2-D and 3-D Solvable Chaos Maps

Shunji Kawamoto

Osaka Prefecture University, Sakai, Osaka, Japan
(E-mail: kawamoto@eis.osakafu-u.ac.jp)

Abstract. It is presented firstly in this paper that one-dimensional (1-D) chaos solutions give 1-D and 2-D solvable chaos maps, which are related to the delayed logistic map and the Smith map. From 2-D general chaos solutions, the most general 2-D solvable chaos map is derived, and it is shown that the map relates to the generalized Hénon map and the Holmes map. Then, 2-D chaotic maps by time-discretizing the Duffing equation and the Van der Pol equation are obtained, which are known to have a strange attractor or the Japanese attractor on the Poincaré section or the phase-plane. Finally, the Lorenz system and the Rössler system are time-discretized, and the 3-D chaotic maps are discussed from the standpoint of a 3-D solvable chaos map derived from general chaos solutions.
Keywords: Chaos map, Chaos solution, Delayed logistic map, Smith map, Hénon map, Holmes map, Duffing equation, Van der Pol equation, Lorenz system, Rössler system, Poincaré section, Strange attractor, Japanese attractor.

1 Introduction

It is well known for the study of nonlinear dynamics that nonlinear difference equations and differential equations have arisen widely in the field of biological, physical, chemical, mechanical and social sciences, and possess a rich spectrum of dynamical behavior as chaos in many respects [1-4]. As an example, a population growth in biology is modeled [5], and has been afforded by a nonlinear difference equation called the logistic map. For one-dimensional (1-D) chaotic maps, a bifurcation diagram of the two parameter quadratic family has been observed [6], and the edge of chaos in the self-adjusting logistic map with a slowly changing parameter has been considered [7]. Moreover, various chaotic sequences have been proposed for the generation of pseudo-random numbers, the synchronization and the application to cryptosystems [8-10].
In the meantime, a family of shapes and many other irregular patterns in nature called fractals has been discussed for the geometric representation as an irregular set consisting of parts similar to the whole [11-13]. However, since the Mandelbrot map is defined by complex functions, it has been pointed out that the physics of fractals is a research subject to be born [14]. As an application, fractal compression has been presented to compress images using fractals [15], and chaotic and fractal dynamics have been widely expanded to physical, chemical, mechanical and electrical observations with the mathematical models [4]. Recently, a construction method of 2-D and 3-D chaotic maps has been proposed, and nonlinear dynamics of the map on the fractal sets have been studied for the physical analogue [16].

Received: 28 March 2016 / Accepted: 15 December 2016

ISSN 2241-0503

In this paper, it is shown that from 1-D chaos solutions, 1-D and 2-D solvable chaos maps are derived in Section 2, which are related to the delayed logistic map and the Smith map [17, 18]. From 2-D general chaos solutions, the most general 2-D solvable chaos map is derived in Section 3, and it is explained that the map relates to the generalized Hénon map [19] and the Holmes map [20]. Then, 2-D chaotic maps are obtained by time-discretizing the Duffing equation and the Van der Pol equation [21], which are known to have a strange attractor or the Japanese attractor [22, 23]. In Section 4, a 3-D solvable chaos map is derived from general chaos solutions, and the time-discretized map of the Lorenz system [24] and the Rössler system [25] are discussed with the 3-D solvable chaos map. The last Section is devoted to conclusions.

2 1-D Solvable Chaos Maps

In this section, we discuss the following three cases of 1-D chaos solutions to find 1-D and 2-D solvable chaos maps.

Case 1

Firstly, from a chaos solution;

$$x_n = \sin^2(C2^n) \tag{1}$$

with $C \neq \pm m\pi/2^l$ and finite positive integers $\{l, m\}$ to the well-known logistic map;

$$x_{n+1} = 4x_n(1-x_n), \tag{2}$$

which is a 1-D solvable chaos map, we have a 2-D solvable chaos map, by introducing a real parameter $\alpha \neq 0$ [16], as

$$x_{n+1} = 4x_n - 4(1-\alpha)x_n^2 - 4\alpha y_n, \tag{3}$$

$$\begin{aligned} y_{n+1} &= 16(1-x_n)^2 y_n \\ &= 16(1-2x_n + y_n)y_n \end{aligned} \tag{4}$$

with

$$y_n \equiv \sin^4(C2^n) = x_n^2, \tag{5}$$

where the condition (5) is used for the order reduction in (4). Here, it is known that the first equation (3) has the same form as the Helleman map, which has been obtained from the motion of a proton in a storage ring with periodic impulses [26].

