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Numerical Methods for Discontinuous Singularly Perturbed Differential Systems

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Abstract. In this paper we study the numerical solution of singularly perturbed systems with a discontinuous right hand side. We will avoid to consider the associate reduced differential system because often this study leads to wrong conclusions. To handle either the stiffness, due to different scales, or the discontinuity of the vector field we will consider numerical method which are semi-implicit and of low order of accuracy.
Keywords: Singularly perturbed differential systems, Filippov discontinuous systems, numerical methods.

1 Introduction

In this paper we study singularly perturbed systems with a discontinuous right hand side. Differential systems of this type appear in several fields (see for instance [7], [8], [14]) and they have attracted a growing interest also from a theoretical point of view (see for instance [13]). Let us consider the singularly perturbed differential system in $\mathbb{R}^n$ given the the following form:

$$\begin{cases} x' = f(x,y), & x(0) = x_0, \quad t \in [t_0, T], \\ \epsilon y' = g(x,y), & y(0) = y_0, \end{cases} \tag{1}$$

where usually $0 < \epsilon \ll 1$, while $x : [0,T] \to \mathbb{R}^{n-m}$ is the *slow* variable, $y : [0,T] \to \mathbb{R}^m$ is the *fast* variable, the vector field f is discontinuous along a surface Σ while g is sufficiently smooth. Let us suppose that the state space $\mathbb{R}^n$ is split into two subspaces R_1 and R_2 by a surface Σ such that $\mathbb{R}^n = R_1 \cup \Sigma \cup R_2$. The surface Σ is implicitly characterized by a scalar *event* function $h : \mathbb{R}^n \to \mathbb{R}$, that is

$$\Sigma = \{(x,y) \in \mathbb{R}^n | \ h(x,y) = 0\} \ , \tag{2}$$

so that the subspaces R_1 and R_2 are

$$R_1 = \{(x,y) \in \mathbb{R}^n | \ h(x,y) < 0\} \, , \ R_2 = \{(x,y) \in \mathbb{R}^n | \ h(x,y) > 0\} \, . \tag{3}$$

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We will assume that $h(x, y)$ is sufficiently smooth and that its gradient $\nabla h(x, y) \neq 0$ for all $(x, y) \in \Sigma$, so that the normal $n(x, y) = \frac{\nabla h(x,y)}{\|\nabla h(x,y)\|}$ to Σ is well defined. In many practical applications, the function h is actually linear (Σ is a plane).

Let us suppose that the vector field f is discontinuous along Σ, that is:

$$f(x, y) = \begin{cases} f_1(x, y) \text{ when } (x, y) \in R_1 \\ f_2(x, y) \text{ when } (x, y) \in R_2 \end{cases},$$

where f_1 is sufficiently smooth on $R_1 \cup \Sigma$ and f_2 is sufficiently smooth on $R_2 \cup \Sigma$.

Let us assume that for $\epsilon = 0$, the algebraic equation $(1.b)$, that is $g(x, y) = 0$, can be solved for y for all x and that this solution (denotated by $y_0(x)$) satisfies the stability condition:

$$\text{Re Spec } \partial_y g(x, y_0(x))) < -\mu < 0 \tag{4}$$

with a uniform decay rate μ (see [12]).

Furthermore, let us assume that for the reduced system

$$x' = \begin{cases} f_1(x, y_0(x)), & \text{when } h(x, y_0(x))) < 0 \\ f_2(x, y_0(x)), & \text{when } h(x, y_0(x)) > 0 \end{cases} \tag{5}$$

the sufficient conditions for the attractivity of the sub-surface

$$\Sigma_0 = \{(x, y) \in \mathbb{R}^n | y = y_0(x) \ , \ h(x, y_0(x)) = 0\} , \tag{6}$$

hold.

