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Weak Regularized Solutions to Stochastic Cauchy Problems*

Irina V. Melnikova[1] and Uljana A. Alekseeva[2]

[1] Ural Federal University, Institute of Mathematics and Computer Sciences, 620083 Ekaterinburg, Lenina av., 51, Russia
(E-mail: Irina.Melnikova@urfu.ru)
[2] Ural Federal University, Institute of Mathematics and Computer Sciences, 620083 Ekaterinburg, Lenina av., 51, Russia
(E-mail: Uljana.Alekseeva@urfu.ru)

Abstract. The Cauchy problem for systems of differential equations with stochastic perturbations is studied. Weak regularized solution are constructed for the case of systems with operators generating R-semigroups; generalized and mild solutions are introduced.
Keywords: white noise process; Wiener process; weak, regularized, generalized and mild solutions; Gelfand-Shilov spaces.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a random space. We consider the Cauchy problem for the systems of differential equations with stochastic perturbations:

$$\frac{\partial X(t,x)}{\partial t} = A\left(i\frac{\partial}{\partial x}\right)X(t,x) + B\mathcal{W}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}, \tag{1}$$

$$X(0,x) = f(x), \tag{2}$$

where $A\left(i\frac{\partial}{\partial x}\right)$ is a matrix operator: $A\left(i\frac{\partial}{\partial x}\right) = \left\{A_{jk}\left(i\frac{\partial}{\partial x}\right)\right\}_{j,\,k=1}^{m}$ generating different type systems in the Gelfand-Shilov classification [3], $A_{jk}\left(i\frac{\partial}{\partial x}\right)$ are linear differential operators in $L_2(\mathbb{R})$ of finite orders; $\mathcal{W} = \{\mathcal{W}(t),\ t \geq 0\}$ is a random process of white noise type in $L_2^n(\mathbb{R})$: $\mathcal{W}(t) = (\mathcal{W}_1(t,x,\omega), \ldots \mathcal{W}_n(t,x,\omega))$, $x \in \mathbb{R}$, $\omega \in \Omega$; B is a bounded linear operator from $L_2^n(\mathbb{R})$ to $L_2^m(\mathbb{R})$; f is an $L_2^m(\mathbb{R})$-valued random variable; $X = \{X(t), t \in [0,T]\}$ is an $L_2^m(\mathbb{R})$-valued stochastic process $X(t) = (X_1(t,x,\omega), \ldots X_m(t,x,\omega))$, $x \in \mathbb{R}$, $\omega \in \Omega$, which is to be determined.

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This problem usually is not well-posed for several reasons. The first one is caused by the fact that the differential operators $A\left(i\frac{\partial}{\partial x}\right)$ generally do not generate semigroups of class C_0 and the corresponding homogeneous Cauchy problem is not uniformly well-posed in $L_2^m(\mathbb{R})$, they generate only some regularized semigroups. By this reason we look for a regularized solution of (1)–(2). The second reason is the irregularity of a white noise process, because of this we need to consider not the original equation (1) but the integrated one, that is an equation in the Ito form with a Wiener process W being a kind of primitive of white noise $\mathcal{W}$:

$$X(t,x) = f(x) + \int_0^t A\left(i\frac{\partial}{\partial x}\right) X(\tau,x)\,d\tau + BW(t,x), \quad t\in[0,T],\ x\in\mathbb{R}. \tag{3}$$

In addition, we can not expect the stochastic inhomogeneity be in the domain of $A\left(i\frac{\partial}{\partial x}\right)$, by this reason we have to explore weak regularized solutions to the integrated problem (3).

2 Necessary definitions and preliminary results

We consider the problem (1)–(2) as an important particular case of the abstract Cauchy problem

$$X'(t) = AX(t) + B\mathcal{W}(t), \quad t\in[0,T], \quad X(0)=f, \tag{4}$$

and the problem (3) as that of the abstract integral one (written as usually in the form of differentials):

$$dX(t) = AX(t)dt + BdW(t), \quad t\in[0,T], \quad X(0)=f, \tag{5}$$

with A being the generator of a regularized semigroup in a Hilbert space H, especially an R-semigroup (see, exp., Melnikova[4], Melnikova and Anufrieva[6]). Thus, we continue investigations of Da Prato[2], Melnikova *et al.* [5], Alshanskiy and Melnikova[1]. We assume in this paper $H = L_2^m(\mathbb{R})$.

