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Newtonian and special-relativistic probability densities for a low-speed system

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Abstract. The Newtonian and special-relativistic predictions for the position and momentum probability densities of a model *low-speed* (i.e., much less than the speed light) dynamical system are compared. The Newtonian and special-relativistic probability densities, which are initially the same Gaussian, are calculated using an ensemble of trajectories. Contrary to expectation, we show that the predictions of the two theories can rapidly disagree completely. This surprising result raises an important fundamental question: which prediction is empirically correct?

INTRODUCTION

It is conventionally believed [1-3] that the predictions of special-relativistic mechanics for the motion of a dynamical system are well approximated by the predictions of Newtonian mechanics for the same parameters and initial conditions if the speed of the system v is *low* compared to the speed of light c ($v \ll c$). However, contrary to expectation, it was shown in recent numerical studies [4-8] that the Newtonian prediction for the trajectory of a low-speed dynamical system can rapidly disagree completely with the special-relativistic prediction.

In this paper, we extend the studies in [4-8] from the comparison of single-trajectory predictions to the comparison of the probability-density predictions calculated from an ensemble of trajectories. The model system we study here is the periodically delta-kicked system previously studied in [4]. Details of the model system and the probability-density calculations are presented next, followed by the results and concluding remarks.

Model System

The periodically delta-kicked system [4] is a one-dimensional Hamiltonian system where a particle is subjected to a sinusoidal potential that is periodically turned on for an instant. The Newtonian equations of motion for this system are easily integrated exactly [9,10] to yield the well-known standard map, which

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maps the dimensionless scaled position X and dimensionless scaled momentum P from just before the nth kick to just before the (n+1)th kick:

$$P_n = P_{n-1} - \frac{K}{2\pi}\sin(2\pi X_{n-1}) \tag{1}$$

$$X_n = (X_{n-1} + P_n) \mod 1 \tag{2}$$

where n = 1,2,…, and K is a dimensionless positive parameter.

The special-relativistic equations of motion are also easily integrated exactly, producing a mapping known as the relativistic standard map [11,12] for the dimensionless scaled position X and dimensionless scaled momentum P from just before the nth kick to just before the (n+1)th kick:

$$P_n = P_{n-1} - \frac{K}{2\pi}\sin(2\pi X_{n-1}) \tag{3}$$

$$X_n = \left(X_{n-1} + \frac{P_n}{\sqrt{1+\beta^2 P_n^2}} \right) \mod 1 \tag{4}$$

where n = 1,2, …, and β, like K, is a dimensionless positive parameter.

The initial probability density is a Gaussian for both position and momentum with means $<X_0>$ and $<P_0>$, and standard deviations σ_{X0} and σ_{P0}:

$$\frac{1}{2\pi\sigma_{X0}\sigma_{P0}}\exp\left[-\frac{\left(X_0 - <X_0>\right)^2}{2\sigma_{X0}^2} - \frac{\left(P_0 - <P_0>\right)^2}{2\sigma_{P0}^2}\right].$$

In each theory, the probability density is calculated using an ensemble of trajectories, where each trajectory is time-evolved using the map. The probability density is first calculated using 10^6 trajectories, where the accuracy of the double-precision calculation is determined by comparison with the quadruple-precision calculation. The probability density is then recalculated using 10^7 trajectories with the same accuracy determination. Finally, the accuracy of the probability density is determined by comparing the 10^6-trajectories calculation with the 10^7-trajectories calculation.

Results

In the example presented here, the means and standard deviations of the initially Gaussian probability density are $<X_0>$ = 0.5, $<P_0>$ = 99.9 and $\sigma_{X0} = \sigma_{P0} = 10^{-10}$. The parameters of the maps are K = 0.9 and $\beta = 10^{-7}$.

Figures 1, 2 and 3 show that the Newtonian and special-relativistic position and momentum probability densities evolve approximately as Gaussians with increasing widths up to at least kick 114.

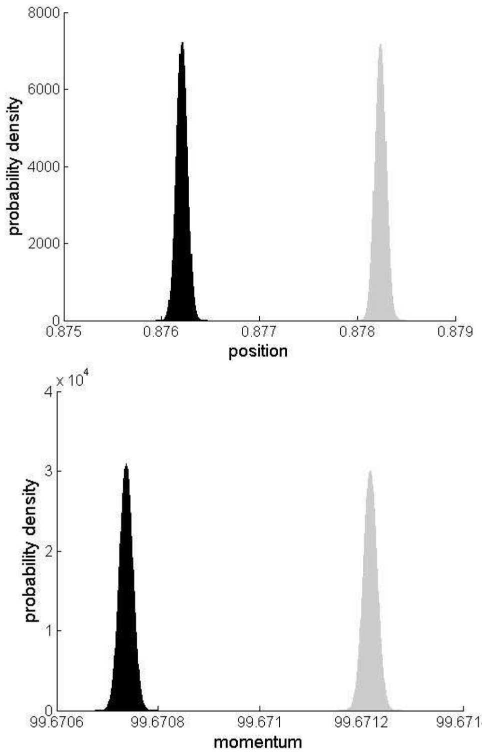

Figure 1. Comparison of Newtonian (grey) and special relativistic (black) position (top plot) and momentum (bottom plot) probability density for kick 80.

