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Smoothing discontinuities in Predator Prey Models

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Abstract. Several modern predator prey like systems as well as a number of simple low dimensional chaotic systems involve discontinuous functions. This causes difficulties in simulating such systems in order to obtain characteristics such as stationary points, local and structural stability, Lyapunov Exponents or limit cycles. These parameters are very important in understanding the models. The distinction between differential equations and discretized maps is also of interest, particularly in two dimensional systems. In this work, we propose to replace discontinuous functions by continuous functions that approximate them in order to facilitate the analysis. For example, the step function can be replaced by an inverse tangent or hyperbolic tangent.

Keywords: Predator-Prey models, Discontinuous functions, Local and Structural Stability, Simulation of Chaotic Systems and Lyapunov Exponents.

1 Introduction

Predator prey systems and their generalizations such as the HANDY(Human and Nature Dynamics) model (Motesharrei *et al*.[1]) or models used in ecological systems (Caia,Z.*et al*[2])or systems that involve general functional responses such as the low dimensional chaotic systems proposed by Sprott and Linz.[3] involve discontinuous functions. In real world application, the biological dynamical systems are usually discontinuous. For example, in many models of renewable resource management, sliding mode feedback control or nonlinear circuits, stabilization favors the use of threshold policies involving discontinuous functions. As far as simulating the fiducial trajectory is concerned, discontinuities do not pose difficulties in integrating the system numerically. However, Lyapunov exponents are usually regarded as invariants that characterize chaotic behavior and their calculation requires varied

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trajectories about a fiducial trajectory (Wolf, A *et al*. [4]) and discontinuities affect the varied trajectory calculation and produce stiffness or instabilities during the numerical integration. This has been pointed out in the case of the signum nonlinearity by Sun and Sprott [5].

There are ways with which some discontinuities can be approximated by continuous functions so that the derivative that is necessary for the varied trajectory can exist. An example is using 2/π arctan(ax) or tanh(ax), a>0 for the step function. A list of similar discontinuous functions and their continuous equivalents are given below.

$$\operatorname{sgn}(x) = \frac{2}{\pi}\tan^{-1}(ax) \quad or \quad \tanh(ax)$$

$$|x| = x\operatorname{sgn}(x)$$

$$\min(a,b) = \frac{a+b}{2} - \left|\frac{a-b}{2}\right|$$

$$\max(a,b) = \frac{a+b}{2} + \left|\frac{a-b}{2}\right|$$

A graph of the arc tangent and hyperbolic tangent functions is presented in Fig 1 where the parameter a can be varied to increase the steepness as much as possible without affecting differentiability**.**

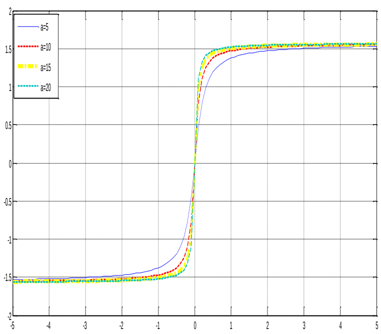	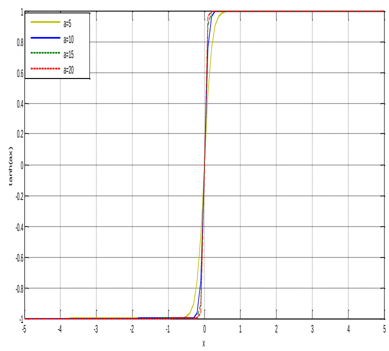
Fig1.a.arctan (ax) function. The graph is generated for the values of parameter a in range from 5 to 20 in increasing amounts of 5.	Fig1.b. tanh(ax) function. The graph is generated for the values of parameter a in range from 0.2 to 1 in increasing amounts of 0.2.

This is important since Sprott [5] correctly claims that the Wolf integration or even attractor reconstruction do not successfully analyze such systems.

In this work, the Sprott and HANDY models, the two examples mentioned above for discontinuities in the dynamical systems will be examined. Existence of chaotic behavior in the Sprott model and its nonexistence in the HANDY model for a range of values of the parameters as evidenced by the maximal Lyapunov Exponent will be presented.