2 Filippov approach

By setting:

$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad F_1(z, \epsilon) = \begin{bmatrix} f_1(z) \\ \frac{1}{\epsilon} g(z) \end{bmatrix}, \qquad F_2(z, \epsilon) = \begin{bmatrix} f_2(z) \\ \frac{1}{\epsilon} g(z) \end{bmatrix}, \tag{7}$$

the singularly perturbed discontinuous system (1) may be rewritten in Filippov's form

$$z' = F(z, \epsilon) = \begin{cases} F_1(z, \epsilon), & \text{when } h(z) < 0 \\ F_2(z, \epsilon), & \text{when } h(z) > 0 \end{cases} \tag{8}$$

with initial condition $z_0 = [x(0), y(0)]^T$.

A solution in the sense of Filippov (see [6]) is an absolutely continuous function $z : [0, T] \to \mathbb{R}^n$ such that $z'(t) \in F(z(t), \epsilon)$ for almost all $t \in [0, T]$, where $F(z(t), \epsilon)$ is the closed convex hull

$$\overline{\text{co}} \{F_1, F_2\} = \{F \in \mathbb{R}^n \ : \ F = (1 - \alpha)F_1 + \alpha F_2, \ \alpha \in [0, 1]\} \ . \tag{9}$$

Now, suppose $z_0 \in R_1$ (that is $h(z_0) < 0$) and assume that the trajectory of the differential system $z' = F_1(z, \epsilon)$ is directed towards Σ and reaches it in a finite time. At this point, one must decide what happens next. Loosely speaking, there are two possibilities: (a) we leave Σ and enter into R_2 (*transversal case*); (b) we remain in Σ with a defined vector (*sliding mode*). Filippov deviced a very powerful theory which helps to decide what to do in this situation and how to define the vector field during the sliding motion.

Let $z \in \Sigma$ and let $n(z) = \frac{\nabla h(z)}{\|\nabla h(z)\|}$ be the normal to Σ at z. Let $n^T(z)F_1(z,\epsilon)$ and $n^T(z)F_2(z,\epsilon)$ be the projections of $F_1(z,\epsilon)$ and $F_2(z,\epsilon)$ onto the normal direction and suppose that $n^T(z)F_1(z,\epsilon) > 0$. We will exclude the case in which we entry Σ in a tangent way, that is $n^T(z)F_1(z,\epsilon) = 0$ at $z \in \Sigma$.

Transversal Intersection. In case in which, at $z \in \Sigma$, we have

$$[n^T(z)F_1(z,\epsilon)] \cdot [n^T(z)F_2(z,\epsilon)] > 0 \ , \tag{10}$$

then we will leave Σ and enter R_2 with $F = F_2$. Any solution of (8) with initial condition not in Σ, reaching Σ at a time t_1, and having a transversal intersection there, exists and is unique.

Sliding Mode. Instead, if, at $z \in \Sigma$, we have

$$[n^T(z)F_1(z,\epsilon)] \cdot [n^T(z)F_2(z,\epsilon)] < 0 \ , \tag{11}$$

then we have a so-called attracting sliding mode through z.

When we have (11) satisfied at $z \in \Sigma$, a solution trajectory which reaches z does not leave Σ, and will therefore have to move along Σ. During the sliding motion the solution will continue along Σ with time derivative F_S given by:

$$F_S(z,\epsilon) = (1-\alpha(z))F_1(z,\epsilon) + \alpha(z)F_2(z,\epsilon)\,. \tag{12}$$

and $\alpha(z)$ such that $F_S(z,\epsilon)$ lies in the tangent plane T_z of Σ at z, that is $n^T(z)F_S(z,\epsilon) = 0$, and this gives

$$\alpha(z) = \frac{n^T(z)F_1(z,\epsilon)}{n^T(z)(F_1(z,\epsilon) - F_2(z,\epsilon))}. \tag{13}$$

Observe that a solution having an attracting sliding mode exists and is unique, in forward time.

