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Nonlinear Dynamics of Two-Dimensional Chaos Map and Fractal Set for Snow Crystal

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Abstract. In this paper, it is shown firstly that the one-dimensional (1-D) generalized logistic map obtained on the basis of a generalized logistic function for population growth has originally a discrete dynamical property. From the 1-D exact chaos solution, 2-D and 3-D chaos maps including the Mandelbrot map and the Julia map in terms of real variables are derived, and 2-D maps related to the Henon map, the Lorenz map and the Helleman map are obtained. Finally, a 2-D chaos map and the fractal set constructed from a 1-D exact chaos solution are considered for the physical analogue of snow crystal, and nonlinear dynamics on the fractal set are discussed by iterating numerically the 2-D map.
Keywords: Chaos map, Fractal set, Logistic map, Mandelbrot map, Julia map, Henon map, Lorenz map, Helleman map, Snow crystal, Nonlinear dynamics.

1 Introduction

For the study of nonlinear phenomena, it is known that simplest nonlinear difference equations have arisen in the field of biological, economic and social sciences, and possess a rich spectrum of dynamical behavior as chaos in many respects [1-3]. A population growth is modeled as a special example, and has been afforded by the nonlinear difference equation called the logistic map. Particularly, for one-dimensional (1-D) chaos maps, a bifurcation diagram of the two parameter quadratic family has been observed [4], and the self-adjusting logistic map with a slowly changing parameter in time have been considered [5]. Moreover, the logistic map with a periodically modulated parameter has been presented [6]. In the meantime, various chaotic sequences have been proposed for the generation of pseudo-random numbers, and for the application to cryptosystems [7-9].
At the same time, a family of shapes and many other irregular patterns in nature called fractals has been discussed for the geometric representation, as an irregular set consisting of parts similar to the whole [10-12]. However, since the Mandelbrot map is defined as a complex map, it has been pointed out that the physics of fractals is a research subject to be born [13]. In addition, chaotic and

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fractal dynamics have been expanded to experimental observations with the mathematical models [14], and fractal compression has been presented to compress images using fractals [15]. Recently, a construction method of 3-D chaos maps has been proposed, and the fractal sets with physical analogue have been shown numerically [16].
In this paper, we derive a generalized logistic map from a generalized logistic function for population growth, and discuss the dynamical behavior of the map in Section 2. Then, by introducing the 1-D exact chaos solution, we construct 2-D and 3-D chaos maps including the Mandelbrot map and the Julia map in terms of real variables, and 2-D maps related to the Henon map, the Lorenz map and the Helleman map are obtained in Section 3. Finally, a 2-D chaos map and the fractal set are considered for the physical analogue of snow crystal, and nonlinear dynamics on the fractal set are discussed by iterating the 2-D map in Section 4. The last Section is devoted to conclusions.

2 A Generalized Logistic Map

Firstly, we introduce a generalized logistic function $P(t)$ as

$$P(t) = \frac{a}{b + e^{-ct}} + d, \tag{1}$$

with the time $t>0$, real constants $\{ a \neq 0, b > 0, c \neq 0, d \geq 0 \}$, and a constant population growth term d. By differentiating (1), we have the first order differential equation;

$$\frac{dP}{dt} = (\frac{bc}{a})P\left[(\frac{a+2bd}{b}) - P\right] - (\frac{cd}{a})(a+bd), \tag{2}$$

and by a variable transformation;

$$X(t) \equiv P(t) / \frac{(a+2bd)}{b}, \tag{3}$$

and the difference method;

$$\frac{dX}{dt} \approx \frac{X_{n+1} - X_n}{\Delta t}, \quad n = 0,1,2,\cdots, \tag{4}$$

with $X_n \equiv X(t)$ and the time step $\Delta t > 0$, we find

$$\frac{X_{n+1} - X_n}{\Delta t} \equiv (\frac{bc}{d})(\frac{a+2bd}{b})X_n(1 - X_n) - (\frac{bcd}{a})(\frac{a+bd}{a+2bd}). \tag{5}$$