2 HANDY (Human and Nature Dynamics) Model

Originally, the HANDY Model is derived from a predator-prey model as indicated in [1].It is a socio economical model that aims to describe accumulated wealth and economic inequality in a predator–prey model of humans and nature. This model divides the society into two parts: commoners $\mathbf{x_c}$ and elites $\mathbf{x_e}$. Natural resources or nature $\boldsymbol{y}$ in which society lives and the wealth $\boldsymbol{w}$ that that society uses and spends is included in the model as additional variables. It is interesting that in this model the wealth is only generated by commoners $\mathbf{x_c}$, however the wealth is not equally distributed between commoners $\mathbf{x_c}$ and elites $\mathbf{x_e}$ such that elites get $\boldsymbol{\kappa}$ (which is the inequality factor) times more than commoners. The governing differential equations of the HANDY model are given below:

$$\dot{x_c} = \beta_c x_c - \alpha_c x_c$$

$$\dot{x_e} = \beta_e x_e - \alpha_c x_e$$

$$\dot{y} = \gamma y(\lambda - y) - \delta x_c y$$

$$\dot{w} = \delta x_c y - C_c - C_e$$

Here β_c and β_e are the birth that rates of commoners and elites, γ is the regeneration rate of nature, λ is the carrying capacity of nature, δ is depletion (or production) factor. The problematic parts in this model come from C_c, C_e and α_c, α_ewhich are discontinuous functions of x_c, x_e $and\ w$:

$$C_c = \min\left(1, \frac{w}{w_{th}}\right).s.x_c$$

$$C_e = \min\left(1, \frac{w}{w_{th}}\right).s.\kappa.x_c$$

$$\alpha_c = \alpha_m + \max(0{,}1 - \frac{C_c}{sx_c})(\alpha_M - \alpha_m)$$

$$\alpha_e = \alpha_m + \max(0{,}1 - \frac{C_e}{sx_e})(\alpha_M - \alpha_m)$$

Here α_M and α_m are maximum and minimum death rates, s is subsistence salary per capita. In addition to this, w_{th} is defined as $w_{th} = \rho x_c + \kappa\rho x_e$, where $is\ the$ threshold wealth per capita It can be seen discontinuous function max () and min () are used for consumption functions C_c, C_e and death rate constants α_c, α_e.

To eliminate the effect of the discontinuous functions we introduce the following modifications to $C_c, C_e, \alpha_c, \alpha_e$: Note that tanh can replace the 2/π arctan function.

$$C_c = \left(\frac{1+\frac{w}{w_{th}}}{2} - \frac{2}{\pi} \arctan\left(\frac{1-\frac{w}{w_{th}}}{2} \right) \right) s x_c$$

$$C_e = \left(\frac{1+\frac{w}{W_{th}}}{2} - \frac{2}{\pi} \arctan\left(\frac{1-\frac{w}{w_{th}}}{2} \right) \right) s \kappa x_e$$

$$\alpha_e = \alpha_m + (\frac{1-\frac{C_e}{s x_e}}{2} + \frac{2}{\pi} \arctan\left(\frac{1-\frac{C_e}{s x_e}}{2} \right))(\alpha_M - \alpha_m)$$

$$\alpha_c = \alpha_m + (\frac{1-\frac{C_c}{s x_c}}{2} + \frac{2}{\pi} \arctan\left(\frac{1-\frac{C_c}{s x_c}}{2} \right))(\alpha_M - \alpha_m)$$

In Reference [1], simulation results for different values of parameters introduced above are given. We also repeated the simulation of this model with the modification that we proposed for the case of the Egalitarian society in [1] and the result is given in Fig 2.

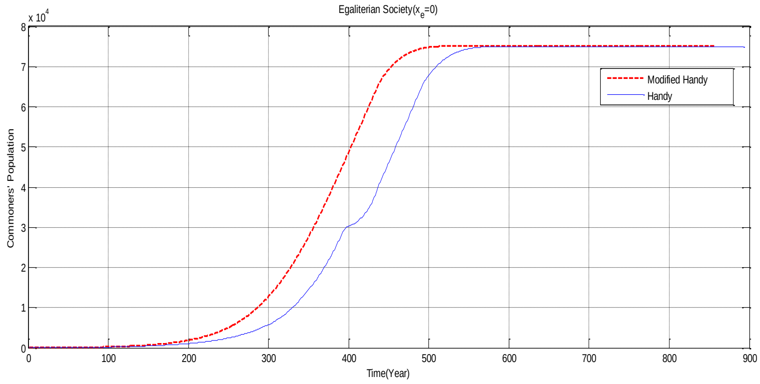

Fig 2a.Simulation results which compare commoners' population result for Egalitarian Society where there is no elite (Soft Landing Solution).