As far as the reduced system (5) is concerned, we have to observe that during the sliding mode the Filippov vector field will be

$$f_S(x) = (1-\alpha_0(x))f_1(x,y_0(x)) + \alpha_0(x)f_2(x,y_0(x))\,. \tag{14}$$

where

$$\alpha_0(x) = \frac{n_x^T(x)f_1(x,y_0(x))}{n_x^T(z)(f_1(x,y_0(x)) - f_2(x,y_0(x)))}. \tag{15}$$

where $n_x(x) = \frac{\nabla h(x,y_0(x))}{\|\nabla h(x,y_0(x))\|}$.

3 An Example

We observe that while Σ_0 is an attractive surface for the solution of the reduced system (5), on the other hand, the trajectories of the singularly perturbed system (1) could transverse the discontinuity surface Σ, or could slide on it for a certain time interval, or could show a periodic or *chattering* behaviour.

As an example of different behaviours between the initial and reduced system, we consider the following system:

$$\begin{cases} x' = -\text{sign}[\theta x + (1-\theta)y], \\ \epsilon y' \quad = x - y \quad , \end{cases} \tag{16}$$

where θ is a real parameter ($\theta \neq 0$) and where the discontinuity surface is the line

$$\Sigma = \left\{(x,y) \in \mathbb{R}^2 |\ h(x,y) = \theta x + (1-\theta)y = 0\right\} \ . \tag{17}$$

A theoretical study of singularly perturbed systems of this kind has been derived in [13]. When $\epsilon = 0$, the reduced system becomes the well known discontinuous system $x' = -\text{sign}[x]$, $x = y$, which has the equilibrium point $(x, y) = (0, 0)$. Such a point is exponentially stable and attractive in finite time. Actually (0,0) is a *pseudo-equilibrium* because it is an equilibrium of (16) which is on the discontinuity surface Σ. Let us denote

$$F_1(x,y,\epsilon) = \begin{bmatrix} 1 \\ \frac{1}{\epsilon}(x-y) \end{bmatrix}, \qquad F_2(x,y,\epsilon) = \begin{bmatrix} -1 \\ \frac{1}{\epsilon}(x-y) \end{bmatrix}, \tag{18}$$

thus, the sliding region will be defined by the points of the line Σ such that $\nabla h^T \cdot F_1 > 0$ and $\nabla h^T \cdot F_2 < 0$, that is the points $(x, y) \in \Sigma$ such that

$$\theta + \frac{1-\theta}{\epsilon}(x-y) > 0 \ , \qquad -\theta + \frac{1-\theta}{\epsilon}(x-y) < 0 \ .$$

Thus, for $\theta > 0$ and $\theta \neq 1$, assuming $y = \frac{\theta}{\theta-1}x$, it follows that the sliding region is defined by

$$-\epsilon\theta < x < \epsilon\theta \ ,$$

this means that there is a small neighborhood of $(0, 0)$, on the discontinuity line Σ, on which the solution of (16) sliding reaches the pseudo-equilibrium.

If $\theta < 0$, then $(0, 0)$ is an unstable pseudo-equilibrium, in particular there is a repelling sliding region near the origin and we have a symmetric exponentially stable periodic orbit around the origin switching between the two different vector fields F_1 and F_2 (see [13] for the details). Thus the dynamics of the perturbed system ($\epsilon > 0$) are close the dynamics of the unperturbed system ($\epsilon = 0$) only in a very weak sense (see [5]) and the reduced system cannot be used to study the perturbed one.

4 Numerical methods

The previous example shows that the study of the reduced stystem ($\epsilon = 0$) could lead to wrong conclusions, in particular certain dynamics of the system could be lost. However, the reduced differential system (5) could be used to approach the discontinuty surface Σ, that is to find an initial point close to Σ from which starting with the numerical solution of the unperturbed differential system.

On the other hand, the numerical solution of discontinuous singularly perturbed problems meets several difficulties. In fact, we need to consider numerical schemes that handle either the discontinuity of the vector field or the stiffness of the solution which arises because of the presence of the small parameter ϵ. To this end we will consider two semi-implicit schemes, one in the class of Predictor-Corrector methods and the other in the class of Rosenbrock methods.