Then, by the variable transformation;

$$x_n \equiv (B_0 / A) X_n, \tag{6}$$

with $A \equiv 1 + \Delta t(c/a)(a+2bd)$ and $B_0 \equiv \Delta t(c/a)(a+2bd)$, we arrive at a 1-D generalized logistic map [16];

$$x_{n+1} = Ax_n(1 - x_n) + B, \tag{7}$$

here

$$A \equiv 1 + \frac{c(a + 2bd)}{a}(\Delta t), \tag{8}$$

$$B \equiv \frac{-bc^2 d(a + bd)(\Delta t)^2}{a[a + c(a + 2bd)(\Delta t)]}, \tag{9}$$

which gives a discrete nonlinear system. If d = 0 in (8) and (9), then (7) yields the logistic map $x_{n+1} = Ax_n(1 - x_n)$, and the map at A = 4.0 has an exact chaos solution $x_n = \sin^2(C2^n)$ with a real constant $C \neq \pm m\pi / 2^l$ and finite positive integers {l, m}. We call the map $x_{n+1} = 4x_n(1 - x_n)$ the kernel chaos map of (7). Therefore, the constant A in (7) denotes a coefficient of the nonlinear term, and B corresponds to the constant population growth term d of (1).

It is interesting to note that the logistic function has been introduced for the population growth of city in a discussion of the discrete numerical data [17], and has found an application to the field of such as biology, ecology, biomathematics, economics, probability and statistics. Therefore, the function (1) has originally a discrete property for population growth, that is, a discrete nonlinear dynamics.

3 2-D and 3-D Chaos Maps

We have the following three cases to find 2-D and 3-D chaos maps from a 1-D exact chaos solution:

Case 1

From an exact chaos solution;

$$x_n = \sin^2(C2^n) \tag{10}$$

with $C \neq \pm m\pi / 2^l$ and finite positive integers {l, m} to the logistic map $x_{n+1} = 4x_n(1 - x_n)$, we have, by introducing a real parameter $\alpha \neq 0$, as

$$\begin{aligned} x_{n+1} &= 4\cos^2(C2^n)\sin^2(C2^n) \\ &= 4(1-\alpha)\cos^2(C2^n)\sin^2(C2^n) + 4\alpha\cos^2(C2^n)\sin^2(C2^n) \\ &= 4(1-\alpha)x_n(1-x_n) + 4\alpha(x_n - \sin^4(C2^n)) \\ &\equiv 4x_n - 4(1-\alpha)x_n^2 - 4\alpha y_n \end{aligned} \tag{11}$$

with

$$y_n \equiv \sin^4(C2^n). \tag{12}$$

Therefore, we get a 2-D kernel chaos map from (10)-(12);

$$x_{n+1} = 4x_n - 4(1-\alpha)x_n^2 - 4\alpha y_n, \tag{13}$$

$$y_{n+1} = 16(1-x_n)^2 y_n, \tag{14}$$

and a generalized 2-D chaos map, according to the construction method [16], as

$$x_{n+1} = a_1(x_n - (1-\alpha)x_n^2 - \alpha y_n) + b_1, \tag{15}$$

$$y_{n+1} = a_2(1-x_n)^2 y_n + b_2, \tag{16}$$

with real coefficients and constants $\{a_1, a_2, b_1, b_2\}$. Here, the first equation (15) has the same form as the Helleman map;

$$x_{n+1} = 2ax_n + 2x_n^2 - y_n, \tag{17}$$

$$y_{n+1} = x_n, \tag{18}$$

with a real coefficient a, which has been obtained from the motion of a proton in a storage ring with periodic impulses [18].
Moreover, from the exact chaos solution (10), we find the following 3-D map;

$$x_{n+1} = (2\cos(C2^n)\sin(C2^n))^2$$

$$\equiv (2y_n z_n)^2, \tag{19}$$

with

$$y_n \equiv \cos(C2^n), \tag{20}$$

$$z_n \equiv \sin(C2^n), \tag{21}$$

and have a 3-D kernel chaos map from (19)-(21) as

$$x_{n+1} = 4x_n y_n^2, \tag{22}$$

$$y_{n+1} = y_n^2 - x_n, \tag{23}$$

$$z_{n+1} = 2y_n z_n. \tag{24}$$

Therefore, we get a generalized 3-D chaos map;

$$x_{n+1} = a_1 x_n y_n^2 + b_1, \tag{25}$$

$$y_{n+1} = a_2(y_n^2 - x_n) + b_2, \tag{26}$$

$$z_{n+1} = a_3 y_n z_n + b_3, \tag{27}$$

which has been discussed in [16], where $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ are real coefficients and constants.