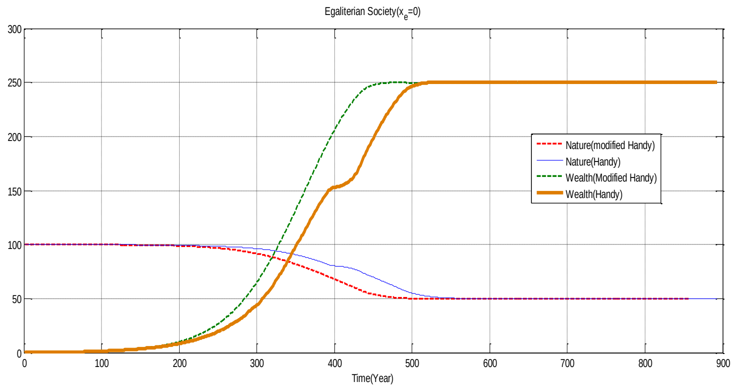

Fig 2b.Simulation results which compare wealth and nature result for Egalitarian Society where there is no elite (Soft Landing Solution).

As it can be seen from the graph, simulation results of the HANDY model are not too much affected and the modification and discontinuity in the model can thus be eliminated.

Attempting to calculate the Lyapunov exponents by the Wolf algorithm where the discontinuities are implemented in the code gives the following rather meaningless result, since the Lyapunov exponents should not depend on the initial conditions. Replacing the discontinuities by their continuous approximations involving the arctan function gives no positive Lyapunov exponents and corrects this defect. This implies the lack of chaotic behavior. The Poincare sections also confirm the lack of chaotic behavior.

initial conditions	**Λ1**	**Λ2**	**Λ3**	**Λ4**
$X_c(0)$=100,$X_e(0)$=0 Y(0)=100, W(0)=0	**0.1988668**	**0.0248482**	**-0.738032**	**-1.2567**
$X_c(0)$=10, $X_e(0)$=0 Y(0)=10,W(0)=0	**0.9456494**	**0.0288546**	**-0.07919**	**-1.0961804**
$X_c(0)$=100, $X_e(0)$=10 Y(0)=100, W(0)=0	**0.0119613 8**	**-0.0114135**	**- 0.140219 1**	**-1.0534976**
$X_c(0)$=100, $X_e(0)$)=10 Y(0)=100, W(0)=100	**0.1245292**	**0.0215223**	**- 0.071940 3**	**-1.2414130**
$X_c(0)$=100, $X_e(0)$= 20 Y(0)=200, W(0)=100	**0.1260577**	**0.019006**	**- 0.087243 4**	**-1.2701920**

Table 1:
Lyapunov Exponents of HANDY System for different initial conditions

3 Sprott Model

The other nonlinear model that Sprott proposed is one of the several that have been proposed by this author. It belongs to a restricted class of three dimensional dynamical systems, referred to as jerky dynamics. These are ordinary differential equations in one scalar real dynamical variable x(t) which are of third order, explicit and autonomous. Their functional form reads $\dddot{x} = J(x, \dot{x}, \ddot{x})$ where $\dddot{x}$, is mechanically the rate of change of acceleration or the jerk function. Under certain restrictions, jerky dynamics can be interpreted as the direct extension of a one-dimensional Newtonian dynamics to spatially or temporally non-local forces, such as radiation reaction.

The nonlinear Sprott system of interest in this work is given below:

$$\dddot{x} + \ddot{x} + x + f(\dot{x}) = 0$$

where $f(\dot{x}) = sign(1 + 4\dot{x})$

When we introduce the following change of variables, we get the following set of equation, suitable for simulations:

$$\dot{x} = y$$
$$\dot{y} = z$$
$$\dot{z} = -y - x - f(1 + 4y)$$

A simulation of the given system yields the graph shown in Fig 3.

