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Numerical calculation of the infinite cluster and its backbone fractal dimension for a square network of percolation

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Abstract. We calculate the fractal dimension of the infinite cluster, minimal path and its backbone for a square network of percolation by using box counting algorithms. We find for d=2, Dmin= 1.136; Df= 1.7197 ; Dbb= 1.63.
The numerical values of these exponents are in good agreement with recent theoretical predictions.
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Keywords: percolation- backbone- infinite cluster- fractal dimension- Monte Carlo

1 Introduction

Usually, the start of percolation theory is associated with a 1957 publication of Broadbent and Hammersley which introduced the name and dealt with it more mathematically, using the geometrical and probabilistic concepts[1]. Percolation is an important area of research in physics (e.g. Stauffer and Aharony,[2]; Bunde and Havlin,[3]; Stanley *et al*.,[4]. Applications of this research are exceedingly broad and range from modeling forest fires to predicting human social phenomena [5] (Solomon *et al*., 2000). Berkowitz and Balberg [6] and Berkowitz and B., Klafter [7] have reviewed applications in groundwater hydrology and soil physics. Sahimi [8] has discussed many other applications. In most of the problems adressed, only one parameter is used; the bond probability or the site occupancy probability. Percolation theory deals with the size of the clusters when this parmeter varies.
In the bond problem, each bond of L is occupied (or open, or working, etc) with probability p or vacant(or blocked, or defective, etc) with probability 1-p. Occupied bonds are connected if they meet at a common site, and a connected

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set of s bonds form a bond cluster of size s. In the site problem, each site is occupied with probability p and vacant with probability 1-p. Occupied sites are connected to form site clusters if they are adjacent through the bonds of L
For an infinite lattice L there is a critical probability $P_c = P_c(b, L)\ or\ P_c(s, L)$ for the bond or site problems such that for $p < p_c$, all clusters wil be finite while for $p > p_c$ there will, with positive probability be an infinite cluster in L. The infinite cluster is called a percolation cluster. We can define the percolation probability, P_∞, as the probability that a site chosen at random, belongs to an infinite percolating network. Then P_∞, vanishes below p_c and is nonzero above p_c; close to $p_c$we can define a "critical exponent" β by postulating $P_\infty \propto (p - p_c)^\beta$ for p slightly above pc. For p>pc in the vicinity of percolation a very large number of sites connected to the infinite cluster are located on what are called dead ends which are connected to the infinite cluster at only one point[9]. If current were to flow only through bonds that connect one side of the system to the other, these dead ends would be avoided. If all dead ends are pruned from the cluster, what remains is called the backbone which coincides with the minimal path.
In percolation theory, the structure of clusters can be described efficiently with the concept of fractal that we describe briefly in this paper.
The main purpose of this work is to implement some algorithms founded on Monte-Carlo methods in order to simulate an infinite percolating network and its backbone and to calculate the fractal dimension of each both.

2 Models

The theory of fractals has become one of the most researched ones in modern times, and there are applications all over society. In technology, in physics, in computer graphics, in art.
Mandelbrot [10] introduced the word "fractals" to describe objects with fractal dimensions D_f smaller than the Euclidean dimensionality d of the underlying lattice or space [11]. In general, two kinds of fractals can be distinguished: deterministic fractals (self-similar) such as the family of the Sierpinski gaskets (Gefen ***et*** al[12] and random fractals (statistically self-similar) such as the percolation clusters[10].
As mentioned in the introduction, the infinite cluster (at a percolation threshold) is a fractal object, of fractal dimensionality [13,14]

$$D_f = d - \frac{\beta_p}{v_p} \qquad (1)$$

Where *d* is the Euclidean dimension of the lattice, v_p, β_p are the critical exponents of the percolation transition [13]
To define D_f, we first review some simple ideas of dimension in ordinary Euclidean geometry. Consider a circular or spherical object with a mass *M* and

radius *r*. If the object radius is increased from *r* to 2*r*, the mass of the object is increased by 2^2 if the object is circular, or by 2^3 if the object is spherical. We can express this relation between mass and length as:

$$M(r) \sim r^{D_f} \quad (2)$$

Where D_f is the dimension of the object.
Equation (2) implies that if the linear dimension of an object is increased by a factor k while preserving its shape, then the mass of this object is increased by k^{D_f} . This mass-length scaling relation is closely related to our intuitive understanding of dimension. Note that if the dimension of the object D_f and the dimension of the Euclidean space in which the object is embedded, *d*, are identical, then the mass density

$$\rho(r) \propto \frac{M(r)}{r^d} \quad (3)$$

Scales as

$$\rho(r) \sim r^0$$

An object whose mass-length relation satisfies (3) with $D_f = d$ is said to be compact
We denote objects as fractals if they satisfy equation (2) with a value of D_f different from the spatial dimension *d*. Note that if an object satisfies (2) with $D_f < d$, its density is not the same for all *r*, but scales as:

$$\rho(r) \sim r^{D_f - d} \quad (4)$$

Because $D_f < d$, a fractal object becomes less dense at larger length scales. The scale dependence of the density is a quantitative measure of the ramified or stringy nature of fractal objects. That is, one characteristic of fractal objects is that they have holes of all sizes.
Another important characteristic of fractal objects is that they look the same over a range of length scales. This property of self-similarity or scale invariance means that if we take part of a fractal object and magnify it by the same magnification factor (r) in all directions, the magnified picture is indistinguishable from the original.
The dimension is one of the most important parameters for the description of a fractal curve. It can be calculated by various methods, which we will recall some in this article [14]. They will all give, theoretically, the same result in the limit. But it is important to note that they are not always equivalent. The accuracy of the results depends greatly on the methods used, and some of them, especially the most commonly held, do not provide evaluation of the fractal dimension with an error of 10% at 20% [15]
The fractal properties of the backbone and the minimal paths, which are particularly important for flow and transport processes, must also be considered.

To characterize a fractal set, one must estimate the fractal dimension D_f. The simplest method of measuring D_f is the so-called box counting method, which can be described as follows. The fractal set is completely covered by non-overlapping spheres (in a general sense) of Euclidean size *r*. The number *N(r)* of such spheres required is then plotted and the following relation is used,

$$N(r) \sim r^{-D_f} \quad (5)$$

As we have said previously, the theory of fractals can be treated as a statistical process and so has been modeled a number of times using Monte-Carlo computer simulations [13,16]. Monte Carlo methods tend to be used when it is infeasible or impossible to compute an exact result with a deterministic algorithm.

We present the results of Monte Carlo calculations of the fractal dimension of an infinite percolating network and its backbone .

3 The algorithms

Calculating the fractal dimension of the backbone (skeleton) of the infinite cluster from a square lattice

Our algorithm is based on a very simple idea. Consider a square lattice with N x N sites. A site can be occupied or empty, so its state can be represented using a boolean variable, say true for occupied or false for empty.

The programs starts with each of the sites “closed” and are represented by a black square. As you can plainly see, all the squares are black

When a square turns blue it’s considered to be “full” which it means that the open site (white) was connected to the top of the grid, or connected to another neighboring site (the 4 immediate squares around it) that were connected to the top of the grid. This connecting continues until the grid percolates.

As sites are continuing to be opened, more and more squares begin to get and blue in response. When the systems percolates, this wave of blue squares will form a path from top of the grid to the bottom of the grid (figure 2). So once more sites are opened, it’s only a matter of time until the grid percolates, or forms a path from the top of the grid to the bottom of the grid.