Case 2

For a chaos solution;

$$x_n = \cos(C2^n), \tag{6}$$

we have a 1-D solvable chaos map;

$$x_{n+1} = 2x_n^2 - 1, \tag{7}$$

which is also called the logistic map, because the solution (1) can be transformed by $\sin^2(C2^n) = (1-\cos(C2^{n+1}))/2$ to the solution (6). From (6), by introducing a real parameter $\alpha \neq 0$, we find a 2-D solvable chaos map [16];

$$x_{n+1} = -\alpha + (1+\alpha)x_n^2 - (1-\alpha)y_n, \tag{8}$$

$$y_{n+1} = 4{x_n}^2 y_n \tag{9}$$

with $y_n \equiv \sin^2(C2^n)$, where the first equation (8) has the same form as the Hénon map [19]. If we put $\alpha = 0$ in (8) and define $y_n \equiv \sin(C2^n)$, then we obtain a 2-D generalized chaotic map; $x_{n+1} = x_n^2 - y_n^2 + k_1$ and $y_{n+1} = 2x_n y_n + k_2$ with parameters $\{k_1, k_2\}$, including the Mandelbrot map and the Julia map in terms of real variables [11][16].

Case 3

Similarly, in the case of a chaos solution;

$$x_n = \sin(C2^n), \tag{10}$$

we have a 1-D delayed solvable chaos map;

$$x_{n+1} = 2x_n(1 - 2x_{n-1}^2), \tag{11}$$

which can be rewritten into a 2-D solvable chaos map as

$$x_{n+1} = 2x_n(1 - y_n), \tag{12}$$

$$y_{n+1} = 2x_n^2 \tag{13}$$

with

$$y_n \equiv 2x_{n-1}^2 = 2\sin^2(C2^{n-1}), \tag{14}$$

where the map (12) and (13) has chaos solutions (10) and (14). Here, it is interesting to note that the time-delayed logistic map;

$$x_{t+1} = \lambda x_t(1 - x_{t-1}) \tag{15}$$

with a real parameter λ, can be rewritten into a 2-D chaotic map as

$$x_{t+1} = \lambda x_t(1 - y_t), \tag{16}$$

$$y_{t+1} = x_t, \tag{17}$$

which is called the Smith map [17], and has been discussed as a simplest population growth model [27]. It is found that the first equation (16) of the Smith map has the same form as the first equation (12) of the 2-D solvable chaos map (12) and (13). Then, the map (16) and (17) has been developed to the following 2-D map [18];

$$x_{t+1} = \lambda x_t(1-(1-b)x_t - by_t), \tag{18}$$

$$y_{t+1} = x_t \tag{19}$$

with parameters $\{\lambda, b\}$, where the first equation (18) has the same form as the second equation (4) of the 2-D solvable chaos map (3) and (4) derived from the chaos solution (1). Here, we call functions (1), (6) and (10) 'chaos function.'

3 2-D Solvable Chaos Maps

By introducing the following general solutions consisting of chaos functions;

$$x_n = a_{11}\cos(C2^n) + a_{12}\sin(C2^n) + b_1, \tag{20}$$

$$y_n = a_{21}\cos(C2^n) + a_{22}\sin(C2^n) + b_2, \tag{21}$$

we have

$$\begin{bmatrix} \cos(C2^n) \\ \sin(C2^n) \end{bmatrix} = A^{-1}\begin{bmatrix} x_n - b_1 \\ y_n - b_2 \end{bmatrix} \tag{22}$$

with

$$A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ |A| \neq 0, \tag{23}$$

and (22) gives

$$\begin{bmatrix} \cos(C2^n) \\ \sin(C2^n) \end{bmatrix} = \begin{bmatrix} A_{11}x_n + A_{12}y_n - (A_{11}b_1 + A_{12}b_2) \\ A_{21}x_n + A_{22}y_n - (A_{21}b_1 + A_{22}b_2 \end{bmatrix}. \tag{24}$$