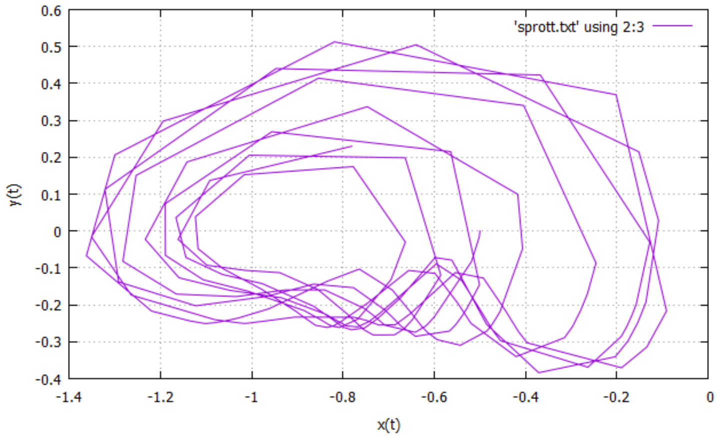

Fig3.Simulation results for Sprott System x(t) vs. y(t) with the initial conditions x(0) = −0.5, y(0) = z(0) = 0

We work on this system to try to find the possible chaotic behavior using the TISEAN package [6]. In Fig 4 it can be seen that delay time for the Sprott system is close to 5. In Fig5, the result of False Nearest neighbors analysis is plotted and from there it can be claimed that the embedding dimension is bigger than 5.

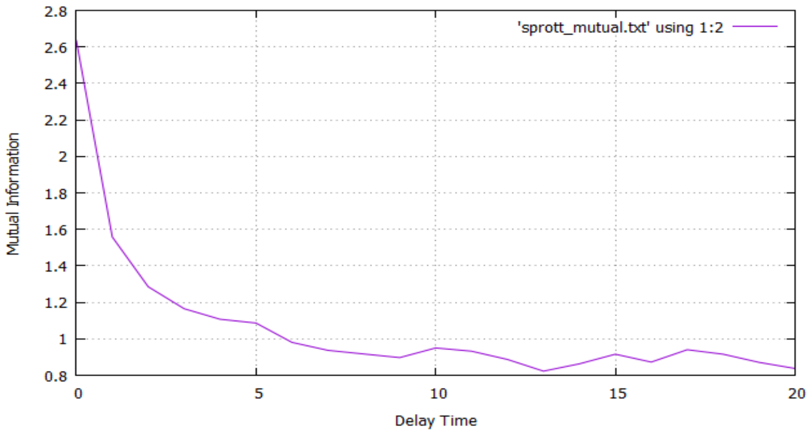

Fig4. Mutual Information

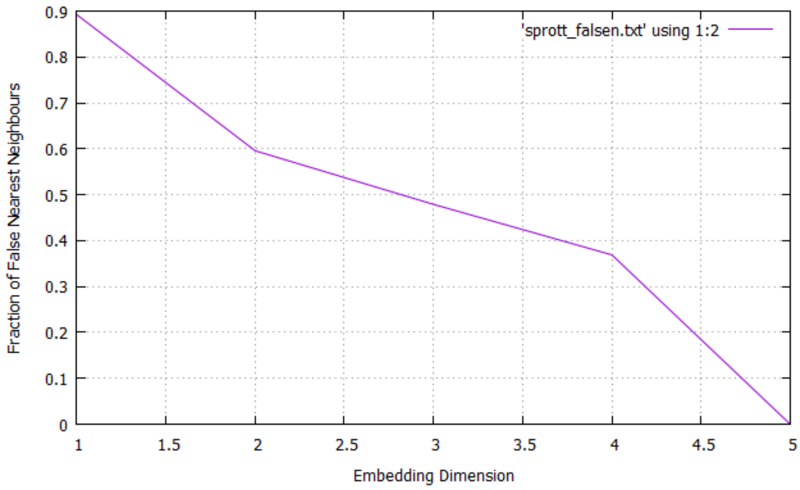

Fig5.False Nearest Neighbors Analysis

We have calculated the largest Lyapunov Exponent for the Sprott system and find its value is 1.34933 which shows that this is system presents chaotic behavior.(Fig6).

