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The dynamics of Hamiltonians with non-integrable normal form

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Abstract. In general Hamiltonian systems are non-integrable but their dynamics varies considerably depending on the question whether the corresponding normal form is integrable or not. We will explore this issue for two and three degrees of freedom systems; additional remarks on Hamiltonian chains can be found in [9]. A special device, the quadratic part of the Hamiltonian $H_2(p, q)$ is used to illustrate the results.
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1 Integrability versus non-integrability

We will consider time-independent Hamiltonian systems, Hamiltonian $H(p, q)$, $p, q \in \mathbb{R}^n$ with $n \geq 2$ degrees of freedom (DOF). A more detailed study is found in [9]. Regarding mechanics, or more generally dynamical systems, Hamiltonian systems are non-generic.
In addition we have that the existence of an extra independent integral besides the energy for two or more degrees of freedom is again non-generic for Hamiltonian systems (shown by Poincaré in 1892, [4] vol. 1).
So the following question is relevant: why would we bother about the integrability of Hamiltonian systems?
We give a few reasons, leaving out the esthetic arguments:

- Symmetries play a large part in mathematical physics models. Symmetries may sometimes induce integrability but more often integrability of the normal forms. An example is discrete (or mirror) symmetry.
- Near-integrability plays a part in many models of mathematical physics where the integrability, although degenerate , can be a good starting point to analyze the dynamics. Integrals of normal forms may help.
- Non-integrability is too crude a category, it takes many different forms. A first crude characterization is to distinguish non-integrable Hamiltonian systems with integrable or non-integrable normal form.

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2 How to pinpoint (non-)integrability?

Looking for a smoking gun indicating integrability there are a few approaches:

1. Poincaré [4] vol. 1:
   A periodic solution of a time-independent Hamiltonian system has two characteristic exponents zero. A second integral adds two characteristic exponents zero except in singular cases. This can be observed (for an explicit Hamiltonian system) as a continuous family of periodic solutions on the energy manifold. Finding such a continuous family can be either a special degeneration of the system or a sign of the existence of an extra integral.
2. Symmetries of course; strong symmetries like spherical or axial symmetry induce extra integrals. Weaker symmetries may or may not induce an integral. An example is studied in [6] where discrete symmetry is explored in two degrees of freedom systems. It is shown for instance that the spring-pendulum displays many degenerations depending on the resonance studied.
3. Degenerations in variational equations or bifurcations are degenerations that often suggest the presence of integrals.

3 Normal forms

There are many papers and books on normalization. A rather complete introduction is [5]. One considers k-jets of Hamiltonians:

$$H(p, q) = H_2 + H_3 + \ldots + H_k,$$

usually in the neighbourhood of stable equilibrium $(p, q) = (0, 0)$. The H_m are homogeneous polynomials in the p, q variables.

An important feature is that $H_2(p, q)(t)$ is an independent normal form integral, see [5]; its physical interpretation is that H_2 is the energy of the linearized equations of motion. The implication of the existence of this integral is that near stable equilibrium, for two DOF, the normal form is for all resonance ratios integrable so that chaos has for two DOF near stable equilibrium generally a smaller than algebraic measure. This explains a lot of analytic and numerical results in the literature (see again [5]).

In general, for more than two DOF, integrability of the normal form can not be expected without additional assumptions. If we find integrability, it restricts the amount of chaos and also of Arnold diffusion.

Example: Braun's parameter family
Two DOF normal forms are integrable but it is still instructive to consider them. An exampe is Braun's family of Hamiltonians:

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + \omega^2 q_2^2) - \frac{a_1}{3} q_1^3 - a_2 q_1 q_2^2.$$