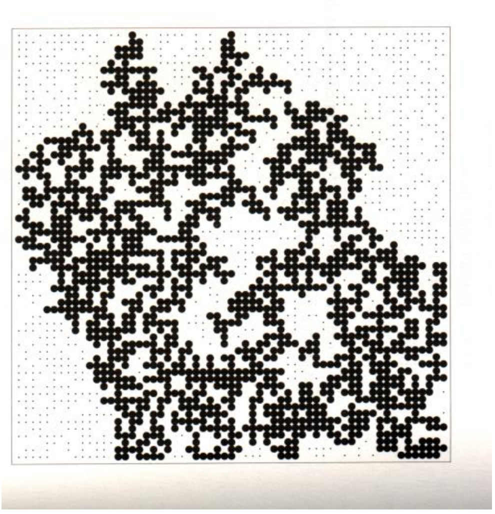


Figure 1: Spanning cluster tenuous and not compact

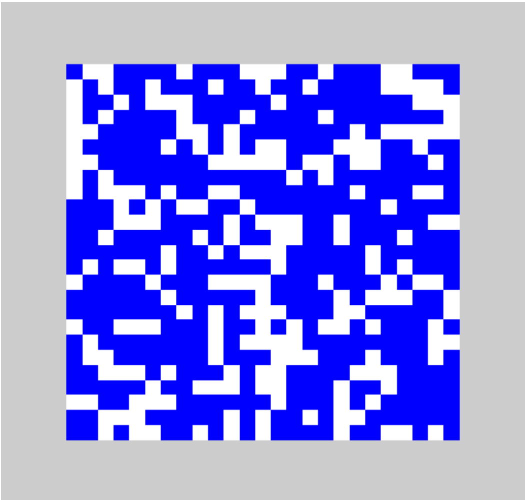


Figure 2: infinite cluster above threshold

Initialize all sites to be blocked.

Repeat the following until the system percolates:

First of all, we randomly generated sites of square lattice by taking a percentage of active sites $p > pc$ (near the percolation threshold) so that we can have finite clusters and the infinite cluster.

First step is to inject the color red in the active sites located in the first line.

A (1, j) = 2 implies A (1, j) = 5

Start condition:

If A (1, j) = 5 and A(2,j)=2 implies A(2,j)=5 and A(1,j) =6

The second step is as follows

If at least one neighbor site 5 is 2, then the site 5 becomes 6 and Site 2 becomes 3.

The instruction is repeated until the last line. The next step is to replace the box 3 per box 5.

The first operation of the second step is repeated until the appearance of box A (a, j) = 3 (the first box in the last line = 3) and box A (a, j) = 3 becomes a box A (a, j) = 6.

Third step is to eliminate all sites that do not have the number 6

If A (i, j) = 5 or A (i, j) = 2, then it becomes A (i, j) = 8, where only the appearance boxes 6.

To eliminate these dead ends we must repeat steps which consist in reverse matrix

A (i, j) = A (a-i +1, a-j +1) and A (i, j) = 6 turns in A (i, j) = 2 and repeat step 1 repeatedly