We have adopted a computational approach in which each particular state of the differential system is integrated with an appropriate numerical method, and the event points, where structural changes in the system occur, are located in an accurate way. In [1], this approach is called an *event driven* method (see also the numerical methods in [2], [3]), and the numerical methods we consider will be effective if there are not too many events.

We will be mainly concerned with developing a numerical procedure which will accomplish the following different tasks:

(i) Integration outside Σ;
(ii) Accurate location of points on Σ reached by a trajectory;
(iii) Check of the transversality or sliding conditions at the points on Σ;
(iv) Integration on Σ (sliding mode);
(v) check of the exit conditions from Σ.

For discretizing the singularly perturbed discontinuous system in (8) we are going to consider schemes (of low order 1) suitable to handle stiff problems. Integration of (8) while the solution remains in R_1 (or R_2) is not different than standard numerical integration of a singularly perturbed differential system (see [10]). Therefore, the only interesting case to consider is when, while integrating the system with F_1 (or F_2), we end up reaching the surface Σ.

Let $z_0 \in R_1$ and consider one step of the implicit Euler method:

$$z_1(\tau) \ = \ z_0 + \tau F_1(z_1(\tau) \ , \epsilon) \ , \tag{19}$$

where $\tau > 0$ is the time step of integration. We suppose that τ is sufficiently small in order to avoid situations in which, in the interval $[0, \tau]$, more than one event point occurs. We have to notice that in order to find $z_1(\tau)$ from (19), we have to solve a nonlinear system of n algebraic equation. Let us suppose that τ is such that

$$h(z_0)h(z_1(\tau)) < 0 \tag{20}$$

that is $z_1(\tau)$ is on the other side of Σ. We observe that in the interval $[0, \tau]$ the function $H(\eta) = h(z_1(\eta))$ changes sign. Thus, we may apply a zero finding routine (for instance the bisection or secant method) to determine $\bar{\tau}$, such that $h(z_1(\bar{\tau})) = 0$, that $z_1(\bar{\tau}) \in \Sigma$. The secant methods gives:

$$\eta_{k+1} = \eta_k - \frac{(\eta_k - \eta_{k-1})}{H(\eta_k) - H(\eta_{k-1})} H(\eta_k), \qquad k \geq 1,$$

with $\eta_0 = 0$, $\eta_1 = \tau$. However, at each iteration of a such routine a nonlinear system of equations must be solved in order to compute the new vector $z_1(\eta_k)$ required in $H(\eta_k)$ and this could be very expensive.

In order to derive a semi-explicit procedure suitable to treat stiff problems, we consider a predictor-corrector method where the predictor is the Euler explicit method and the corrector is the Euler implicit method, that is

$$\begin{cases} z_1^{(0)}(\tau) \;=\; z_0 + \tau F_1(z_0 \;, \epsilon) \;, \\ z_1(\tau) \;=\; z_0 + \tau F_1(z_1^{(0)}(\tau) \;, \epsilon) \;, \end{cases} \tag{21}$$

which is equivalent to the explicit formula:

$$z_1(\tau) \;=\; z_0 + \tau F_1(z_0 + \tau F_1(z_0 \;, \epsilon) \;, \epsilon) \;. \tag{22}$$

Now, if (20) holds, a simple scalar non linear equation must be solved to find the step size $\bar{\tau}$ for which $z_1(\bar{\tau})$ is on Σ.

A different method we could employ is the semi-explicit Rosenbrock method of order 1:

$$z_1 \;(\tau) = \; z_0 + \; \tau \; t_0, \tag{23}$$

where the vector t_0 is given by

$$[I - \tau J_{F_1}(z_0)] \; t_0 \;=\; F_1(z_0, \epsilon) \;, \tag{24}$$

and where $J_{F_1}(z_0)$ denotes the Jacobian matrix of F_1 at z_0.

