 . \tag{26}$$

It is now necessary to investigate the following three cases:
a) $u_0:=k\le k_0-1\sum u_{0k}w_k\in P_{k_0}\left(H_0^1(\Omega)\right):=H_{k_0}$; *b)* $u_0:=k\ge k_0\sum u_{0k}w_k\in Q_{k_0}\left(H_0^1(\Omega)\right):=H_{-k_0}$ and *c)* $u_0:=k\ge 1\sum u_{0k}w_k$, when c_1) $\|Q_{k_0}u_0\|_2<\|P_{k_0}u_{02}\|_2$ and c_2) $\|Q_{k_0}u_0\|_2\ge\|P_{k_0}u_0\|_2$, separately.

Consider *a*): in this case we have
$|u_k(t)|^2=\exp\left\{-2\left(\lambda_kt-t0\int r^\rho(\tau)\,d\tau\right)\right\}|u_{0k}|^2$ for any $k=1,...,k_0-1$, thus , $u_k(t)=0$ for $k\ge k_0$ since $r_0^2=r(0)^2:=k\le k_0-1\sum(u_{0k})^2$ and $r(t)^2:=k\le k_0-1\sum|u_k(t)|^2$.
On the other hand, in this case $\left(\lambda_kt-t0\int r^\rho(\tau)\,d\tau\right)<0$ as $r_0^\rho>\lambda_k$ for each $k=1,...,k_0-1$ and $r^\rho(t)$ increases as $t\uparrow\infty$.

Now let us consider case b), i.e. $r_0^2=r(0)^2:=k\ge k_0\sum(u_{0k})^2$.
Then

$$|u_k(t)|^2=\exp\left\{-2\left(\lambda_kt-t0\int r^\rho(\tau)\,d\tau\right)\right\}|u_{0k}|^2$$

for any $k\ge k_0$,
giving rise to $u_k(t)=0$ for $k=1,2,...,k_0-1$, since $\left(\lambda_kt-t0\int r(\tau)^\rho d\tau\right)>0$ as $r_0^\rho<\lambda_k$ for each $k\ge k_0$ and $r^\rho(t)$ decreases as $t\uparrow\infty$. It follows from continuity that $\|u\|_2(t)=r(t)<r_0$ for all $t>0$ and $k\ge k_0$ for any solution to the problem

$$\frac{1}{2}\frac{d}{dt}|u_k(t)|^2+(\lambda_k-r(\tau)^\rho)|u_k(t)|^2=0,\ |u_k(0)|^2=|u_{0k}|^2 .$$

The above results show that for the cases $\lambda_{k_0}>r^\rho(0)>\lambda_{k_0-1}$ and c) we need to consider the space decomposition $H:=H_{k_0}\oplus H_{-k_0}=H_0^1(\Omega)$ $(P_{k_0}(H):=H_{k_0},\ Q_{k_0}(H):=H_{-k_0})$, by means of which one analyze the intrinsic behavior of the corresponding solutions. Let the space $H:=H_0^1(\Omega)$ be representable, if $r_0^2:=\left\|u_{0k_0}^-\right\|_{H_{k_0}}^2+\left\|u_{0k_0}^+\right\|_{H_{-k_0}}^2$, in the vector form

$$H:=\left\{u=\left(u_{k_0}^+,u_{k_0}^-\right):\ u_{k_0}^+\in H_{k_0},u_{k_0}^-\in H_{-k_0}\right\}.$$

Then the following proposition follows directly from the above discussion.

Proposition 3. *Under the above conditions if* $\left\|u_{0k_0}^-\right\|_{H_{-k_0}}^2>\left\|u_{0k_0}^+\right\|_{H_{k_0}}^2$ *and if the rate of decrease of the norm* $\left\|u_{k_0}^-(t)\right\|_{H_{-k_0}}^2$ *is greater than the rate of* *decrease of the norm* $\left\|u_{k_0}^{+}(t)\right\|_{H_{k_0}}^{2}$, *then there exists* $\widetilde{t}>0$ *such that* $|u_k(t)|$ *decreases for* $t\geq\widetilde{t}$. *On the other hand, if* $\left\|u_{0k_0}^{-}\right\|_{H_{-k_0}}^{2}<\left\|u_{0k_0}^{+}\right\|_{H_{k_0}}^{2}$ *and the rate of increase rate of the norm* $\left\|u_{k_0}^{+}(t)\right\|_{H_{k_0}}^{2}$ *is greater than the rate of decrease of* $\left\|u_{k_0}^{-}(t)\right\|_{H_{-k_0}}^{2}$, *then there exists* $\bar{t}>0$ *such that* $|u_k(t)|$ *increases for* $t\geq\bar{t}$.

Now we will proceed to investigation of the behavior of solutions that start at a fixed function with the norm fixed by the number $r_0:\lambda_{k_0-1}<r_0^{\rho}<\lambda_{k_0}$. As $H=H_{k_0}\oplus H_{-k_0}$ and for each $u\in H$ there holds the decomposition $u(t)=u_{k_0}^{+}(t)+u_{k_0}^{-}(t)$ for any $t>0$, then it is enough to study the case when $\left\|u_{0k_0}^{-}\right\|_{H_{-k_0}}\gg\left\|u_{0k_0}^{+}\right\|_{H_{k_0}}$. Indeed if we set $\|u_0\|=r_0$ and r_0 satisfies the inequality $\lambda_{k_0-1}<r_0^{\rho}<\lambda_{k_0}$, then each of solutions to the problem (14) - (15) satisfies one of the following statements:

1. If u_0 lies in H_{k_0} or in a small neighborhood of the subspace H_{k_0}, then $|u_k(t)|\uparrow\infty$ as $t\uparrow\infty$ for $k=\overline{1,k_0-1}$, moreover since in this case $r(t)$ increases gradually and so in time is greater than each λ_k for $k\geq k_0$, i.e. in this case $|u_k(t)|$ gradually increases for all k;
2. If u_0 lies in H_{-k_0} or in a small neighborhood of the subspace H_{-k_0}, then $|u_k(t)|\downarrow 0$ as $t\uparrow\infty$ for $k\geq k_0$, moreover since in this case $r(t)$ decreases gradually so that in time it is less than each λ_k, i.e. in this case $|u_k(t)|$ gradually decreases for all k;
3. If $u_0\in H$ such that $\left\|u_{0k_0}^{-}\right\|_{H_{-k_0}}\approx\left\|u_{0k_0}^{+}\right\|_{H_{k_0}}$, then there is a relation between of $u_{0k_0}^{-}(t)$ and $u_{0k_0}^{+}$ such that the behavior of the $u_k(t)$ is chaotic for all k for which $u_k(0)\neq 0$.
4. If $u_0\in H$ such that $\left\|u_{0k_0}^{-}\right\|_{H_{-k_0}}$ and $\left\|u_{0k_0}^{+}\right\|_{H_{k_0}}$ are different, then the solutions are still connected by some relations.

As the Claims 1. and 2. were proved above, we need only study Claim 3. Consider the representation of the formal solutions (22) to the problem (21):

$$|u_k(t)|^2=\exp\left\{-2\left(\lambda_k t-t0\int r(\tau)^{\rho}\,d\tau\right)\right\}|u_{0k}|^2,\quad k=1,2,....$$

It is known that $|u_k(t)|$ increases for $k:1\leq k\leq k_0-1$ and decreases for $k\geq k_0$ by virtue of Proposition 3, depending on the difference $\lambda_k-r(0)^{\rho}$. But as $r^2(t):=\left\|u_{k_0}^{-}\right\|_{H_{k_0}}^{2}(t)+\left\|u_{k_0}^{+}\right\|_{H_{-k_0}}^{2}(t)$ holds in a vicinity $t=0$, $r(t)$ can change depending on the behavior of $|u_k(t)|$ as $k\geq k_0$ and $k\leq k_0-1$ vary; consequently, the corresponding subspaces of H change, i.e. $r(t)^{\rho}$ can become greater than λ_{k_0} or less than λ_{k_0-1}. This variation of $r(t)$ is very complicated, as the variation depends on relations among the behaviors of $u_k(t)$ in the case when $k\geq k_0$ and $k\leq k_0-1$, which may give rise to chaos.

We now investigate Claim 4 with an eye toward the question of whether or not there is an attractor for the operator resolving the problem (14) - (15). Toward this end, we will consider the following system of differential equations

$$E_k(u):=\frac{d}{dt}\langle u(t),w_k\rangle+\langle\nabla u(t),\nabla w_k\rangle(t)-\|u\|_2^{\rho}(t)\langle u(t),w_k\rangle=\langle h,w_k\rangle,$$

where $\{w_k(x)\}_{k=1}^{\infty}$ are eigenfunctions of the Laplacian $-\Delta$ in $H_0^1(\Omega)$ corresponding to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, respectively, by virtue of the imposed conditions.

Whence, it follows that

$$E_k(u) := \frac{d}{dt}u_k(t) + \lambda_k u_k(t) - \|u\|_2^{\rho}(t)\, u_k(t) = h_k(t), \quad k = 1, 2, ...$$

As our aim is the investigation of the behavior of solutions of the problem under the condition $u_0 \in S_{r_0}^{H_0^1(\Omega)}(0)$, $\lambda_{k_0-1} < r_0^{\rho} < \lambda_{k_0}$ it is enough to study the homogeneous equation case. So we assume $h(t,x) = 0$. Then for the problem (14) - (15) we obtain the following problem Cauchy

$$E_k(u) = \frac{d}{dt}u_k(t) + \lambda_k u_k(t) - \|u\|_2^{\rho}(t)\, u_k(t) \ = 0, \tag{27}$$

$$\langle u(t), w_k\rangle \mid_{\ t=0} \ = u_k(t) \mid_{\ t=0} \ = u_{0k}, \quad k = 1, 2, ..., k_0 - 1 \tag{28}$$

Whence, for some $t_0 > 0$ for $t \in [0, t_0)$ we have $\|u\|_2^{\rho}(t) \le r_0^{\rho} + \varepsilon < \lambda_{k_0}$, for some $\varepsilon > 0$. Indeed, we have from (18) - (19) that

$$\frac{d}{dt}u_k(t) + (\lambda_k - \|u\|_2^{\rho}(t))u_k(t) = 0, \quad u_k(0) = u_{0k},$$

which leads to the formal solution of the Cauchy problem

$$u_k(t) = \exp\left\{-0t\int(\lambda_k - r^{\rho}(\tau))d\tau\right\}u_{0k} = \exp\left\{-\lambda_k t + 0t\int r^{\rho}(\tau)\, d\tau\right\}u_{0k} \tag{29}$$

Hence, if $\lambda_{k_0-1} < r_0^{\rho}$, then $u_k(t,x)$ increases in the vicinity of zero if $u_{0k}(x) > 0$ for $k = 1, 2, ..., k_0 - 1$ and the part $\|P_{k_0}u_0\|_2$ is sufficiently greater than $\|Q_{k_0}u_0\|_2$.