4 Results

4.1 Results for visualize the minimal path and estimation of its fractal dimension

For square lattice with 60x60 sites

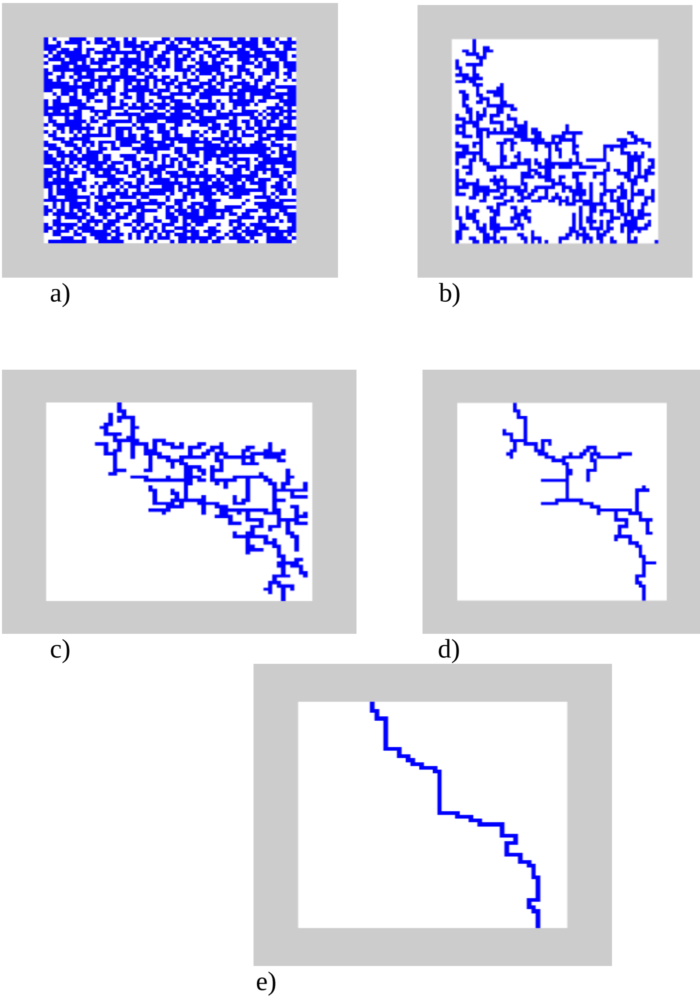
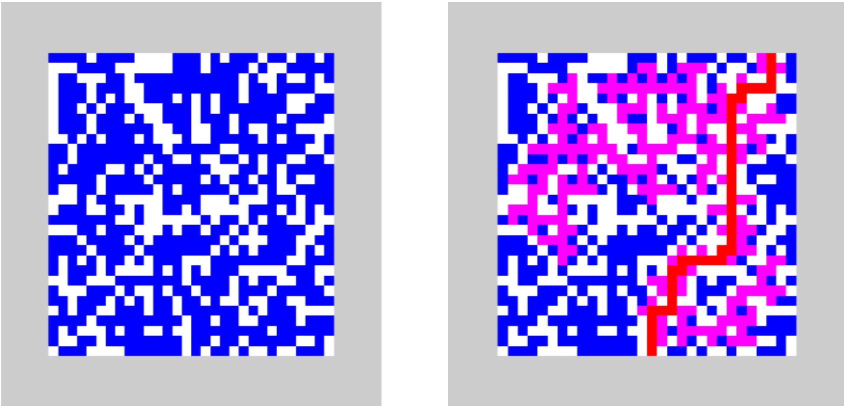


Figure 3: a) above threshold Visualization percolation cluster. b) and c) elimination of certain sites of infinite cluster d) appearance of backbone e) appearance of minimal path . A= 60, Nbr =105 ; Dmin=1.136

Using the box-counting dimension we found Dmin =1.136

4.2 Results for visualize the backbone and estimation the fractal dimension from the percolation threshold for a square lattice percolation.

For A=30



a) Infinite cluster b) appearance of backbone

Figure 4: visualization of Backbone for A=30,N=259,Dbb=1.63

We found Dbb =1.63

4.3 Results for visualize the Infinite cluster and estimation the fractal dimension from the percolation threshold for a square lattice percolation

Network 200x200 size

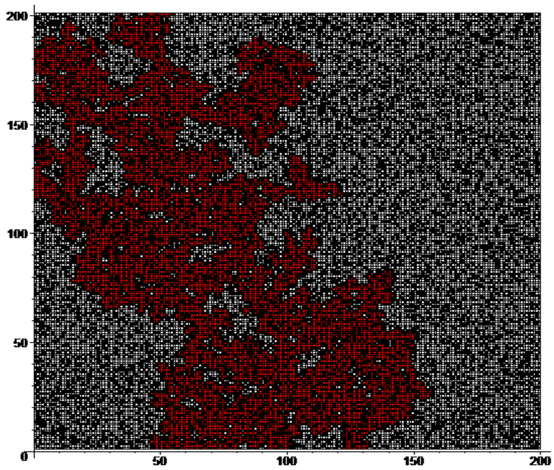


Figure 5: visualization of percolation cluster A=200 ; Df=1.7197

For network 200x200 sizes, we found, the fractal dimension Df=1.7197

Conclusions

In this paper, we have calculated the dimension fractal of the infinite cluster, the backbone and the minimal path for square lattice of sites, using box counting algorithm. Our results for 2D, are agree with some authors

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