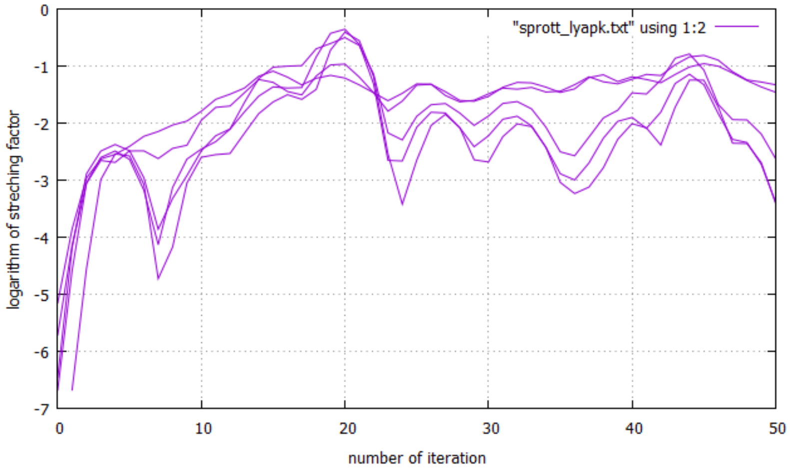

Fig 6.Maximal Lyapunov Exponent

The Lyapunov spectrum obtained by direct use of the Wolf simulation gives the following values displayed in Table2, thus confirming the presence of chaotic behavior.

Λ_1	Λ_2	Λ_3
0.4360020	0.4235678	-0.08662078
Table 2.Lyapunov Exponents for the given Sprott System		

We also calculated the Lyapunov exponents of the Sprott system by changing the discontinuous function with both tanh(a*x) and 2/π arctan(a*x) functions and we got the following results in Table 3 and Table 4.

A	Λ_1	Λ_2	Λ_3
10	0.4190020	0.4075673	-0.07841198
8	0.4112314	0.4000936	-0.071223512
6	0.3898712	0.39634	-0.42424063
4	0.3877567	0.3745321	-0.21213898
2	0.3764682	0.3675292	-0.00162423
1	0.3701262	0.35325125	0.01261420
Table 3.Lyapunov Exponent for Modified Sprott System (by **2/π arctan(a*x))**			

As it can be seen from Table 3 and 4, changing discontinuous function in Sprott System with tanh(a*x) and arctan(a*x) functions gives similar results and both choices give satisfactory results since both lead to results that are close to the Lyapunov Exponents of the original system.

A	Λ_1	Λ_2	Λ_3
10	0.4260020	0.4135678	-0.07662078
8	0.4120014	0.4012876	-0.075662708
6	0.3998712	0.3987625	-0.45376685
4	0.3877567	0.3766513	-0.24567602
2	0.3775645	0.3688565	-0.00156670
1	0.3847020	0.3635767	0.01423355

Table 4.Lyapunov Exponent for Modified Sprott System (by **tanh(a*x))**

Conclusions

It has been demonstrated that in a number of models on discontinuous dynamical models, difficulties have been encountered in calculating Lyapunov exponents. Sprott acknowledges these difficulties and has an approach parallel to that used in this work, while Handy and similar ecological models do not mention chaotic behavior or Lyapunov exponents, since in most cases only a stable limit cycle (or otherwise) is sufficient. Nevertheless, an attempt has been made to replace discontinuities by functions that nearly approximate it and bypass these difficulties. In the original form of the discontinuous Handy model, the Wolf algorithm gives meaningless results while replacing the discontinuities by the arc tangent function corrects this defect, while not affecting the trajectory seriously. It is clear that the system is not chaotic; this fact can also be confirmed by looking at the trajectories in their paper. The Sprott system can also be handled by replacing the discontinuities by the arc tangent function. One can adjust the steepness of the step function by changing the free parameter. The change also helps alleviate possible stiffness problems in the numerical integration of trajectories coming from discontinuous derivatives. Thus, a solution to the differentiability problem in discontinuous models has been proposed. This should aid other dynamical systems calculations with such models such as central manifolds, normal forms or bifurcation analysis which relies on the existence of derivatives.

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