Let $u_0 \in H_0^1(\Omega)$ and $\|u_0\|_2 = r_0$, then the above expression implies that the behavior of the solution $u_k(t)$ depends on the relationship between r_0 and λ_k and also between $\|P_{k_0}u_0\|_2$ and $\|Q_{k_0}u_0\|_2$.

Consider the behavior of $|u_k(t)|$ for all $k = 1, 2, ...,$ in the case when $\exists\lambda_{k_0}$: $\lambda_{k_0-1} < r_0^{\rho} < \lambda_{k_0}$. We have the following system of equations

$$0 = \frac{d}{dt}u_k(t) + \lambda_k u_k(t) - r(t)^{\rho}\, u_k(t) =$$

$$= \frac{d}{dt}u_k(t) + (\lambda_k - r(t)^{\rho})\, u_k(t), \ u_k(0) = u_{0k}, k = 1, 2, ..., \tag{30}$$

where $u(t,x) := k = 1\infty\sum u_k(t)\, w_k(x)$. It is easy to see that this system of equations is of interest only for the cases $k \ge k_0$, $k \le k_0 - 1$ and $k = k_0$ separately. If $k \ge k_0$ this part of the system has a solution that is unique as $t \le t_2(k, r_0)$ for some $t_2(k, r_0) > 0$ if $\|Q_{k_0}u_0\|_2$ is sufficiently greater than $\|P_{k_0}u_0\|_2$. Formally we can determine the solution of each equation from (21) in the following form:

$$u_k(t) = \exp\left\{-\left(\lambda_k t - t0\int r(\tau)^{\rho}\, d\tau\right)\right\}u_{0k}\ . \tag{31}$$

Thus, considering the expression (21) for $k \leq k_0 - 1$, it follows from (22) that

$$|u_k(t)| = \exp\left\{-\left(\lambda_k t - t0\int r(\tau)^\rho \, d\tau\right)\right\}|u_{0k}| \geq \exp\{(r(0)^\rho - \lambda_k)t\}|u_{0k}|,$$

as $r(t)^\rho > \lambda_k$ for $1 \leq k \leq k_0 - 1$ and some $t > 0$. Consequently, the sequence $|u_k(t)|$ increases for each $k : 1 \leq k \leq k_0 - 1$, if the part $\|P_{k_0}u_0\|_2$ is sufficiently greater than $\|Q_{k_0}u_0\|_2$, that renders the increase of $r(t)$.

Consider the representation of the formal solutions (22) to the problem (21):

$$u_k(t) = \exp\left\{-\left(\lambda_k t - t0\int r(\tau)^\rho \, d\tau\right)\right\}u_{0k}, \quad k = 1, 2, ... \tag{32}$$

It is known that $|u_k(t)|$ increases for $k : 1 \leq k \leq k_0 - 1$ and decreases for $k \geq k_0$ by virtue of Proposition 3, depending on the difference $\lambda_k - r(0)^\rho$ and the relation between $\|P_{k_0}u_0\|_2$ and $\|Q_{k_0}u_0\|_2$. Thus, concerning the system (22) for k we need to study the expression $\lambda_k - r(0)^\rho$ which is negative owing to the conditions imposed. But the behavior of functions $|u_k(t)|$ cannot exactly explain the behavior of functions $u_k(t)$, and also the behavior of the solution $u(t, x)$. Consequently, we need to study the behavior of functions $u_k(t)$ in greater detail.

So, from the expression (32) under the corresponding relation between $\|P_{k_0}u_0\|_2$ and $\|Q_{k_0}u_0\|_2$ is clear that if $u_{0k} \geq 0$ ($u_{0k} \leq 0$) then $u_k(t) \geq 0$ ($u_k(t) \leq 0$) and in addition if $\lambda_k - r(0)^\rho > 0$ then in the case $u_{0k} \geq 0$ we see that $u_k(t)$ decreases, in the case $u_{0k} < 0$ we see that $u_k(t)$ increases, but $|u_k(t)|$ will decreases at least in some vicinity of zero. And next let $\lambda_k - r(0)^\rho < 0$, then when $u_{0k} \geq 0$ we see that $u_k(t)$ increases in the case $u_{0k} < 0$ we see that $u_k(t)$ decreases, but $|u_k(t)|$ increases at least in a vicinity of zero. If $\lambda_k - r(0)^\rho = 0$ then $u_k(t)$ does not vary at least in some vicinity of zero.

Thus, we obviously need to investigate the behavior of $r(t)$ for various initial function $u_0(x)$ in the case when $\|u_0\| = r_0$. So as $\lambda_k - r_0^\rho > 0$, the functions $|u_k(t)|$ decreases and converge to zero when $t \nearrow \infty$ for $k \geq k_0$, where $k_0 \geq 1$ is such that $\lambda_{k_0} - r_0^\rho > 0$ and $\lambda_{k_0-1} - r_0^\rho \leq 0$. Therefore, the behavior of $r(t)$ essentially depends on the selections of $u_k(0)$ for $1 \leq k \leq k_0 - 1$.

It is clear from the above analysis that need to consider the expression $u(t, x) = k \geq 1\sum u_k(t) w_k(x)$ for the solution and the expression $u_0(x) = k \geq 1\sum u_{0k} w_k(x)$ for initial data. Let $r_0 > 0$, then there exists $k_0 \geq 2$ such that $\lambda_{k_0} > r_0^\rho \equiv r(0)^\rho > \lambda_{k_0-1}$ and $\|u_0\| = r_0$.