Figure 1 shows that, for both position and momentum, the Newtonian and special-relativistic probability densities are still close to one another on the whole at kick 80. The centers of the Newtonian and special-relativistic probability densities are displaced from each other in the figure because of the very small scale required for the horizontal axis to see the very narrow densities.

By kick 89, Figure 2 shows that, for both position and momentum, although the centers of the Newtonian and special-relativistic probability densities are still close, the Newtonian probability density is significantly wider and shorter than the special-relativistic probability density.

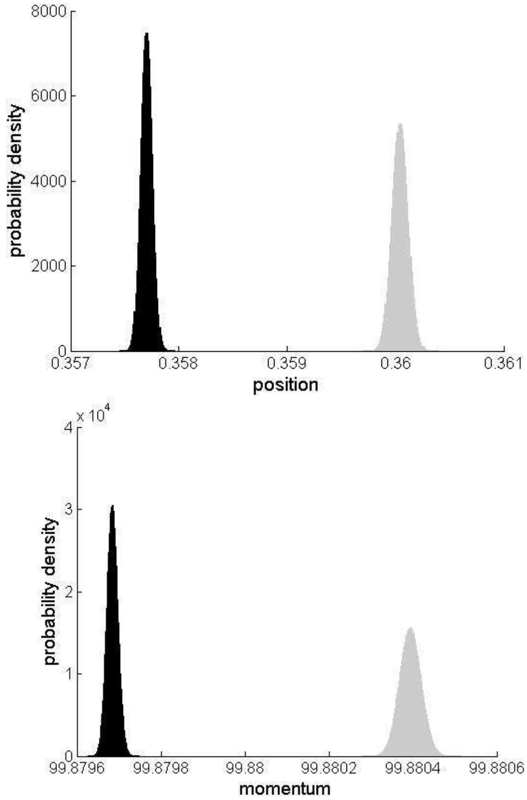

Figure 2. Comparison of Newtonian (grey) and special relativistic (black) position (top plot) and momentum (bottom plot) probability density for kick 89.

At kick 114, Figure 3 shows that not only are the widths and heights of the Newtonian and special-relativistic probability densities completely different for both position and momentum, the centers of the position probability densities are also completely different.

In summary, the three figures show that, although the mean speed of the system remains low, only 0.001% the speed of light, the Newtonian position and momentum probability densities disagree completely with the corresponding special-relativistic probability densities from kick 89 onwards.

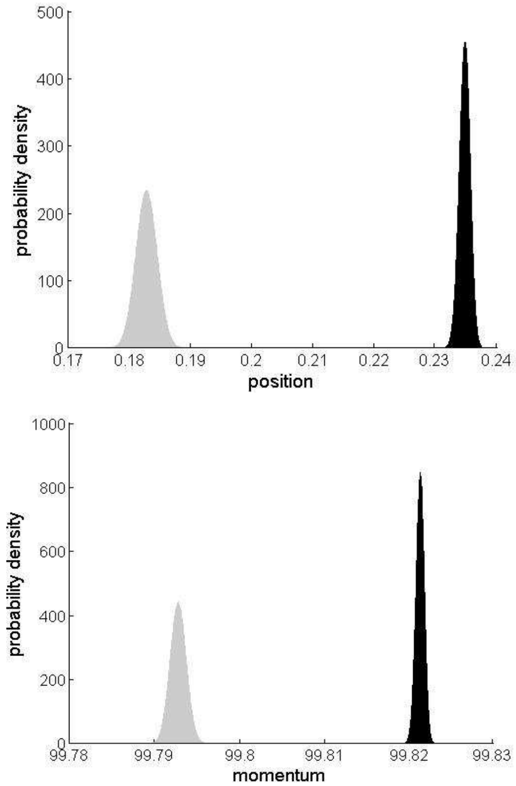

Figure 3. Comparison of Newtonian (grey) and special relativistic (black) position (top plot) and momentum (bottom plot) probability density for kick 114.

Concluding remarks

We have shown that, contrary to expectation, the Newtonian and special-relativistic probability-density predictions for a low-speed dynamical system can rapidly disagree completely.

Our result raises an important fundamental question: When Newtonian and special-relativistic mechanics predict completely different probability densities for a *low-speed* dynamical system, which of the two predictions is empirically correct? Since special relativity has survived many experimental tests in the high speed regime, it would be very strange indeed if the theory is invalid for low speed motion. If special relativity is also empirically correct at low speed as we expect, then it must be used, instead of the standard practice of using Newtonian theory, to correctly calculate the probability density for a low-speed dynamical system.

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