Then, from general solutions (20) and (21), we find

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A\begin{bmatrix} \cos^2(C2^n) - \sin^2(C2^n) \\ 2\cos(C2^n)\sin(C2^n) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{25}$$

and obtain the following map by substituting (24) into the rhs of (25);

$$x_{n+1} = ax_n^2 + bx_n y_n + cy_n^2 + dx_n + ey_n + f, \tag{26}$$

$$y_{n+1} = a'x_n^2 + b'x_n y_n + c'y_n^2 + d'x_n + e'y_n + f', \tag{27}$$

which is a 2-D solvable chaos map with general chaos solutions (20) and (21), where real parameters $\{a, b, c, d, e, f, a', b', c', d', e', f'\}$ are given by (A1) and (A2) in Appendix. Here, it is interesting to note that a most general quadratic map

$$x_{i+1} = f + ax_i + by_i + cx_i^2 + dx_i y_i + ey_i^2, \tag{28}$$

$$y_{i+1} = f' + a'x_i + b'y_i + c'x_i^2 + d'x_i y_i + e'y_i^2 \tag{29}$$

depending on twelve parameters $\{f, a, b, c, d, e, f', a', b', c', d', e'\}$, has been proposed as the generalized Hénon map, and the Jacobian is known to have a constant if some relations are satisfied by these parameters [19].
In addition, for a set of general solutions consisting of chaos functions;

$$x_n = a_{11}\cos(C2^n) + b_1, \tag{30}$$

$$y_n = a_{22}\sin^2(C2^n) \tag{31}$$

with a function $\sin^2(C2^n)$ in (31) and nonzero parameters $\{a_{11}, a_{22}, b_1\}$, we find the following map, from (30), (31) and the condition $(x_n - b_1)^2 / a_{11}^2 + (y_n / a_{22}) = 1$, as

$$x_{n+1} = -\left(\frac{2a_{11}}{a_{22}}\right) y_n + (a_{11} + b_1), \tag{32}$$

$$y_{n+1} = \frac{4}{a_{11}^2}(-2b_1 x_n y_n + x_n^2 y_n + b_1^2 y_n), \tag{33}$$

which is a 2-D solvable chaos map with general chaos solutions (30) and (31), and has the third-order nonlinear term in (33). On the other hand, the Holmes map [20] has been suggested as

$$x_{n+1} = y_n, \tag{34}$$

$$y_{n+1} = -bx_n + dy_n - y_n^3 \tag{35}$$

with parameters $\{b, d\}$, which has some of the features of a negative stiffness Duffing oscillator [28], and it is found that the map has a similar form to the map (32) and (33) with general chaos solutions (30) and (31).
For nonlinear differential equations of the second order, we discuss firstly the forced Duffing equation given by

$$\ddot{x} + \delta\ddot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t), \tag{36}$$

where $x = x(t)$ is the displacement at time t, $\dot{x}$ is the velosity, and $\ddot{x}$ is the acceleration. The coefficients $\{\alpha, \beta, \gamma, \delta\}$ are real constants, and the rhs of (36) gives a periodic driving force. Here, it should be emphasized that randam oscillations occuring in a nonlinear electric circuit, which is equivalent to (36), have been considered, and the points of orbit on the phase-plane are known to give the Japanese attractor representing the random oscillations [22][23]. Then, the differential equation (36) is rewritten into a 2-D form as

$$\frac{dx}{dt} = y, \tag{37}$$

$$\frac{dy}{dt} = -\alpha x - \beta x^3 - \delta y + \gamma \cos(\omega t), \tag{38}$$

and by the difference method;

$$\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{\Delta t}, \frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t} \tag{39}$$

with $x_n \equiv x(t)$, $y_n \equiv y(t)$ and the time step $\Delta t > 0,$ we find the following 2-D map in the case of $\gamma = 0$ from (37)-(39);

$$x_{n+1} \equiv x_n + (\Delta t) y_n, \tag{40}$$

$$y_{n+1} \equiv -\alpha(\Delta t) x_n - \beta(\Delta t) x_n^3 + (1 - (\Delta t)\delta) y_n, \tag{41}$$

which has a similar form to the Holmes map (34) and (35) with a cubic term, and to the 2-D solvable chaos map (32) and (33) with general chaos solutions (30) and (31).