Case 2

For an exact chaos solution;

$$x_n = \cos(C2^n), \tag{28}$$

we have the following derivation by introducing a real parameter $\alpha \neq 0$ as

$$\begin{aligned} x_{n+1} &= \cos^2(C2^n) - \sin^2(C2^n) \\ &= \cos^2(C2^n) - (1-\alpha)\sin^2(C2^n) - \alpha\sin^2(C2^n) \\ &\equiv -\alpha + (1+\alpha)\cos^2(C2^n) - (1-\alpha)y_n, \end{aligned} \tag{29}$$

with

$$y_n \equiv \sin^2(C2^n). \tag{30}$$

Then, from (28)-(30), we obtain the kernel chaos map as

$$x_{n+1} = -\alpha + (1+\alpha)x_n^2 - (1-\alpha)y_n, \tag{31}$$

$$y_{n+1} = 4x_n^{\ 2} y_n, \tag{32}$$

and a generalized 2-D chaos map;

$$x_{n+1} = a_1(-\alpha + (1+\alpha)x_n^2 - (1-\alpha)y_n) + b_1, \tag{33}$$

$$y_{n+1} = a_2 x_n^{\ 2} y_n + b_2, \tag{34}$$

with real coefficients and constants $\{a_1, a_2, b_1, b_2\}$, where the first equation (33) has the same form as the Henon map [19];

$$x_{n+1} = 1 - ax_n^2 + y_n, \tag{35}$$

$$y_{n+1} = bx_n, \tag{36}$$

with real coefficients $\{a, b\}$, which has been introduced as a simplified model of the Poincare section of the Lorenz model, and is known as one of the most studied maps for dynamical systems.

Here, it is interesting to note that if we define $y_n \equiv \sin(C2^n)$ with $\alpha = 0$ in (29),

we find a generalized 2-D chaos map;

$$x_{n+1} = a_1(x_n^2 - y_n^2) + b_1, \tag{37}$$

$$y_{n+1} = a_2 x_n y_n + b_2, \tag{38}$$

where the case of (a_1, a_2, b_1, b_2)=(1.0, 2.0, x_0, y_0) or (1.0, 2.0, k_1, k_2) with initial values {x_0, y_0} and real parameters {k_1, k_2} corresponds to the Mandelbrot map or the Julia map in terms of real variables, respectively [16].

Case 3

Similarly, for another exact chaos solution;

$$x_n = \sin(C2^n), \tag{39}$$

we have the following derivation;

$$\begin{aligned} x_{n+1} &= 2\cos(C2^n)\sin(C2^n), \\ x_{n+2} &= 2(\cos^2(C2^n) - \sin^2(C2^n))x_{n+1} \\ &= 2x_{n+1}(1-2x_n^2), \end{aligned} \tag{40}$$

$$x_{n+1} \equiv 2x_n(1-2y_n), \tag{41}$$

with

$$y_n \equiv x_{n-1}^2. \tag{42}$$

Then, we find a 2-D kernel chaos map from (41) and (42) as

$$x_{n+1} = 2x_n(1-2y_n), \tag{43}$$

$$y_{n+1} = x_n^2, \tag{44}$$

and a generalized 2-D chaos map;

$$x_{n+1} = a_1(x_n - 2x_n y_n) + b_1, \tag{45}$$

$$y_{n+1} = a_2 x_n^2 + b_2, \tag{46}$$

with real coefficients and constants {a_1, a_2, b_1, b_2}. It is interesting to note that the first equation (45) has the same form as the 2-D Lorenz map [20, 21];

$$x_{n+1} = (1+ab)x_n - bx_n y_n, \tag{47}$$

$$y_{n+1} = bx_n^2 + (1-b)y_n, \tag{48}$$

with real coefficients {a, b}, which is known to have chaotic dynamics.

Thus, it is found that the 2-D chaos maps derived from 1-D exact chaos solutions (10), (28) and (39) include the Mandelbrot map and the Julia map, and are related to the Helleman map, the Henon map and the 2-D Lorenz map, which give chaotic behaviors and nonlinear dynamics.