Definition 1. Let A be a closed operator and R be a bounded linear operator in $L_2^m(\mathbb{R})$ with a densely defined R^{-1}. A strongly continuous family $S := \{S(t),\ t\in[0,\tau)\}$, $\tau\le\infty$, of bounded linear operators in $L_2^m(\mathbb{R})$ is called an *R-regularized semigroup* (or *R-semigroup*) generated by A if

$$S(t)Af = AS(t)f, \quad t\in[0,\tau), \quad f\in\operatorname{dom}A, \tag{6}$$

$$S(t)f = A\int_0^t S(\tau)f\,ds + Rf, \quad t\in[0,\tau), \quad f\in L_2^m(\mathbb{R}). \tag{7}$$

The semigroup is called *local* if $\tau<\infty$.

Definition 2. Let Q be a symmetric nonnegative trace class operator in $L_2^n(\mathbb{R})$. An $L_2^n(\mathbb{R})$-valued stochastic process $\{W(t),\ t\ge 0\}$ is called a Q-Wiener process if

(W1) $W(0)=0$ $P_{\text{a.s.}}$;

(W2) the process has independent increments $W(t)-W(s)$, $0\le s\le t$, with normal distribution $\mathcal{N}(0,(t-s)Q)$;

(W3) $W(t)$ has continuous trajectories $P_{\text{a.s.}}$

Definition 3. Let $\{\mathcal{F}_t, t \le \infty\}$ be a filtration defined by W. An $L_2^m(\mathbb{R})$-valued $\mathcal{F}_t$-measurable process $X = \{X(t), t \in [0,T]\}$ is called a *weak R-solution* of the problem (3) with $A\left(i\frac{\partial}{\partial x}\right)$ generating an R-semigroup $\{S(t),\ t \in [0,\tau)\}$ in $L_2^m(\mathbb{R})$ if the following conditions are fulfilled:

1) for each $t \in [0,T]$, $k = \overline{1,m}$, $\int_0^t \|X_k(\cdot,\tau)\|_{L_2(\mathbb{R})}\, d\tau < \infty$ $P_{\text{a.s.}}$;

2) for each $g \in dom\, A^*$, X satisfies the weak regularized equation:

$$\langle X(t), g\rangle = \langle Rf, g\rangle + \int_0^t \langle X(\tau), A^* g\rangle\, d\tau + \langle RBW(t), g\rangle\ P_{\text{a.s.}}, \quad t \in [0,T]. \quad (8)$$

It is proved by Melnikova and Alshanskiy[1] that a weak R-solution of the abstract stochastic Cauchy problem (5) with densely defined A being the generator of an R-semigroup and W being a Q-Wiener process exists and is unique. In the case of the problem (3) this result is as follows.

Theorem 1. *Let $\{W(t), t \ge 0\}$ be a Q-Wiener process in $L_2^n(\mathbb{R})$ and $A\left(i\frac{\partial}{\partial x}\right)$ be the generator of an R-semigroup $\{S(t), t \in [0,\tau)\}$ in $L_2^m(\mathbb{R})$ satisfying the condition*

$$\int_0^t \|S(\tau)B\|_{HS}^2\, d\tau < \infty, \quad (9)$$

where $\|\cdot\|_{HS}$ is the norm in the space of Hilbert-Schmidt operators acting from the space $Q^{\frac{1}{2}} L_2^n(\mathbb{R})$ to $L_2^m(\mathbb{R})$. Then for each $\mathcal{F}_0$-measurable $L_2^m(\mathbb{R})$-valued random variable f

$$X(t) = S(t)f + \int_0^t S(t-\tau)B\, dW(\tau), \quad t \in [0,T], \quad (10)$$

is the unique weak R-solution of (5).

We see in (10) that the main part of constructing an R-solution is constructing an R-semigroup generated by A. It is not an easy task to construct R-semigroups generated by given operators A in the general case. But for differential operators $A\left(i\frac{\partial}{\partial x}\right)$ such semigroups can be constructed and we describe a way to do this in the present paper.