Secondly, we consider the forced Van der Pol oscillator [21] given by

$$\ddot{x} + x - \varepsilon(1 - x^2)\dot{x} = E \sin(\Omega t), \tag{42}$$

which represents a model for a simple vacuum tube oscillator circuit with a nonlinear damping term, where $x = x(t)$ is the proposition coordinate function of the time t, and $\{\varepsilon \neq 0, E, \Omega\}$ are the system parameters. Equation (42) is rewritten into a 2-D form;

$$\frac{dx}{dt} = \varepsilon\left(x - \frac{1}{3}x^3 - y\right), \tag{43}$$

$$\frac{dy}{dt} = \frac{1}{\varepsilon}x - \frac{E}{\varepsilon}\sin(\Omega t), \tag{44}$$

and by the difference method (39), we find the 2-D map in the case of E=0 as

$$x_{n+1} \equiv (1+\varepsilon(\Delta t))x_n - \frac{1}{3}\varepsilon(\Delta t)x_n^3 - \varepsilon(\Delta t)y_n, \tag{45}$$

$$y_{n+1} \equiv \frac{1}{\varepsilon}(\Delta t)x_n + y_n, \tag{46}$$

which has also a similar form to the map (40) and (41) obtained from the Duffing equation, to the Holmes map (34) and (35), and to the 2-D solvable chaos map (32) and (33) with general chaos solutions (30) and (31).

4 3-D Solvable Chaos Maps

In this section, we begin with 3-D general solutions consisting of chaos functions;

$$x_n = a_{11}\cos(C2^n) + a_{12}\sin(C2^n) + a_{13}\sin^2(C2^n) + b_1, \tag{47}$$

$$y_n = a_{21}\cos(C2^n) + a_{22}\sin(C2^n) + a_{23}\sin^2(C2^n) + b_2, \tag{48}$$

$$z_n = a_{31}\cos(C2^n) + a_{32}\sin(C2^n) + a_{33}\sin^2(C2^n) + b_3, \tag{49}$$

and can derive a 3-D solvable chaos map with general chaos solutions (47)-(49). Here, as a simple set of chaos solutions, we introduce from (47)-(49);

$$x_n = a_{11}\cos(C2^n) + b_1, \tag{50}$$

$$y_n = a_{22}\sin(C2^n) + b_2, \tag{51}$$

$$z_n = a_{33}\sin^2(C2^n), \tag{52}$$

and have the following conditions from (50)-(52) as

$$(\frac{1}{a_{11}})^2(x_n - b_1)^2 + (\frac{1}{a_{33}})z_n = 1, \tag{53}$$

$$(\frac{1}{a_{22}})^2(y_n - b_2)^2 = (\frac{1}{a_{33}})z_n \tag{54}$$

with real parameters $\{a_{11} \neq 0, a_{22} \neq 0, a_{33} \neq 0, b_1, b_2\}$. Thus, we find from (50)-(52) and the condition (53);

$$x_{n+1} = -(\frac{2a_{11}}{a_{33}})z_n + (a_{11} + b_1), \tag{55}$$

$$y_{n+1} = (\frac{2}{a_{11}})(x_n y_n - b_2 x_n - b_1 y_n) + (\frac{2b_1}{a_{11}} + 1)b_2, \tag{56}$$

$$z_{n+1} = 4(\frac{1}{a_{11}})^2 (x_n - b_1)^2 z_n$$

$$= 4z_n(1 - (\frac{1}{a_{33}})z_n), \tag{57}$$

which is a 3-D solvable chaos map with general chaos solutions (50)-(52), and the second-order nonlinear terms are included in (56) and (57), respectively. Here, it should be noted that the condition (53) is applied to the order reduction in (57).