Now, if (20) holds, in the zero finding routine, instead of (23), we may consider the continuous extension of the Rosenbrock method

$$z_1 \;(\sigma\tau) = \; z_0 + \; \sigma\tau \; t_0, \quad \sigma \in (0, 1). \tag{25}$$

where the vector t_0 is again given by (24) but is independent on σ, according to the theory of continuous extensions.

An advantage of (23) with respect (21) is that the former does not require the evaluation of the vector field F_1 above Σ, and this property could be necessary in certain discontinuous models.

Once we have a point $\bar{z}$ on Σ, we need to decide if we will need to cross Σ or slide on Σ, that is we will check if

$$[n^T(\bar{z})F_1(\bar{z}, \epsilon)] \cdot [n^T(\bar{z})F_2(\bar{z}, \epsilon)] > 0 \;, \tag{26}$$

or

$$[n^T(\bar{z})F_1(\bar{z}, \epsilon)] \cdot [n^T(\bar{z})F_2(\bar{z}, \epsilon)] < 0 \;, \tag{27}$$

[recall we are supposing that $[n^T(\bar{z})F_1(\bar{z}, \epsilon)] > 0$].

If (26) is satisfied, then we change the vector field and continue to integrate the system:

$$z'(t) = F_2(z(t), \epsilon), \quad z(\bar{\tau}) = \bar{z} \;, \tag{28}$$

by using the same numerical method used to reach Σ.

5 Integration on Σ

Instead, if (27) is satisfied then we enter an attractive sliding mode, thus we need to integrate the differential Filippov system:

$$z'(t) = F_S(z(t), \epsilon), \quad z(\bar{\tau}) = \bar{z} \ , \tag{29}$$

where with F_S we indicate the standard Filippov vector field (12).

Since F_S is a linear convex combination of F_1 and F_2, to integrate (29) we will employ the same method used to reach Σ, that is (21) or (23) where the vector field F_1 is now replaced by F_S.

Now, one step of the Rosenbrock method becomes $z_1(\tau) = z_0 + \tau t_0$, with

$$[I - \tau J_{F_S}(z_0)] \ t_0 \ = \ F_S(z_0, \epsilon) \tag{30}$$

where $J_{F_S}(z_0)$ denotes the Jacobian matrix of F_S at $z_0 \in \Sigma$. Because of the form of F_S, this Jacobian matrix J_{F_S} could be very expensive to evaluate and a free-Jacobian procedure has to be used in the solution of the linear system (30) by means of iterative or Krylov type procedures (see [11]).

We observe that when we integrate on Σ, usually, the numerical solution given by (21) or (23) leaves the surface Σ and a projection is necessary to return on Σ. The projection on Σ may be done in the standard way (e.g., see [4], [9]). If $\hat{z}$ is a point close to Σ, then the projected vector $z = P(\hat{z})$ on Σ is the solution of the following constrained minimization problem

$$\min_{z \in \Sigma} g(z) \ , \quad g(z) = \frac{1}{2}(\hat{z} - z)^T (\hat{z} - z) \ .$$

By using the Lagrange's multiplier's method, we have to find the root of

$$G(z, \lambda) = \begin{pmatrix} \nabla g(z) + \lambda \nabla h(z) \\ h(z) \end{pmatrix} \ , \qquad \lambda \in \mathbb{R} \ ,$$

and we can apply Newton's method to find the root of $G(z, \lambda) = 0$.

On the other hand, if Σ is flat, that is $h(z) = a^T z + b$ linear with respect to z, then the numerical solution given by (21) lies on Σ while the one obtained by (23) does not.