Our methods are based on investigations of the differential systems:

$$\frac{\partial u(t,x)}{\partial t} = A\left(i\frac{\partial}{\partial x}\right) u(t,x), \quad t \in [0,T],\ x \in \mathbb{R}, \quad (11)$$

provided by the generalized Fourier transform technique in [3]. So, let us apply the Fourier transform to the system (11) and consider the dual one:

$$\frac{\partial \widetilde{u}(t,s)}{\partial t} = A(s)\widetilde{u}(t,s), \quad t \in [0,T],\ s \in \mathbb{C}. \quad (12)$$

Let the functions $\lambda_1(\cdot), \dots, \lambda_m(\cdot)$ be characteristic roots of the system (12) and $\Lambda(s) := \max\limits_{1\le k\le m} \Re\lambda_k(s), s \in \mathbb{C}$. Then solution operators of (12) have the following estimation

$$e^{t\Lambda(s)} \le \left\| e^{tA(s)} \right\|_m \le C(1+|s|)^{p(m-1)} e^{t\Lambda(s)}, \qquad t \ge 0,\ s \in \mathbb{C}. \quad (13)$$

Definition 4. A system (11) is called

1) *correct by Petrovsky* if there exists such a $C > 0$ that $\Lambda(\sigma) \le C,\ \sigma \in \mathbb{R}$;

2) *conditionally–correct* if there exist such constants $C > 0,\ 0 < h < 1,\ C_1 > 0$ that $\Lambda(\sigma) \le C|\sigma|^h + C_1,\ \sigma \in \mathbb{R}$;

3) *incorrect* if the function $\Lambda(\cdot)$ grows for real $s = \sigma$ in the same way as for complex ones: $\Lambda(\sigma) \le C|\sigma|^{p_0} + C_1,\ \sigma \in \mathbb{R}$.

Finally, note that the operator $i\frac{\partial}{\partial x}$ is self-conjugate in $L_2(\mathbb{R})$: $\left(i\frac{\partial}{\partial x}\right)^* = i\frac{\partial}{\partial x}$. Hence the differential operator of (1) has the following conjugate one

$$A^*\left(i\frac{\partial}{\partial x}\right) = \left\{\overline{A_{kj}}\left(i\frac{\partial}{\partial x}\right)\right\}_{k,j=1}^m ,$$

obtained of $\{A_{jk}\left(i\frac{\partial}{\partial x}\right)\}_{j,k=1}^m$ by replacing components with conjugate operators and by further transposition.

3 Construction of R-semigroups generated by $A\left(i\frac{\partial}{\partial x}\right)$

Since for the problem (12) solution operators of multiplication by $e^{tA(\cdot)}, t \ge 0$, generally have an exponential growth (13), one can not obtain propagators of the problem (11) in the framework of the classical inverse Fourier transform. That is why we introduce an appropriate multiplier $K(\cdot)$ into the inverse Fourier transform:

$$G_R(t,x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\sigma x} K(\sigma) e^{tA(\sigma)}\, d\sigma, \tag{14}$$

providing the uniform convergence of this integral with respect to $t \in [0,T]$ in $L_2^m(\mathbb{R}) \times L_2^m(\mathbb{R}) =: \mathbf{L}_2^m$. For this purpose we require $K(\cdot)e^{tA(\cdot)} \in \mathbf{L}_2^m$.

The matrix-function $G_R(t,x)$ obtained in (14) is a regularized Green function. If its convolution with f is well-defined, then the convolution gives a regularized solution of (11). In addition to the above condition, we introduce $K(\cdot)$ providing

$$\int_{-\infty}^{\infty} e^{i\sigma x} K(\sigma) e^{tA(\sigma)} \widetilde{f}(\sigma)\, d\sigma \in L_2^m(\mathbb{R}), \quad t \in [0,T], \tag{15}$$

for each $\widetilde{f} \in L_2^m(\mathbb{R})$. These conditions hold, for example, if $K(\cdot)e^{tA(\cdot)} \in \mathbf{L}_2^m$ and is bounded.

Now we show that the family of convolution operators with $G_R(t,x)$:

$$(S(t)f)(x) := G_R(t,x) * f(x), \quad t \in [0,\tau), \tag{16}$$

forms a local R-semigroup in $L_2^m(\mathbb{R})$ for any $\tau < \infty$. To begin with, we verify the strong continuity property of the family $\{S(t), t \in [0,T]\}, T < \infty$: for arbitrary $f \in L_2^m(\mathbb{R})$ we show that $\|S(t)f - S(t_0)f\|_{L_2^m(\mathbb{R})} \to 0$ as $t \to t_0$.

$$\|S(t)f - S(t_0)f\|^2_{L_2^m(\mathbb{R})} =$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\sigma x} K(\sigma) \left[e^{tA(\sigma)} \widetilde{f}(\sigma) - e^{t_0 A(\sigma)} \widetilde{f}(\sigma) \right] d\sigma \right)^2 dx.$$