On the other hand, the Lorenz system has been well considered as a simplified mathematical model for atmospheric convection [24] and an identical model for instabilities of the single mode laser [29], and is known to have a strange attractor for certain parameter values and initial conditions. The model is a system of three ordinary differential equations given by

$$\frac{dx}{dt} = -\sigma x + \sigma y, \tag{58}$$

$$\frac{dy}{dt} = \rho x - xz - y, \tag{59}$$

$$\frac{dz}{dt} = xy - \beta z, \tag{60}$$

where $\{x, y, z\}$ are the system state variables, which are proportional to the circulatory fluid flow velocity, the temperature difference and the distortion of the virtical temperature profile, respectively. The coefficients $\{\sigma, \rho, \beta\}$ are the system dimensionless parameters. By the difference method;

$$\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{\Delta t}, \frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}, \frac{dz}{dt} \approx \frac{z_{n+1} - z_n}{\Delta t} \tag{61}$$

with $x_n \equiv x(t)$, $y_n \equiv y(t)$, $z_n \equiv z(t)$ and the time step $\Delta t > 0$, we have from (58)-(60) as

$$x_{n+1} \equiv (1 - \sigma(\Delta t))x_n + \sigma(\Delta t)y_n, \tag{62}$$

$$y_{n+1} \equiv \rho(\Delta t)x_n - (\Delta t)x_n z_n + (1 - (\Delta t))y_n, \tag{63}$$

$$z_{n+1} \equiv (\Delta t)x_n y_n + (1 - \beta(\Delta t))z_n, \tag{64}$$

which has the second-order nonlinear terms in (63) and (64). It is interesting to note that the map (62)-(64) has a similar form to the 3-D solvable chaos map

(55)-(57) with general chaos solutions (50)-(52). Moreover, the Rössler system [25] is given by

$$\frac{dx}{dt} = -y - z, \tag{65}$$

$$\frac{dy}{dt} = x + ay, \tag{66}$$

$$\frac{dz}{dt} = b + z(x - c), \tag{67}$$

with system parameters $\{a,b,c\}$, which has one nonlinear term in (67), and is known to have a strange attractor in the 3-D space. Then, we find the map by the difference method (61) as;

$$x_{n+1} \equiv x_n - (\Delta t)(y_n + z_n), \tag{68}$$

$$y_{n+1} \equiv (\Delta t)x_n + (1 + a(\Delta t))y_n, \tag{69}$$

$$z_{n+1} \equiv (\Delta t)z_n(x_n - c) + z_n + (\Delta t)b, \tag{70}$$

where if we approximate the nonlinear term $x_n z_n$ in (63) by y_n^2 and put a linear term z_n for the y_n^2 by the condition (54), then we find the linear equation (69). Here, it is notable that the Rössler system (65)-(67) is obtained by simplifying the Lorenz system (58)-(60), and is known to be useful in modeling chemical reactions. In a recent study, a simpler 3-D system given by

$$\frac{dx}{dt} = y, \tag{71}$$

$$\frac{dy}{dt} = -x + yz, \tag{72}$$

$$\frac{dz}{dt} = 1 - y^2, \tag{73}$$

has been considered, and is shown to have a chaotic attractor [30]. By the difference method (61), we have the map;

$$x_{n+1} \equiv x_n + (\Delta t)y_n, \tag{74}$$

$$y_{n+1} \equiv -(\Delta t)x_n + y_n + (\Delta t)y_n z_n, \tag{75}$$

$$z_{n+1} \equiv (\Delta t)(1 - y_n^2) + z_n, \tag{76}$$

which has a quadratic nonlinear term in (76), and has a similar form to the 3-D solvable chaos map (55)-(57) with general chaos solutions (50)-(52).