Theorem 1. *Let us assume Σ given by $h(z) = a^T z + b$, and suppose that $z_0 \in \Sigma$. Then z_1 given by (21) lies on Σ while z_1 given by (23) does not.*

Proof. Let us consider the numerical solution

$$z_1 \ = \ z_0 + \tau F_S(z_0 + \tau F_S(z_0 \ , \epsilon) \ , \epsilon) \ . \tag{31}$$

We notice that the predicted vector $z_0 + \tau F_S(z_0 \ , \epsilon)$ remains on Σ since it has been obtained by an explicit method which preserves linear invariants (see [9]). Thus, it follows that

$$a^T z_1 + b = a^T [z_0 + \tau F_S(z_0 + \tau F_S(z_0, \epsilon), \epsilon)] + b = a^T z_0 + b = 0 \ ,$$

since $a^T F_S(z_0 + \tau F_S(z_0, \epsilon)) = 0$ being a^T the normal vector of Σ.

Now, we would like to see if $a^T z_1 + b = 0$ when z_1 is the numerical solution obtained by (23). Then it follows that

$$a^T z_1 + b = a^T(z_0 + \tau[I - \tau J(z_0)]^{-1} F_S(z_0, \epsilon)) + b =$$

$$= a^T z_0 + b + \tau a^T [I - \tau J(z_0)]^{-1} F_S(z_0, \epsilon) \ ,$$

thus z_1 is on Σ only if $a^T[I - \tau J(z_0)]^{-1} F_S(z_0, \epsilon)$. We observe that $a^T F_S(z_0, \epsilon) = 0$, and that for τ sufficiently small we have

$$[I - \tau J]^{-1} = I + \tau J + \frac{\tau^2}{2} J^2 + \frac{\tau^3}{6} J^3 + \dots$$

thus z_1 is on Σ if and only if $JF_S = F_S$, that in general is not true.

Thus, usually, to remain on Σ a projection on it is required. While we integrate on Σ, we will monitor if we have to continue sliding on it, or if we need to leave Σ. Once the point z_1 on Σ has been computed, we need to check if the sliding condition

$$[n^T(z_1) F_1(z_1, \epsilon)] \cdot [n^T(z_1) F_2(z_1, \epsilon)] < 0 \ , \tag{32}$$

is satisfied or if this product changes sign, that is

$$[n^T(z_1) F_1(z_1, \epsilon)] \cdot [n^T(z_1) F_2(z_1, \epsilon)] > 0 \ , \tag{33}$$

If (32) holds then we continue to integrate on Σ. On the other hand, if (33) holds then we have to determine $\bar{\tau}$ (and hence $z_1(\bar{\tau})$) such that the previous product vanishes. Thus, starting with $z_1(\bar{\tau})$, we exit the surface Σ with vector field $F_2(z_1(\bar{\tau}),\ \epsilon)$.

6 Numerical tests

In this section we report the numerical simulations of some singularly perturbed discontinuous systems, obtained by using the numerical methods studied. We will report the results obtained by `Matlab` codes using both the predictor-corrector method in (21) and the Rosenbrock method in (23) with sufficiently small time step τ.

Example 1. Here we consider the numerical solution of the system in (16), with $\epsilon = 0.001$, by means of the numerical methods proposed in the previous section. Figure 1 concerns with the case $\theta > 0$ (we have taken $\theta = 0.9$ and denoted by '*' the initial value). We can see that the numerical solution first crosses the discontinuity surface Σ (denoted by the red color), then begins to slide on Σ until to reach the pseudoequilibrium $(0, 0)$.

Figure 2 concerns with the case $\theta < 0$ ($\theta = -0.9$). We can see that the numerical solution tends to an exponentially stable periodic orbit around the origin while the vector field switches between the two different vector fields F_1

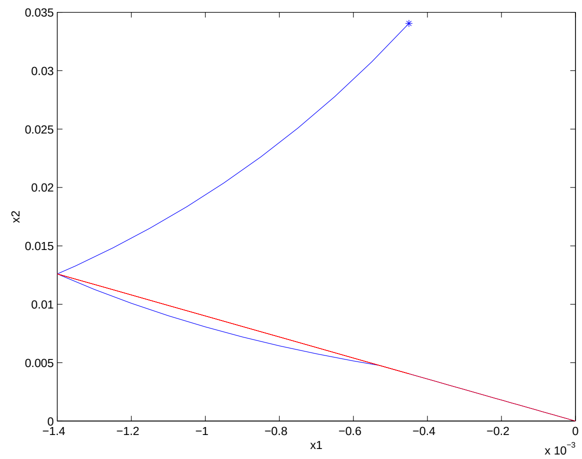

Fig. 1. Example 1. Case $\theta = 0.90$.