4 Nonlinear Dynamics for Snow Crystal

According to the approach presented in Section 3, we introduce a 1-D exact chaos solution;

$$x_n = \cos(C6^n), \tag{49}$$

to the kernel chaos map;

$$x_{n+1} \equiv f(x_n, y_n), \tag{50}$$

with

$$y_n \equiv \sin(C6^n), \tag{51}$$

$$x_n^2 + y_n^2 = 1, \tag{52}$$

and find a generalized 2-D chaos map as

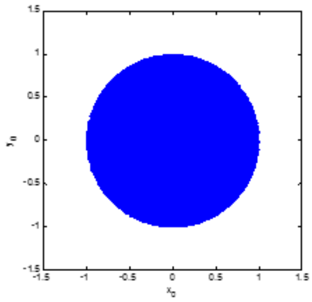
$$x_{n+1} = a_1(x_n^6 - 15x_n^4 y_n^2 + 15x_n^2 y_n^4 - y_n^6) + k_1, \tag{53}$$

$$y_{n+1} = a_2(6x_n^5 y_n - 20x_n^3 y_n^3 + 6x_n y_n^5) + k_2, \tag{54}$$

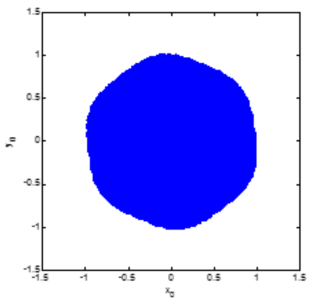
with real coefficients and constants $\{a_1, a_2, k_1, k_2\}$.
Then, the fractal set is defined by

$$M = \{x_0, y_0 \in \mathbf{R} \mid \lim_{n\to\infty} x_n, y_n < \infty\}, \tag{55}$$

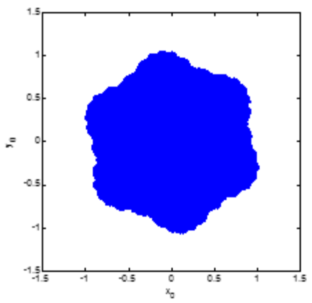
where $\{x_0, y_0\}$ are initial values, and the fractal sets are illustrated in Figure 1, which depend on the constant parameters $\{k_1, k_2\}$. The fractal set (a) of Figure 1 gives a circle under the condition (52), and (b)-(e) show how the fractal set (a) grows as a physical analogue toward a natural snow crystal (f), which is a six-cornered dendrite-type depending on the temperature and the saturation in environment [22, 23]. Here, for calculating the fractal set *M*, we introduce an iteration number *n*=300 to obtain each element of *M* under the convergence condition $x_n^2 + y_n^2 < 4.0$ for the map (53) and (54), and the numerical calculation software MATLAB.
Each fractal set illustrated in Figure 1 is a set of initial value point (x_0, y_0) defined by (55) under the condition $x_n^2 + y_n^2 < 4.0$. For nonlinear dynamics of the 2-D chaos map (53) and (54), orbits of (x_n, y_n) governed by the map are calculated and shown on the fractal set of initial values in Figure 2, where (a); *n*=0, 1 illustrates orbits from each initial point (x_0, y_0) to the (x_1, y_1), (b); *n*=0, 1, 2, 3 from (x_0, y_0) to (x_3, y_3), and (c); *n*=0, 1, ..., 5 from (x_0, y_0) to (x_5, y_5), for all the initial points. It is found that the orbits are complex, and seem like colleding of water molecules. Here, the orbits show that we have (x_1, y_1) as the case of *n*=0 from (53); $x_1=f(x_0, y_0)+k_1$ and (54); $y_1=g(x_0, y_0)+k_2$, (x_2, y_2) from



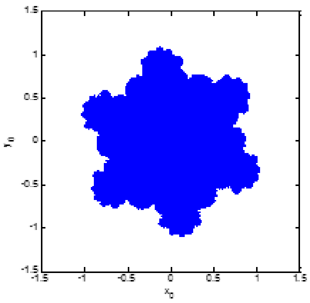
(a) (a_1, a_2, k_1, k_2)=(1.0, 1.0, 0.0, 0.0)



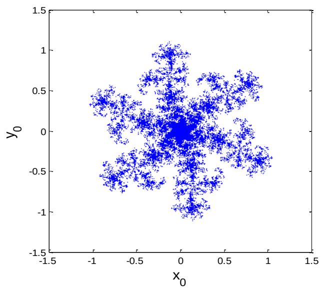
(b) (a_1, a_2, k_1, k_2)=(1.0, 1.0, 0.1, 0.1)



(c) (a_1, a_2, k_1, k_2)=(1.0, 1.0, 0.3, 0.3)