Using the orthogonal splitting $u(t) = P_{k_0}u(t) + Q_{k_0}u(t)$, we obtain the following expressions:

$$u_0(x) = k \geq 1\sum u_{0k} w_k(x) = k_0 - 1 \geq k \geq 1\sum u_{0k} w_k(x) + k \geq k_0\sum u_{0k} w_k(x)$$

and

$$u(t, x) = k \geq 1\sum u_k(t) w_k(x) =$$

$$= k_0 - 1 \geq k \geq 1\sum u_k(t) w_k(x) + k \geq k_0\sum u_k(t) w_k(x). \tag{33}$$

There exist $t_1 > 0$ and $t_2 > 0$ such that $P_{k_0}u(t)$ of (33) can increases in $(0, t_1)$ and $Q_{k_0}u(t)$ of (33) decreases in $(0, t_2)$ if any of their terms are nonnegative functions. Let $\min\{t_1, t_2\} = t_1$, then when $t > t_1$ these summands can behave quite differently. Here are the possibilities: 1) the velocity $\|u\|(t)$ becomes greater than r_0 for $t \geq t_1$, moreover $r(t)^\rho > \lambda_{k_0}$ for $t > t_1$, so the orthogonal splitting $u = P_{k_0}u + Q_{k_0}u$ changes and becomes, at least, $u = P_{k_0+1}u + Q_{k_0-1}u$; 2) $Q_{k_0}u$ decreases of to a point where $\|u\|(t)$ is smaller than r_0 for $t \geq t_1$, moreover $r(t)^\rho < \lambda_{k_0-1}$ for $t > t_1$, so the orthogonal splitting $u = P_{k_0}u + Q_{k_0}u$ changes and becomes, at least, $u = P_{k_0-1}u + Q_{k_0+1}u$; 3) there exist a $t_0 \geq t_1$ and an $R_0 \geq \|P_{k_0}u\| \geq R_1 > 0$ such that beginning at t_0 the changes of $P_{k_0}u$ and $Q_{k_0}u$ become such that

$$r^2(t) = \|u(t)\|^2 = k_0 - 1 \geq k \geq 1 \sum |u_k(t)|^2 + k \geq k_0 \sum |u_k(t)|^2$$

satisfies $R_1 \leq r(t) \leq R_0$ for $t \geq t_0$.

Consider the case 1). In this case we have the following possibilities: *a)* $P_{k_0}u$ increases with such velocity that $\|u\|(t) \nearrow \infty$, which can takes place when $u_0(x)$ is chosen in the vicinity of the subspace H_{k_0} (this scenario is studied in Proposition 3); *b)* rate of growth of $P_{k_0}u$ diminishes beginning at time t and the function $u(t,x)$ behaves as in case 3, which we will explain in what follows. The case 2) has have 2 variants: *a')* $Q_{k_0}u$ decreases with such velocity that $\|u\|(t) \searrow 0$ leading to the inequality $r(t)^\rho < \lambda_1$, which can take place when $u_0(x)$ is chosen in the vicinity of the subspace H_{-k_0} (this variant is also studied in Proposition 3); *b')* rate of decrease of $Q_{k_0}u$ diminishes beginning at some time t and leading to case 1)b).

Consequently, it remains only to investigate case 3). It is clear that this case can occur when $P_{k_0}u_0$ assumes both positive and negative values. Therefore, we consider special initial datum and try to explain case 3 for such functions. So, let $P_{k_0}u_0 = u_{0k_0-1}w_{k_0-1}$, i.e. $u_0(x) = u_{0k_0-1}w_{k_0-1}(x) + Q_{k_0}u_0(x)$ and $\|u_0\| = r_0$. Then we obtain the following: $u_{k_0-1}(t)$ changes with such way that $|u_{k_0-1}(t)|$ increases with t and $|u_{k_0-1}(t)|^\rho \longrightarrow \lambda_{k_0-1}$ when $t \nearrow \infty$; moreover, $\|Q_{k_0}u(t)\|$ decreases with increasing t and therefore $\|Q_{k_0}u(t)\| \longrightarrow 0$ when $t \nearrow \infty$. Hence, $\|u\|^\rho(t) \searrow \lambda_{k_0-1}$ as $t \nearrow \infty$. In other words, the increase of $\|P_{k_0}u\|$ and decrease of $\|Q_{k_0}u(t)\|$ compensate for each other in such a way that this process leads to the case described above.

Thus, it not is difficult to see that in order to obtain the above result, we need to select u_{0k_0-1} in the vicinity of the subspace H_{-k_0}, which depends on the given $r_0 : r_0^\rho \in (\lambda_{k_0-1}, \lambda_{k_0})$. Accordingly it follows in the case when $P_{k_0}u_0$ increases, the corresponding u_{0k}, $1 \leq k \leq k_0 - 1$, must be chosen as done previously. Moreover, in this case there is a λ_{j_0} such that $\|P_{k_0}u(t)\| \nearrow \lambda_{j_0} = \inf\{\lambda_k \mid 1 \leq k \leq k_0 - 1,\ u_{0k} \neq 0\}$ when $t \nearrow \infty$.

Therefore, there exists a "double cone" with the "vertex at zero" that contains the subspace H_{-k_0} and all elements are contained in some neighborhood of H_{-k_0}. In addition, the maximal distance between of the elements of this subset and the subspace H_{-k_0} depends on the given r_0. Now, we denote this subset by $\widetilde{H} \subset H_0^1$. It follows from this definition that any subset of $\widetilde{H} \cap \left\{ B_{r_1}^{H_0^1}(0) - B_{r_2}^{H_0^1}(0),\ r_1 > r_2 > 0 \right\}$ converges to a set, which we can define as $H_{k_0} \cap B_{\lambda_{j_0}}^{H_0^1}(0)$, where r_1, r_2 are some numbers with $\lambda_{k_1-1} \leq r_2^\rho < \lambda_{k_1}$

and there is a $\lambda_{j_1} = \inf\{\lambda_k \mid 1 \leq k \leq k_1 - 1,\ u_{0k} \neq 0\}$ and $k_1 = k_1(r_1)$. This shows that the obtained set is subset of a finite-dimension space and it is local attractor in some sense.