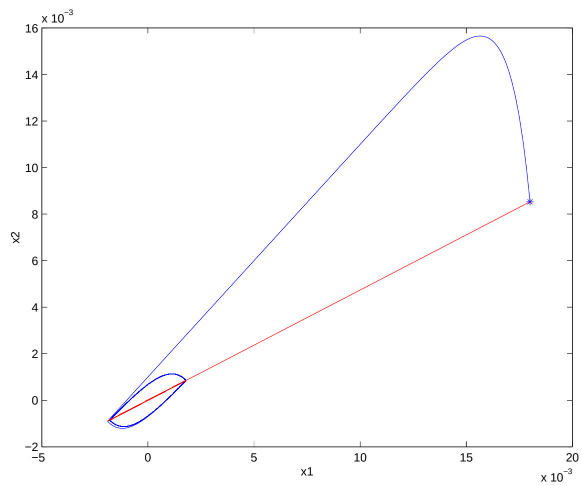

Fig. 2. Example 1. Case $\theta < 0$.

and F_2. In Figure 3 we have reported the exponentially stable periodic solution of the system.

Example 2. Let us consider the following discontinuous differential system:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{cases} \mu x_1 - \omega x_2 - (x_1^2 + x_2^2)x_1 \\ \omega x_1 + \mu x_2 - (x_1^2 + x_2^2)x_2 \end{cases} \quad , \quad \text{when } h(x_1, x_2) \geq 0 \tag{34}$$

or

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{cases} 1 \\ 0 \end{cases} \quad , \quad \text{when } h(x_1, x_2) < 0 \tag{35}$$

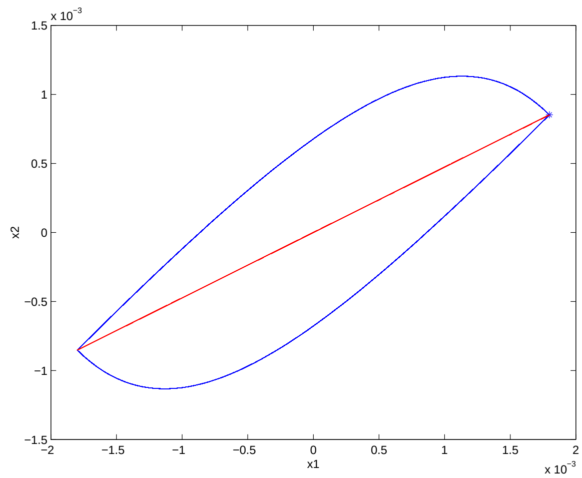

Fig. 3. Example 1. Case $\theta < 0$: stable periodic solution.

[μ and ω positive constants] while the switching line is given by $h(x_1, x_2) = x_1 + 1$, therefore $\nabla^T h(x) = [1\ 0]$. Using our notation, we have:

$$f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad f_2 = \begin{bmatrix} \mu x_1 - \omega x_2 - (x_1^2 + x_2^2)x_1 \\ \omega x_1 + \mu x_2 - (x_1^2 + x_2^2)x_2 \end{bmatrix} , \tag{36}$$

and observe that $\nabla^T h \cdot f_1 = 1 > 0$. Hence, when $\mu > 1$, the attractive sliding region S_R is the segment on the line $x_1 = -1$ for which $\nabla^T h \cdot f_2 < 0$, that is $S_R = \{(-1, x_2) \in \mathbb{R}^2 | -\mu - \omega x_2 + (1 + x_2^2) < 0\}$. In Figure 4 we report the exponentially stable periodic solution of (35) obtained for $\mu = 1.5$ and $\omega = 1$ by our numerical methods.