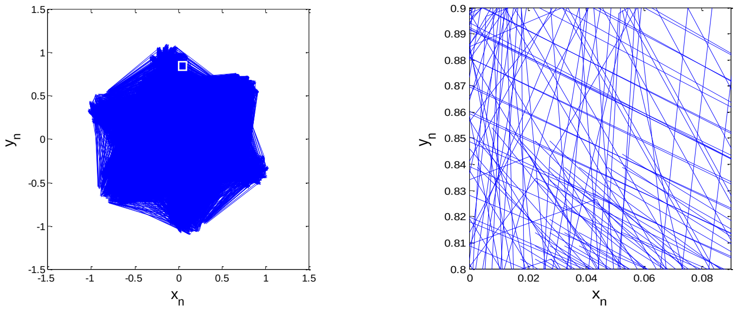
(d) (a_1, a_2, k_1, k_2)=(1.0, 1.0, 0.5, 0.5)



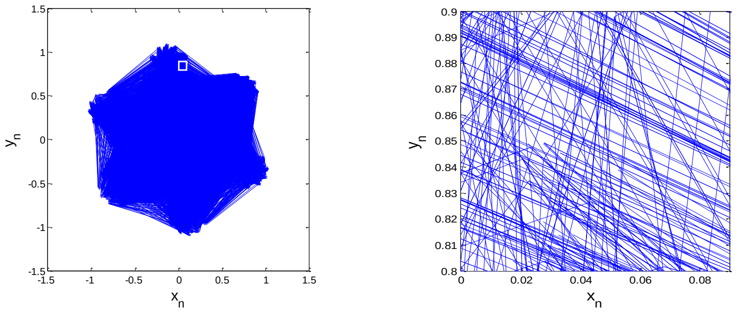
(e) (a_1, a_2, k_1, k_2)
=(1.0, 1.0, 0.58158, 0.58158)

(f) natural snow crystal [23]

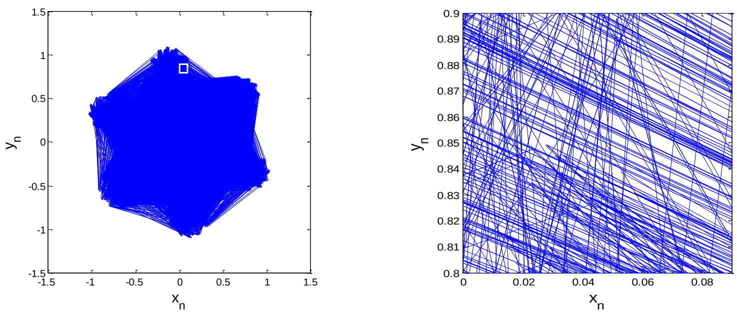
Fig. 1. Fractal sets of the map (53) and (54) for snow crystal.



(a) Orbits for *n*=0, 1 and of a small framed region.



(b) Orbits for *n*=0, 1, 2, 3 and of a small framed region.



(c) Orbits for *n*=0, 1, 2, 3, 4, 5 and of a small framed region.

Fig. 2. Orbits of (x_n, y_n) with n=0, 1, 2, 3, 4, 5 given by the map (53) and (54) on the fractal set (Figure 1 (e)).

$x_2=f(x_1, y_1)+k_1$ and $y_1=g(x_0, y_0)+k_2$, ..., and (x_{300}, y_{300}) from $x_{300}=f(x_{299}, y_{299})+k_1$ and $y_{300}=g(x_{299}, y_{299})+k_2$, under the condition $x_n^2 + y_n^2 < 4.0$. Then, we get one element of the fractal set, and find that $\{(x_1, y_1), (x_2, y_2), ..., (x_{300}, y_{300})\}$ are other initial value points satisfying the condition $x_n^2 + y_n^2 < 4.0$ for the fractal set.

Thus, if the orbits shown in Figure 2 correspond to the dynamics of water molecules colliding with other ones in natural snow crystal, the map (53) and (54) may present the discrete nonlinear dynamics.

Conclusions

We have derived firstly the 1-D generalized logistic map, and have discussed that the map has originally a discrete numerical property for population growth of city. Then, from the 1-D chaos solution, 2-D maps related to the Henon map, the 2-D Lorenz map and the Helleman map, which have chaotic dynamics, have been derived. Furthermore, the 2-D chaos map (53) and (54) gives the fractal set for snow crystal, and orbits of the map on the fractal set have been numerically calculated. As a result, it is found that the 2-D chaos maps derived from 1-D exact chaos solutions have discrete nonlinear dynamics, and may express physical analogues with chaotic property as physics.

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