Conclusions

In this paper, we have derived 1-D, 2-D and 3-D solvable chaos maps by introducing chaos functions, and have considered the delayed logistic map, the Smith map, the Hénon map, the Holmes map, the Duffing equation, the Van der Pol equation, the Lorenz system and the Rössler system with strange attractors or the Japanese attractor. Then, it is found that the chaos functions play a keyrole for chaotic behaviors of the well-discussed maps, the differential equations and the chaotic systems, because the functions have non-periodicity and sensitivity on initial values. Therefore, chaos functions may underlye in nonlinear phenomena with chaotic dynamics, such as population growth in biology, mechanical vibration, oscillation in nonlinear electric circuit, atmospheric convection and chemical reaction, as many of the sciences.

Appendix

We find the parameters of (26) and (27) by substituting (24) into (25) as

$$\begin{aligned}
a &= a_{11}(A_{11}^2 - A_{21}^2) + 2a_{12}A_{11}A_{21},\\
b &= 2a_{11}(A_{11}A_{12} - A_{21}A_{22}) + 2a_{12}(A_{11}A_{22} + A_{12}A_{21}),\\
c &= a_{11}(A_{12}^2 - A_{22}^2) + 2a_{12}A_{12}A_{22},\\
d &= 2[a_{11}(-A_{11}(A_{11}b_1 + A_{12}b_2) + A_{21}(A_{21}b_1 + A_{22}b_2))\\
&\qquad + a_{12}(-A_{11}(A_{21}b_1 + A_{22}b_2) - A_{21}(A_{11}b_1 + A_{12}b_2))],\\
e &= 2[a_{11}(-A_{12}(A_{11}b_1 + A_{12}b_2) + A_{22}(A_{21}b_1 + A_{22}b_2))\\
&\qquad + a_{12}(-A_{12}(A_{21}b_1 + A_{22}b_2) - A_{22}(A_{11}b_1 + A_{12}b_2))],\\
f &= a_{11}[(A_{11}b_1 + A_{12}b_2)^2 - (A_{21}b_1 + A_{22}b_2)^2]\\
&\quad + 2a_{12}(A_{11}b_1 + A_{12}b_2)(A_{21}b_1 + A_{22}b_2) + b_1,
\end{aligned} \tag{A1}$$

and

$$\begin{aligned}
a' &= a_{21}(A_{11}^2 - A_{21}^2) + 2a_{22}A_{11}A_{21},\\
b' &= 2a_{21}(A_{11}A_{12} - A_{21}A_{22}) + 2a_{22}(A_{11}A_{22} + A_{12}A_{21}),\\
c' &= a_{21}(A_{12}^2 - A_{22}^2) + 2a_{22}A_{12}A_{22},\\
d' &= 2[a_{21}(-A_{11}(A_{11}b_1 + A_{12}b_2) + A_{21}(A_{21}b_1 + A_{22}b_2))\\
&\qquad + a_{22}(-A_{11}(A_{21}b_1 + A_{22}b_2) - A_{21}(A_{11}b_1 + A_{12}b_2))],\\
e' &= 2[a_{21}(-A_{12}(A_{11}b_1 + A_{12}b_2) + A_{22}(A_{21}b_1 + A_{22}b_2))\\
&\qquad + a_{22}(-A_{12}(A_{21}b_1 + A_{22}b_2) - A_{22}(A_{11}b_1 + A_{12}b_2))],\\
f' &= a_{21}[(A_{11}b_1 + A_{12}b_2)^2 - (A_{21}b_1 + A_{22}b_2)^2]\\
&\quad + 2a_{22}(A_{11}b_1 + A_{12}b_2)(A_{21}b_1 + A_{22}b_2) + b_2,
\end{aligned} \tag{A2}$$

where parameters $\{A_{11}, A_{12}, A_{21}, A_{22}\}$ in (A1) and (A2) are given by $\{a_{11}, a_{12}, a_{21}, a_{22}\}$ as shown in (23).