Thus, we have proved the following result.

Theorem 1. *Let all the imposed above conditions hold and $u_0 \in H_0^1(\Omega)$ whose norm $\|u_0\| = r_0$ satisfies the inequality $\lambda_{k_0-1} < r_0^\rho < \lambda_{k_0}$. Then each solution to the problem (14) - (15) satisfies one of the following properties:*

1. If u_0 lies in H_{k_0} or in a sufficiently small neighborhood of the subspace H_{k_0}, then $|u_k(t)| \uparrow \infty$ as $t \uparrow \infty$ for $k = \overline{1, k_0 - 1}$, moreover since in this case $r(t)$ increases and gradually it will be greater than each λ_k for $k \geq k_0$, i.e. in this case $|u_k(t)|$ will gradually increase for all k;

2. If u_0 lies H_{-k_0} or in a small neighborhood of the subspace H_{-k_0}, then $|u_k(t)| \downarrow 0$ as $t \uparrow \infty$ for $k \geq k_0$, moreover since in this case $r(t)$ decreases and gradually it will be less than each λ_k, i.e. in this case $|u_k(t)|$ gradually decreases for all k;

3. If $u_0 \in H$, $\|P_{k_0}u_0\| \ll \|Q_{k_0}u_0\|$ and if there are small numbers $\delta(\lambda_{k_0}) > \epsilon(\lambda_{k_0}) > 0$ such that for the Hausdorff distance

$$\epsilon \leq d\left(H_{-k_0}; \left\{u_{0k}^+ \mid k = \overline{1, k_0 - 1}\right\}\right) \leq \delta \tag{34}$$

holds, then the behavior of the $u(t,x)$ is chaotic for sufficient large t. And also, if $\left\|u_{0k_0}^-\right\|_{H_{-k_0}} \approx \left\|u_{0k_0}^+\right\|_{H_{k_0}}$, then there is a relationship between $u_{0k_0}^-(t)$ and $u_{0k_0}^+$ for which the behavior of the $u_k(t)$ is chaotic for all k satisfying $u_k(0) \neq 0$

Remark 1. If 3. of the above theorem obtains, then the following claim is reasonable: for any λ_{k_0} there is a subset $B_{\lambda_{k_0}} \subset H_0^1(\Omega)$ for which (34) holds and for any $u_0 \in B_{\lambda_{k_0}}$ the corresponding solution $u(t)$ satisfies the condition

$$\lambda_{j_0} \leq \|u\|_2^\rho(t) < \lambda_{k_0} \quad for any t > 0,$$

and

$$\lim_{t\uparrow\infty} \|u(t)\|_2^\rho = \lambda_{j_0} = \inf\{\lambda_k \mid 1 \leq k \leq k_0 - 1,\ u_{0k} \neq 0\}$$

then there is an absorbing chaotic set in $L^2(\Omega)$.

4 Problem (1) - (2): the case (β)

Let the mapping g have the local nonlinearity as in the case (β), i.e. now we consider the following problem:

$$\frac{\partial u}{\partial t} - \Delta u - \mu |u|^\rho u = h(x), \quad (t,x) \in (0,T) \times \Omega, \tag{35}$$

$$u(0,x) = u_0(x) \in H_0^1(\Omega), \quad u\big|_{[0,T)\times\partial\Omega} = 0. \tag{36}$$

Here $\mu, \rho > 0$ are some numbers, $\Omega \subseteq R^n$ is a domain with sufficiently smooth boundary $\partial\Omega$ or $\Omega := R^n$, $h \in L^2(\Omega)$.

Rewriting (35) in the form

$$\frac{\partial u}{\partial t} = \Delta u + \mu \left|u\right|^{\rho} u + h\left(x\right) := \left(\Delta + \mu \left|u\right|^{\rho}\right) u + h\left(x\right),$$

we can easily express the formal solution to this problem in the form

$$u\left(t\right) = \exp\left\{ t\Delta + \mu t0 \int \left|u\left(\tau\right)\right|^{\rho} d\tau \right\} \left(u_0 + h\right). \tag{37}$$

Whence, we obtain the estimate

$$\left\|u\right\|_2\left(t\right) \leq \exp\left\{ t0 \int \left(-\lambda_1 + \mu \left\|u\right\|_{2\rho}^{\rho}\left(\tau\right)\right) d\tau \right\} \left(\left\|u_0\right\|_2 + \left\|h\right\|_2\right)$$

for $u\left(t\right) \in H_0^1$ for a.e. $t \geq 0$, where λ_1 is first eigenvalue of the operator $-\Delta : H_0^1 \longrightarrow H^{-1}$. Moreover, it is not difficult to see that from (35) - (36) one can obtain the following problem
$\frac{1}{2}\frac{d}{dt}\left\|u\right\|_2^2\left(t\right) = -\left\|\nabla u\right\|_2^2\left(t\right) + \mu \left\|u\right\|_{\rho+2}^{\rho+2}\left(t\right) + \left\langle h, u\right\rangle\left(t\right),$
$\left\|u\right\|_2^2\left(0\right) = \left\|u_0\right\|_2^2.$