Let us split the inner integral into the three integrals:

$$\int_{|\sigma|\geq N} e^{i\sigma x} K(\sigma) e^{tA(\sigma)} \widetilde{f}(\sigma)\, d\sigma - \int_{|\sigma|\geq N} e^{i\sigma x} K(\sigma) e^{t_0 A(\sigma)} \widetilde{f}(\sigma)\, d\sigma$$
$$+ \int_{|\sigma|\leq N} e^{i\sigma x} K(\sigma) \left[e^{tA(\sigma)} - e^{t_0 A(\sigma)} \right] \widetilde{f}(\sigma)\, d\sigma. \tag{17}$$

Note that the functions $h_N(x,t) := \int_{|\sigma|\geq N} e^{i\sigma x} K(\sigma) e^{tA(\sigma)} \widetilde{f}(\sigma)\, d\sigma$ and

$$g_N(x,t) := \int_{|\sigma|\leq N} e^{i\sigma x} K(\sigma) \left[e^{tA(\sigma)} - e^{t_0 A(\sigma)} \right] \widetilde{f}(\sigma)\, d\sigma$$

are elements of $L_2^m(\mathbb{R})$ for all $t \in [0,T]$ as the inverse Fourier transform of the functions from $L_2^m(\mathbb{R})$

$$\widetilde{h}_N(\sigma,t) = \begin{cases} 0, & |\sigma| \leq N, \\ K(\sigma) e^{tA(\sigma)} \widetilde{f}(\sigma), & |\sigma| > N, \end{cases}$$

and $\widetilde{g}_N(\sigma,t) = K(\sigma) e^{tA(\sigma)} \widetilde{f}(\sigma) - \widetilde{h}_N(\sigma,t)$, respectively. Further, since $K(\cdot) e^{tA(\cdot)} \in \mathbf{L}_2^m$ and $\widetilde{f}(\cdot) \in L_2^m(\mathbb{R})$, the integral (15) is convergent uniformly with respect to $x \in \mathbb{R}$ and $t \in [0,T]$, then for any $\varepsilon > 0$

$$|h_N(x,t)| < \varepsilon/4, \quad x \in \mathbb{R},\ t \in [0,T],$$

by the choice of N. So, sum of absolute values of the first two integrals in (17) is less than $\varepsilon/2$. Now fix N. Since $\left(e^{(t-t_0)A(\sigma)} - 1\right) \to 0$ as $t \to t_0$ uniformly with respect to $\sigma \in [-N,N]$, we can take

$$|g_N(x,t)| < \varepsilon/2, \quad x \in \mathbb{R},\ t \in [0,T].$$

To obtain the estimate for

$$\|S(t)f - S(t_0)f\|^2_{L_2^m(\mathbb{R})} = \frac{1}{4\pi^2} \int_{\mathbb{R}} \left(h_N(x,t) - h_N(x,t_0) + g_N(x,t) \right)^2 dx$$

we consider the difference $h_N(x,t) - h_N(x,t_0) =: \Delta_N(x,t,t_0),\ t,t_0 \in [0,T]$, as a single function, then $\Delta_N(\cdot,t,t_0) \in L_2^m(\mathbb{R})$ and for a fixed N by the choice of t_0, $|\Delta_N(x,t,t_0)| < \varepsilon/2, \quad x \in \mathbb{R}$. In these notations we have:

$$4\pi^2 \|S(t)f - S(t_0)f\|^2_{L_2^m(\mathbb{R})} =$$
$$= \int_{\mathbb{R}} \Delta_N^2(x,t,t_0)\, dx + 2\int_{\mathbb{R}} \Delta_N(x,t) g_N(x,t,t_0)\, dx + \int_{\mathbb{R}} g_N^2(x,t)\, dx.$$

On the way described above one can show that every of these three integrals is an infinitesimal value. That is the integrals over the infinite intervals $|x| > M$

are small by the choice of M because of their uniform convergence with respect to $t \in [0,T]$. Integrals on compacts $[-M,M]$ are small because the integrands are small, that provided by the sequential choice of M and $t \in [0,T]$. This completes the proof that operators of the family (16) are strongly continuous.