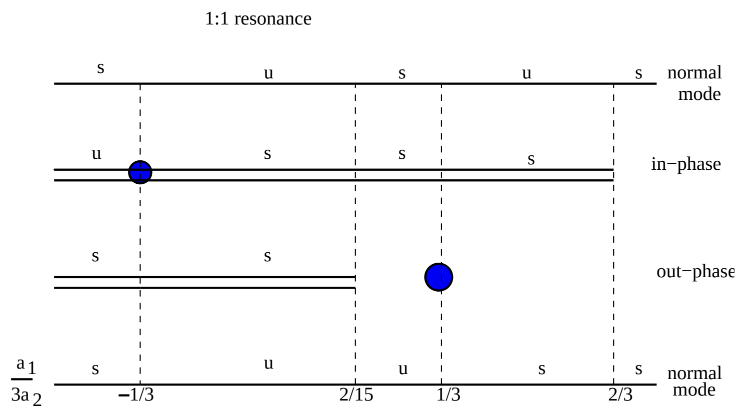

Fig. 1. Periodic solutions obtained from the normal form to cubic terms of Braun's family of discrete-symmetric Hamiltonians. The horizontal lines correspond with isolated periodic solutions on the energy manifold, dots at $-1/3$ (the Hénon-Heiles case) and at $1/3$ correspond with continuous families of periodic solutions.

The analysis is given in [8] and summarized in [5]; consider for instance $\omega = 1$, a_1 and $a_2 \neq 0$ are parameters. The normal form to cubic terms produces two normal modes and, depending on the parameters, two families of in-phase periodic solutions, two families of out-phase periodic solutions and for specific parameter values two continuous families of periodic solutions on the energy manifold; see fig. 1. Normalizing to quartic terms the continuous family at $a_1/(3a_2) = -1/3$ (the Hénon-Heiles Hamiltonian) breaks up into separate periodic solutions; the continuous family at $a_1/(3a_2) = 1/3$ persists, this Hamiltonian before normalization is already integrable.

4 Three degrees of freedom

Genuine first-order resonances are characterized by its normal form. Apart from the three actions, this contains at least two independent combination angles. We have for three DOF:

- 1 : 2 : 1 resonance
- 1 : 2 : 2 resonance
- 1 : 2 : 3 resonance
- 1 : 2 : 4 resonance

A basic analysis of the normal forms to cubic order $\bar{H} = H_2 + \bar{H}_3$ yields short-periodic solutions and integrals. The use of integrals gives insight in the geometry of the flow, enables possible application of the KAM-theorem and may produce measure-theoretic restrictions on chaos.

5 Integrability of normal forms

The normal form has two integrals, H_2 and $\bar{H}$ (or $\bar{H}_3$). Is there a third integral? To establish (non-)integrability we have:

- Ingenious inspection of the normal form or obvious signs of integrability, see van der Aa and F.V. [7].
- Extension into the complex domain and analysis of singularities, see Duistermaat [2].
- Applying Shilnikov-Devaney theory to establish the existence of a transverse homoclinic orbit on the energy manifold, see Hoveijn and F.V. [3].
- Using Ziglin-Morales-Ramis theory to study the monodromy group of a particular nontrivial solution; this study may lead to non-integrability. This involves the variational equation and the characteristic exponents in the spirit of Poincaré. In an extension one introduces the differential Galois group associated with a particular solution; if it is non-commutative, the system is non-integrable. See Christov [1].

5.1 The genuine first-order resonances

A remarkable result is that the normal form to cubic terms of the $1:2:2$ resonance is integrable with quadratic third integral, see [7]. We have that $p_1 = q_1 = 0$ corresponds with an invariant manifold of the normal form; the manifold consists of a continuous set of periodic solutions and is a degeneration according to Poincaré with 4 characteristic exponents zero. The calculation of the normal form to quartic terms produces a break-up of this continuous set into six periodic solutions on the energy manifold.

It was shown in [2] that the normal form to cubic terms of the $1:2:1$ resonance is non-integrable. This was shown by singularity analysis in the complex domain. A different approach was used in [1] where it was shown that for a particular solution the monodromy group is not Abelian; this precludes that the normal form is integrable by meromorphic integrals.

Non-integrability was shown in [1] for the $1:2:4$ resonance in a similar way. One identifies a particular solution in the $(p_1, q_1) = (0, 0)$ submanifold; the local monodromy group is not Abelian which precludes integrability.

The case of the $1:2:3$ resonance is different. The analysis in [3] shows that a complex unstable normal mode (p_2, q_2) is present. The normal form contains an invariant manifold N defined by $H_2 = E_0, \bar{H}_3 = 0$. N contains an invariant ellipsoid, also homoclinic and heteroclinic solutions. They are forming an organizing center producing a horseshoe map and chaos in $H_2 + \bar{H}_3 + \bar{H}_4$. So, the normal form contains only two integrals.

Later, Christov [1] showed by algebraic methods that $H_2 + \bar{H}_3$ is already non-integrable, but the consequences for the dynamics are not yet clear. Technically, this is his most complicated case.