4.1 Existence of a Solution

First, we consider the case $h\left(x\right) = 0$. Then from (4) we obtain the estimate

$$\left\|u\right\|_2^2\left(t\right) \leq \exp\left\{ t0 \int \left(-\lambda_1 + \mu c_0 \left\|u\right\|_{\rho+2}^{\rho}\left(\tau\right)\right) d\tau \right\} \left\|u_0\right\|_2^2 \ ,$$

where $c_0 > 0$ is the constant of the embedding inequality for $L^{\rho+2}\left(\Omega\right) \subset L^2\left(\Omega\right)$. This inequality shows that we can study problem (35) - (36) using the previous approach from Section 2 in the case of the ball $B_{r_0}^X\left(0\right)$ for $r_0 < \left|\frac{\lambda_1}{\mu c_0 c_1}\right|^{\frac{1}{\rho}}$, where c_1 is the constant of the embedding inequality for $H_0^1 \subset L^{\rho+2}\left(\Omega\right)$. In the other words, we can study the solvability only locally for $u_0 \in B_{r_0}^{H_0^1}\left(0\right)$ and $0 \leq \rho \leq \frac{4}{n-2}$.

Now, we deal with problem (4) in the other way. Let $\rho < \frac{4}{n-2}$, then the following interpolation inequality (G-N-S inequality) holds

$$\left\|u\right\|_{\rho+2} \leq c \left\|u\right\|_2^{1-\theta} \left\|\nabla u\right\|_2^{\theta}, \quad \theta = \frac{\rho n}{2\left(\rho+2\right)},\ \rho < \frac{4}{n-2},$$

for $u\left(t\right) \in H_0^1$ and $n \geq 3$. Thus, we get

$$\left\|u\right\|_{\rho+2}^{\rho+2}\left(t\right) = \left\|u\right\|_{\rho+2}^{\rho}\left(t\right) \left\|u\right\|_{\rho+2}^{2}\left(t\right) \leq c \left\|u\right\|_{\rho+2}^{\rho}\left(t\right) \left\|u\right\|_2^{2\left(1-\theta\right)} \left\|\nabla u\right\|_2^{2\theta},$$

where

$$2\theta = \frac{\rho n}{\left(\rho+2\right)} < 2 \quad \textit{and} \quad 2\left(1-\theta\right) = \frac{4-\rho\left(n-2\right)}{\left(\rho+2\right)} < 2$$

and consequently

$$\|u\|_{\rho+2}^{\rho+2}(t) \leq C(\varepsilon, c)\|u\|_{\rho+2}^{\rho\frac{2(\rho+2)}{4-\rho(n-2)}}(t)\|u\|_2^2(t)+\varepsilon\|\nabla u\|_2^2(t). \tag{38}$$

Thus if ρ satisfies the inequality $0<\rho\leq\frac{4}{n}$, then $\rho\frac{2(\rho+2)}{4-\rho(n-2)}\leq\rho+2$, where $\varepsilon>0$ is the small parameter.

Taking into account the inequality (38) and equation (4), we find that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2(t)=-\|\nabla u\|_2^2(t)+\mu\|u\|_{\rho+2}^{\rho+2}(t)\leq-\|\nabla u\|_2^2(t)+$$

$$\mu C(\varepsilon, c)\|u\|_{\rho+2}^{\rho\frac{2(\rho+2)}{4-\rho(n-2)}}(t)\|u\|_2^2(t)+\varepsilon\|\nabla u\|_2^2(t)=$$

$$-(1-\varepsilon)\|\nabla u\|_2^2(t)+\mu C(\varepsilon, c)\|u\|_{\rho+2}^{\rho\frac{2(\rho+2)}{4-\rho(n-2)}}(t)\|u\|_2^2(t).$$

Thus, if we choose $\varepsilon=\varepsilon_0>0$ a fixed small number, then we find, owing to the previous approach, that

$$\frac{d}{dt}\|u\|_2^2(t)\leq\left[-(1-\varepsilon_0)\lambda_1+\mu C(\varepsilon_0, c)\|u\|_{\rho+2}^{\rho\frac{2(\rho+2)}{4-\rho(n-2)}}(t)\right]\|u\|_2^2(t),$$

or

$$\|u\|_2^2(t)\leq\exp\left\{-t0\int\left[(1-\varepsilon_0)\lambda_1+\mu C(\varepsilon_0, c)\|u(\tau)\|_{\rho+2}^{\rho\frac{2(\rho+2)}{4-\rho(n-2)}}\right]d\tau\right\}\|u\|_2^2(0). \tag{39}$$

Consequently, in this case we need to study the problem under the variants (i) - the case when $\|u_0\|_{\rho+2}^{\rho+2}<\frac{1-\varepsilon_0}{\mu C(\varepsilon_0,c)}\lambda_1$; and (ii) - the case when $\|u_0\|_{\rho+2}^{\rho+2}\geq\frac{1-\varepsilon_0}{\mu C(\varepsilon_0,c)}\lambda_1$, separately. It follows from the above estimate that in the case (i) we can prove of the existence theorem on the ball $B_{r_0}^X(0)$ using the approach in Section 2 (or using of the method of compactness [22,23], since it is not difficult to see the operator generated by the problem (35) - (36) is weakly compact).

Hence, (39) implies that$\|u\|_2(t)$ decreases if $u_0\in B_{r_0}^{H_0^1}(0)$ in the case when $r_0<\left[\frac{1-\varepsilon_0}{\mu C_0C(\varepsilon_0,c)}\lambda_1\right]^{\frac{1}{2(\rho+2)}}$ (here $C_0>0$ is the constant of the embedding theorem). Therefore, $\|u\|_H(t)\leq\|u_0\|_H$ for $t\geq0$, i.e. the following statement is verified.