Next, we show that the obtained operators commute with $A\left(i\frac{\partial}{\partial x}\right)$ on $f \in domA\left(i\frac{\partial}{\partial x}\right)$. By properties of convolution, a differential operator may be applied to any components of convolution, so we apply $A\left(i\frac{\partial}{\partial x}\right)$ to $f \in domA\left(i\frac{\partial}{\partial x}\right)$:

$$A\left(i\frac{\partial}{\partial x}\right)(S(t)f)(x) = G_R(t,x) * A\left(i\frac{\partial}{\partial x}\right)f(x) = S(t)A\left(i\frac{\partial}{\partial x}\right)f(x).$$

Hence, the equality (6) holds. In conclusion, we show the R-semigroup equation (7). For an arbitrary $f \in domA\left(i\frac{\partial}{\partial x}\right)$ consider the equality:

$$\frac{\partial}{\partial t}(S(t)f)(x) = \frac{\partial}{\partial t}\left[G_R(t,x) * f(x)\right] = \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{i\sigma x}K(\sigma)e^{tA(\sigma)}\widetilde{f}(\sigma)\,d\sigma.$$

Since the integral converges uniformly with respect to $t \in [0,T]$, we can differentiate under the integral sign:

$$\frac{\partial}{\partial t}(S(t)f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\sigma x}K(\sigma)e^{tA(\sigma)}A(\sigma)\widetilde{f}(\sigma)\,d\sigma.$$

The condition $f \in domA\left(i\frac{\partial}{\partial x}\right)$ provides $A(\cdot)\widetilde{f}(\cdot) \in L_2^m(\mathbb{R})$, hence the inverse Fourier transform of $A(\sigma)\widetilde{f}(\sigma)$ is $A\left(i\frac{\partial}{\partial x}\right)f(x)$ and

$$\frac{\partial}{\partial t}(S(t)f)(x) = G_R(t,x) * A\left(i\frac{\partial}{\partial x}\right)f(x) =$$

$$= A\left(i\frac{\partial}{\partial x}\right)\left[G_R(t,x) * f(x)\right] = A\left(i\frac{\partial}{\partial x}\right)(S(t)f)(x).$$

Integration with respect to t gives the equality

$$(S(t)f)(x) - (S(0)f)(x) = \int_0^t A\left(i\frac{\partial}{\partial x}\right)(S(\tau)f)(x)\,d\tau.$$

Since $A\left(i\frac{\partial}{\partial x}\right)$ is closed in $L_2^m(\mathbb{R})$ and differentiable functions are dense there, this equality holds for any $f \in L_2^m(\mathbb{R})$:

$$(S(t)f)(x) - (S(0)f)(x) = A\left(i\frac{\partial}{\partial x}\right)\int_0^t (S(\tau)f)(x)\,d\tau, \quad t \in [0,T].$$

Put operator R in $L_2^m(\mathbb{R})$ equal to $S(0)$, then by the strong continuity property,

$$Rf(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\sigma x}K(\sigma)\widetilde{f}(\sigma)\,d\sigma.$$

So, we have an R-semigroup generated by $A\left(i\frac{\partial}{\partial x}\right)$ constructed in $L_2^m(\mathbb{R})$.

Now for all types of systems (11) – correct by Petrovsky, conditionally-correct and incorrect – we introduce appropriate correcting functions $K(\sigma)$ as follows:

- for systems correct by Petrovsky we take $K(\sigma) = \frac{1}{(1+\sigma^2)^{d/2+1}}$, where $d = p(m-1)$,
- for conditionally-correct systems we take $K(\sigma) = e^{-a|\sigma|^h}$, where $a > const{\cdot}T$,
- for incorrect systems — $K(\sigma) = e^{-a|\sigma|^{p_0}}$, where $a > const \cdot T$.

4 Some remarks on generalized solutions and solutions of quasi–linear equations

In the previous section we have studied R-solutions to the problem (5) with differential operators $A\left(i\frac{\partial}{\partial x}\right)$ that are generators of R-semigroups in $H = L_2^m(\mathbb{R})$, and we focused ourselves on the construction of these R-semigroups. If not a regularized, but a genuine solution of the problem is needed, then we have to construct the solution in spaces, where operator R^{-1} is bounded.