References

1. R. M. May. Biological populations with non-overlapping generations: Stable points, stable cycles, and chaos. *Science* 15: 645-646, 1974.
2. T. Y. Li and J. A. Yorke. Period three implies chaos. *American Mathematics Monthly* 82: 985-992, 1975.
3. R. M. May. Simple mathematical models with very complicated dynamics. *Nature* 261: 459-467, 1976.
4. F. C. Moon. *Chaotic and Fractal Dynamics.* Wiley, New York, 1992.
5. P. F. Verhulst. Mathematical researches into the law of population growth increase. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles–Lettres de Bruxelles* 18: 1-42, 1845.
6. E. Barreto, B. R. Hunt, C. Grebogi and J. A. Yorke. From high dimensional chaos to stable periodic orbits: The structure of parameter space. *Phys. Rev. Lett.* 78 : 4561-4564, 1997.
7. P. Melby, J. Kaidel, N. Weber and A. Hubler. Adaptation to the edge of chaos in the self-adjusting logistic map. *Phys. Rev. Lett.* 84: 5991-5993, 2000.
8. L. M. Pecora and T. L. Carroll. Synchronization in chaos systems. *Phys. Rev. Lett.* 64: 821-824, 1990.
9. G. Perez and H. A. Cerdeira. Extracting message masked by chaos. *Phys. Rev. Lett.* 74: 1970-1973, 1995.
10. G. D. V. Wiggeren and R. Roy. Optical communication with chaotic waveforms. *Phys. Rev. Lett.* 81: 3547-3550, 1998.
11. B. B. Mandelbrot. *The Fractal Geometry of Nature.* Freeman, San Francisco, 1982.
12. H. Peitgen and P. Richter. *The Beauty of Fractals.* Springer, New York, 1986.
13. H. Peitgen, H. Jurgens and D. Saupe. *Chaos and Fractals – New Frontiers of Science.* Springer, New York, 1992.
14. L. P. Kadanoff. Where's the physics? *Phys. Today* 39: 6-7, 1986.
15. M. F. Barnsley. *Fractals Everywhere.* Academic Press, New York, 1993.
16. N. H. Tuan Anh, D. V. Liet and S. Kawamoto. *Handbook of Applications of Chaos Theory*, ed. by C. H. Skiadas and C. Skiadas: 83-91. Chapman and Hall/CRC Press, 2016.
17. J. M. Smith. *Mathematical Ideas in Biology*. Cambridge University Press, Cambridge, 1968.
18. R. M. May. Chaos and the dynamics of biological populations. *Proc. R. Soc. London* A413: 27-44, 1987.
19. M. Hénon. A two-dimensional mapping with a strange attractor. *Communications in Mathematical Physics* 50: 69-77, 1976.
20. P. J. Holmes. A nonlinear oscillator with a strange attractor. *Philos. Trans. R. Soc. London* A292: 419-448, 1979.
21. B. Van der Pol and J. Van der Mark. Frequency demultiplication. *Nature* 120 (3019): 363-364, 1927.
22. Y. Ueda. Random phenomena resulting from nonlinearity: In the system described by Duffing's equation. *Int. J. Non-linear Mechanics* 20: 481-491, 1985: Translated from *Trans. IEEJ* 98-A:167-173, 1978.

23. Y. Ueda. Randomly transitional phenomena in the system governed by Duffing's equation. *J. Statistical Physics* 20: 181-196, 1979.

24. E. N. Lorenz. Deterministic nonperiodic flow. *J. Atoms. Sciences* 20 (2): 130-141, 1963.

25. O. E. Rössler. An equation for continuous chaos. *Physics Letters* 57A (5): 397-398, 1976.

26. R. H. G. Helleman. Self-generated chaotic behavior in nonlinear mechanics. *Fundamental Problems in Statistical Mechanics* 5: 165-233, North-Holland, Amsterdam, 1980.

27. D. G. Aronson, M. A. Chory. G. R. Hall and R. P. McGehee. Bifurcations from an invariant circle for two-parameter families of maps of the plane: A computer-assisted study. *Communications in Math. Phys*. 83: 303-354, 1982.

28. F. C. Moon and P. J. Holmes. A magneto elastic strange attractor. *J. Sound Vibration* 65 (2): 275-296, 1979.

29. H. Haken. Analogy between higher instabilities in fluids and lasers. *Physics Letters* 53A (1): 77-78, 1975.

30. J. C. Sprott. Some simple chaotic flows. *Phys. Rev.* E 50 (2): 647-650, 1994.