Theorem 2. *Let $h(x)=0$ and the following conditions be satisfied: (1) $0<\rho\leq\frac{4}{n}$, $\|u_0\|_{\rho+2}^{\rho+2}\leq C_0\|u_0\|_{H_0^1}^{\rho+2}<\frac{1-\varepsilon_0}{\mu C(\varepsilon_0,c)}\lambda_1$, $r_0<\left[\frac{1-\varepsilon_0}{\mu C_0C(\varepsilon_0,c)}\lambda_1\right]^{\frac{1}{2(\rho+2)}}$ (here $C_0>0$ is as above); or (2) $0<\rho\leq\frac{4}{n-2}$ and $u_0\in B_{\widehat{r}_0}^{H_0^1}(0)$ for $\widehat{r}_0<\left|\frac{\lambda_1}{\mu c_0c_1}\right|^{\frac{1}{\rho}}$ (here $c_0>0$ and $c_1>0$ are such as in the starting part of this section).*

Then problem (35) - (36) is solvable in $B_{r_0}^X(0)$ (or $B_{\widehat{r}_0}^X(0)$) for all $u_0\in B_{r_0}^{H_0^1}(0)$ (or $u_0\in B_{\widehat{r}_0}^{H_0^1}(0)$), moreover, the solutions $u(t)$ are contained in the closed ball $B_{r_0}^{H_0^1}(0)$ (or $B_{\widehat{r}_0}^{H_0^1}(0)$) for $t\geq0$.

Remark 2. It should noted that if we consider the problem (35) - (36) for $h(x) \neq 0$, for example, as in the case (1) then using the same approach we get

$$\left\langle \frac{\partial u}{\partial t} - \Delta u - \mu |u|^{\rho} u, u \right\rangle = \frac{1}{2}\frac{d}{dt} \|u\|_2^2(t) + \|\nabla u\|_2^2(t) - \mu \|u\|_{\rho+2}^{\rho+2}(t) \leq \|h\|_Y \|u\|_{H_0^1}$$

and it is not difficult to see that here, as in the case (1), we need to determine the balls $B_{r_1}^X(0)$ in X and $B_{R_1}^Y(0)$ in Y (here $Y = H^{-1}$ or $L^2(\Omega)$), where r_1 and R_1 depend on $(\lambda_1, \rho, \mu, C_0)$ as well as each other. However, we shall not go into this case here.

Consequently, we consider here only the case $h(x) = 0$, so that we need to study the posed problem in the variants (i) - the case when $\|u_0\|_{\rho+2}^{\rho+2} < \frac{1-\varepsilon_0}{\mu C(\varepsilon_0, c)}\lambda_1$; and (ii) - the case when $\|u_0\|_{\rho+2}^{\rho+2} \geq \frac{1-\varepsilon_0}{\mu C(\varepsilon_0, c)}\lambda_1$, separately.

4.2 Blow-up Solutions

Consider the case (ii). Denote the functional

$$F(t) := \frac{1}{2}\|\nabla u\|_2^2(t) - \frac{\mu}{\rho+2}\|u\|_{\rho+2}^{\rho+2}(t)$$

for which

$$\frac{d}{dt}F(t) = \langle \nabla u, \nabla u_t \rangle - \mu \langle |u|^{\rho} u, u_t \rangle = -\Omega \int \Delta u u_t -$$

$$\mu\Omega \int |u|^{\rho} u u_t = -\Omega \int |u_t|^2 = -\|u_t\|_H^2, \quad F(t)|_{\ t=0} = F(0),$$

then we get

$$F(t) = F(0) - t0\int \|u_t\|_H^2 = \frac{1}{2}\|\nabla u_0\|_H^2 - \frac{\mu}{\rho+2}\|u_0\|_{\rho+2}^{\rho+2} - t0\int \|u_t\|_H^2 .$$

Now consider the derivative of the functional $G(t) = \|u\|_2^2(t)$

$$\frac{1}{2}\frac{d}{dt}G(t) = \langle u_t, u \rangle = \langle \Delta u + \mu |u|^{\rho} u, u \rangle = -\|\nabla u\|_2^2(t) + \mu \|u\|_{\rho+2}^{\rho+2}(t) = -2F(t) +$$

$$\frac{\rho\mu}{\rho+2}\|u\|_{\rho+2}^{\rho+2}(t) = -\|\nabla u_0\|_2^2 + \frac{2\mu}{\rho+2}\|u_0\|_{\rho+2}^{\rho+2} + 2t0\int \|u_t\|_2^2 + \frac{\rho\mu}{\rho+2}\|u\|_{\rho+2}^{\rho+2}(t)$$

i.e.

$$\frac{1}{2}\frac{d}{dt}G(t) = \frac{1}{2}\frac{d}{dt}\|u\|_2^2(t) = -2F(0) + 2t0\int \|u_t\|_2^2 + \frac{\mu\rho}{\rho+2}\|u\|_{\rho+2}^{\rho+2}(t) \geq$$

$$-2F(0) + \frac{c_1\mu\rho}{\rho+2}\|u\|_2^{\rho+2}(t), \quad G(0) = \frac{1}{2}\|u_0\|_H^2 .$$

So we obtain the problem

$$\frac{d}{dt}G(t) \geq -4F(0) + \frac{2c_1\mu\rho}{\rho+2}G(t)^{\frac{\rho+2}{2}}, \quad G(0) = \|u_0\|_2^2$$

which shows the finite-time blow-up of solutions of the posed problem. Indeed, there are two variants or $F(0) \leq 0$ or $F(0) > 0$. If $F(0) \leq 0$ then for $G(t)$ we obtain

$$\frac{d}{dt}G(t) \geq \frac{2c_1\mu\rho}{\rho+2}G(t)^{\frac{\rho+2}{2}}, \quad G(0) = \|u_0\|_2^2. \tag{40}$$

Clearly, if $\rho > 0$, the solutions of the problem (40) have finite-time blow-up.