How difficult it is to construct R-semigroups in general, we have noted. Constructing the required spaces in the general case, the same challenge. Nevertheless, in the case of the differential operators $A\left(i\frac{\partial}{\partial x}\right)$ suitable spaces can be chosen among those constructed by Gelfand[3] on the basis of the generalized Fourier transform technique. If to take f being an $L_2^m(\mathbb{R})$-valued random variable, for systems correct by Petrovsky we can construct a generalized solution $X(t,\cdot,\omega) = (X_1(t,\cdot,\omega), \dots X_m(t,\cdot,\omega))$, $t \in [0,T]$, $\omega \in \Omega$, in $\mathcal{S}' \times \cdots \times \mathcal{S}'$, where $\mathcal{S}'$ is known as the space of tempered distributions. For conditionally-correct systems these are spaces $\left(\mathcal{S}^{\alpha,\mathcal{A}}_{\beta,\mathcal{B}}\right)'$ of distribution increasing exponentially with order $1/\beta$ dual to $\mathcal{S}^{\alpha,\mathcal{A}}_{\beta,\mathcal{B}}$ — the space of all infinitely differentiable functions satisfying the condition: for any $\varepsilon > 0, \delta > 0$

$$|x^k \varphi^{(q)}(x)| \le C_{\varepsilon,\,\delta}(\mathcal{A}+\varepsilon)^k(\mathcal{B}+\delta)^q k^{k\alpha} q^{q\beta}, \qquad k, q \in \mathbb{N}_0, \quad x \in \mathbb{R},$$

with a constant $C_{\varepsilon,\,\delta} = C_{\varepsilon,\,\delta}(\varphi)$. And for incorrect systems the required space is $\mathcal{Z}'$, that is dual to the space $\mathcal{Z}$ of all entire functions $\varphi(\cdot)$ of argument $z \in \mathbb{C}$, satisfying the condition

$$|z^k \varphi(z)| \le C_k e^{b|y|}, \qquad k \in \mathbb{N}_0, \quad z = x + iy \in \mathbb{C},$$

with some constants $b = b(\varphi)$, $C_k = C_k(\varphi)$.

Now consider the Cauchy problem for a quasi–linear equation:

$$dX(t) = AX(t)dt + F(t,X)dt + BdW(t), \; t \in [0,T], \quad X(0) = f, \qquad (18)$$

with A being the generator of an R-semigroup in a Hilbert space H, in particular with $A = A\left(i\frac{\partial}{\partial x}\right)$ generating one of the constructed R-semigroups in $H = L_2^m(\mathbb{R})$. Here $F(t,X)$ is a nonlinear term satisfying the following conditions:

(F1) $\|F(t,y_1) - F(t,y_2)\|_H \le C\|y_1 - y_2\|_H$, $t \in [0,T]$, $y_1, y_2 \in H$ (the Lipschitz condition);

(F2) $\|F(t,y)\|_H^2 \le C\|1+y\|_H^2$, $t \in [0,T]$, $y \in H$ (the growth condition).

Let us introduce a definition of a mild R-solution for the quasi–linear Cauchy problem (18). In the sense of this paper terminology it will be a strong solution.

Definition 5. An H-valued $\mathcal{F}_t$-measurable process $\{X(t), t \in [0,T]\}, X(t) = X(t,\omega), \omega \in \Omega$, is called a *mild R-solution* of the problem (18) with A generating an R-semigroup $S := \{S(t),\ t \in [0,\tau)\}$ if

1) $\int_0^T \|X(\tau)\|_H \, d\tau < \infty \ P_{\text{a.s.}}$;

2) for each $t \in [0,T]$, $X(t)$ satisfies the following equation

$$X(t) = S(t)f + \int_0^t S(t-s)F(s,X(s))\,ds + \int_0^t S(t-s)B\,dW(s)\,ds \ P_{\text{a.s.}} \quad (19)$$

A unique mild R-solution to (18), in particular to the problem with $A = A\left(i\frac{\partial}{\partial x}\right)$ and with F satisfying the conditions (F1)–(F2), can be constructed by the method of successive approximations, similarly to the case of strongly continuous semigroups considered by Da Prato[2] and Ogorodnikov[8].

As for mild solutions, they can be obtained only in spaces, where operator R^{-1} is defined, and similarly to the case of the linear problem above, these spaces must be special spaces of generalized functions or even more general spaces, where nonlinear operations on generalized functions are possible. That is the problem for further investigations. The beginning to the investigations of generalized solutions to quasi-linear problems

$$X'(t) = AX(t) + F(t,X) + B\mathcal{W}(t), \quad t \geq 0, \quad X(0) = f,$$

was laid in the paper Melnikova and Alekseeva[7] due to construction of abstract stochastic Colombeau spaces.

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