6 Discussion and consequences

In two DOF the Hamiltonian normal form is integrable to any order; this restricts the chaos near stable equilibrium to exponentially small sets between the invariant tori.

For three and more DOF, the situation is more complicated. If the normal form is integrable, chaos is restricted to sets that are algebraically small with respect to the small parameter that scales the energy with respect to stable equilibrium. We would like to distinguish between various kinds of non-integrability near equilibrium. The chaos is usually localized near homoclinic intersections of stable and unstable manifolds.
The phenomenon is most striking after a Hamiltonian-Hopf bifurcation of a periodic solution has taken place, see for the bifurcation diagram fig. 2.

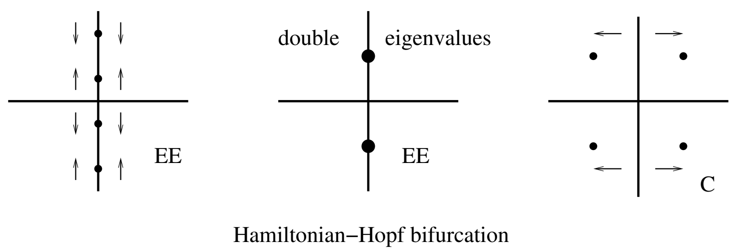

Fig. 2. As a parameter varies, eigenvalues on the imaginary axis become coincident and then move into the complex plane.

Consider the following explicit examples of the $1:2:3$ resonance:

$$H(p,q) = \frac{1}{2}(p_1^2+q_1^2) + (p_2^2+q_2^2) + \frac{3}{2}(p_3^2+q_3^2) + H_3(p,q),$$

$$H_3(p,q) = -q_1^2(a_2q_2+a_3q_3) - q_2^2(c_1q_1+c_3q_3) - bq_1q_2q_3.$$

We will consider two cases. If $a_2 > b$, analysis of the normal form shows that the (p_2, q_2) normal mode is unstable of type HH (hyperbolic-hyperbolic or 4 real eigenvalues). If $a_2 < b$ the (p_2, q_2) normal mode is unstable of type C (complex eigenvalues). The $H_2(p,q)(t)$ time series is shown in figs. 3 and fig 4. Both time series display chaotic behavior, but the case of instability C involves the Devaney-Shilnikov bifurcation producing strong chaotic behaviour; for more information see [3].

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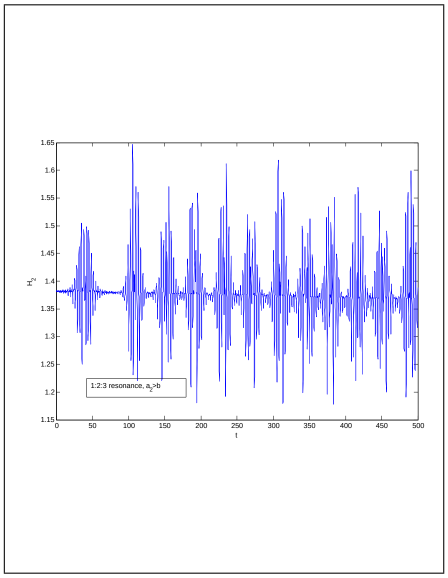

Fig. 3. The $H_2(p,q)(t)$ time series in the case of normal mode instability HH.

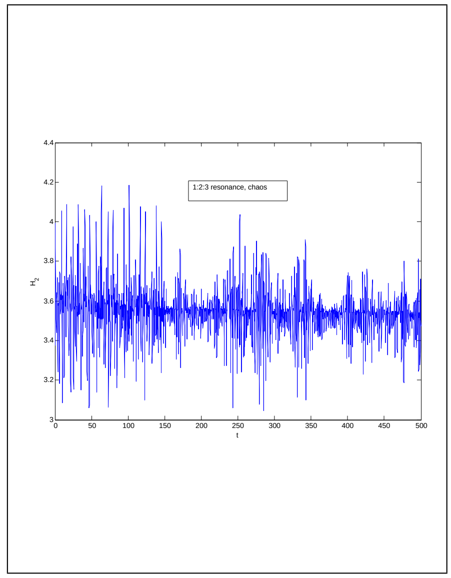

Fig. 4. The $H_2(p,q)(t)$ time series in the case of normal mode instability C.

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