Accordingly, we have proved the following result.

Theorem 3. *Let the initial function $u_0 \in H_0^1 \cap L^{\rho+2}(\Omega)$ satisfy the condition*

$$\frac{1}{2}\|\nabla u_0\|_2^2 - \frac{\mu}{\rho+2}\|u_0\|_{\rho+2}^{\rho+2} \leq 0.$$

Then if the local solution of problem (14) - (15) is sufficiently smooth, this solution has a finite-time blow-up in H.

4.3 Solution Behavior

Now consider (35) - (36) in the general case. For a detailed investigation of the behavior of the solutions, we will use the same approach as in Section 3.

Let the eigenfunctions of the operator $-\Delta : H_0^1 \longrightarrow H^{-1}$ be as in Section 3, then any function $v \in H_0^1$ has the representation $v(t,x) := k = 1\infty\sum v_k(t)\, w_k(x)$.

Thus, we have $\frac{\partial u}{\partial t} - \Delta u - \mu |u|^\rho u = 0; \quad u(0,x) = u_0,$

$\Longrightarrow$

$$\sum_{j=1}^{\infty}\left(\frac{du_j(t)}{dt} + \lambda_j u_j(t)\right) w_j(x) - \mu \left|\sum\nolimits_{j=1}^{\infty} u_j(t)\, w_j(x)\right|^\rho \sum\nolimits_{j=1}^{\infty} u_j(t)\, w_j(x) = 0. \tag{41}$$

Now, if $\rho \leq \frac{2}{n-2}$, then $|u|^\rho u := v_u(t,x,\rho) \in L^2(\Omega)$ for all $u \in H^1$. Consequently, we have $v_u(t,x,\rho) = k = 1\infty\sum v_u^k(t,\rho)\, u_k(t)\, w_k(x) \in H$. It is clear that the variation of $v_u^k(t,\rho)$ depends on t in R_+, as at any point t the function $v_u^k(t,\rho) \Longrightarrow \zeta_k(t)$. Moreover, as we saw in the previous section the variation of $u(t,x)$ depends on the relationship between u_0 and λ_k.

Consequently, we can rewrite the problem (41) in the form

$$j = 1\infty\sum\left(\frac{du_j(t)}{dt} + \lambda_j u_j(t) - \mu v_u^j(t,\rho)\, u_j(t)\right) w_j(x) \equiv 0$$

$$u_j(0) = u_{0j}, \quad j = 1, 2, ...$$

Then, if we take into account that the system $\{w_j(x)\}_{j=1}^{\infty}$ is an orthogonal basis in $L^2(\Omega)$, the problem (41) is equivalent to the system of the problems

$$\frac{du_j(t)}{dt} + \lambda_j u_j(t) - \mu v_u^j(t,\rho)\, u_j(t) = 0, \tag{42}$$

$$u_j\left(0\right) = u_{0j}, \quad j = 1, 2, ...$$

So to study the behavior of solutions of the problem (35)- (36,) it is enough to study the system of problems (42) with solutions

$$u_j\left(t\right) = \exp\left\{-t0\int\left(\lambda_j - \mu v_u^j\left(t,\rho\right)\right)d\tau\right\}u_{0j}, \quad j = 1, 2, ...$$

Thus, we obtain the flow which determines the solution via these formal expressions. Now, if $S(t)$ is the flow determined by problem (42), we can rewrite $S(t)$ as $S(t) := S(t + c_u(t,\rho))$, as far as $S(t)$ depends on x via u.

$$u_j\left(t\right) = e^{-\left(t0\int\left(\lambda_j - \mu v_u^j\left(t,\rho\right)\right)d\tau\right)}u_0. \tag{43}$$

Here we get that the behavior of the solutions depends not only on the time, yet depends strongly on the location of points x in Ω belonging to the solution support as far as the quantity of $v_u^j\left(t,\rho\right)$ depends of the locally variations of quantity of u in Ω. Consequently, (43) demonstrates us that this case of the spatiotemporal states essentially differs from the case (5) - (6). In other words, for fixed $j \in Z_+$ there are possible two choices: either $\lambda_j < \mu v_u^j\left(t,\rho\right)$ or $\lambda_j \geq \mu v_u^j\left(t,\rho\right)$, depending on the state of $(t,x) \in R_+ \times \Omega$.

Consequently, to explain the possible variants in this case we can argue as in Section 3 to obtain the following result.

Proposition 4. *Suppose* $\rho \leq \frac{2}{n-2}$, $u_0 \in B_{r_0}^{H_0^1}(0)$ *and there are subsets with nonzero measure in* Ω *on which* $\lambda_{i_0+1} > u_0\left(x\right) \geq \lambda_{i_0}$, *for some* $i_0 \geq 1$. *Then, the behavior of the solution depends sensitively on the variation of* $v_u^i\left(t,\rho\right)$, *i.e. it has spatio-temporal chaotic dynamics.*

Thus, we can conclude, that as in the previous problem, chaotic dynamics occur, but unlike the previous case, the chaos is spatio-temporal.

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