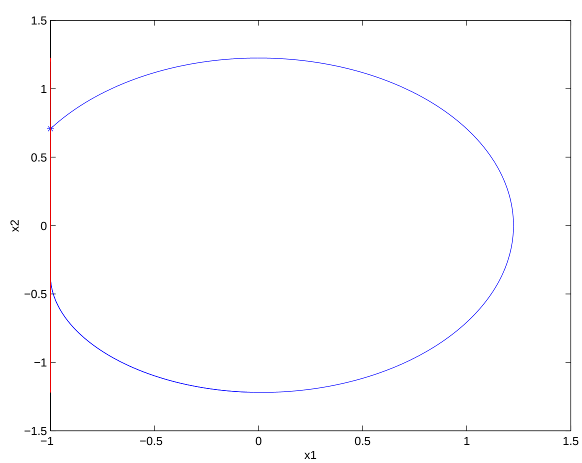

Fig. 4. Example 2. Stable periodic solution.

Now, let us consider the singularly perturbed discontinuous system:

$$\begin{pmatrix} x_1' \\ x_2' \\ \epsilon x_3' \end{pmatrix} = \begin{cases} x_1 - \omega x_2 - (x_1^2 + x_2^2)x_1 \\ \omega x_1 + \mu x_2 - (x_1^2 + x_2^2)x_2 \\ \epsilon[\mu x_1 - \omega x_2 - (x_1^2 + x_2^2)x_1] + x_1 - x_3 \end{cases} , \; h(x_1, x_2, x_3) \geq 0 \quad (37)$$

while

$$\begin{pmatrix} x_1' \\ x_2' \\ \epsilon x_3' \end{pmatrix} = \begin{cases} 1 \\ 0 \\ \epsilon[\mu x_1 - \omega x_2 - (x_1^2 + x_2^2)x_1] + x_1 - x_3 \end{cases} , \; h(x_1, x_2, x_3) < 0 \quad (38)$$

where the last component of the vector field is continuous while the previous two components are discontinuous with respect the line:

$$\Sigma = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | \; h(x_1, x_2, x_3) = \theta x_1 + (1 - \theta)x_3 = 0\right\} \; . \quad (39)$$

The reduced system ($\epsilon = 0$) is the one in (34)-(35). A theoretical study of the system (37)-(38) may be found in [13]. In Figure 5 we report the periodic solution of the singularly perturbed system (37)-(38) for $\epsilon = 0.01$, $\mu = 1.5$, $\omega = 1$ and assuming a positive value of the parameter θ ($\theta = 0.5$). A zoom of the solution near the sliding segment of the reduced system may be seen in Figure 6. Instead, in Figure 7 the periodic solution of (37)-(38) with $\theta = -0.5$ is shown, while in Figure 8 we show the chattering behaviour of the solution near the sliding segment of the reduced system.

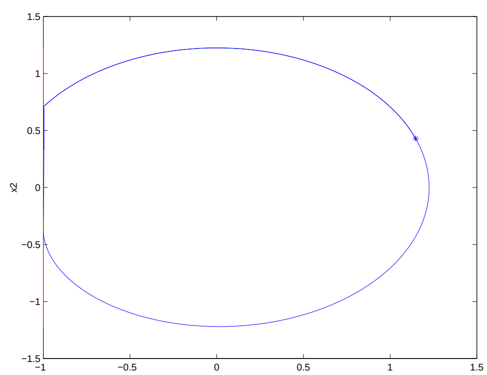

Fig. 5. Example 2. Case $\theta = 0.5$. Periodic solution.

7 Conclusions

In this paper we have studied the numerical solution of singularly perturbed systems with a discontinuous right hand side avoiding to consider the associate reduced differential system, because often this study leads to wrong conclusions. To handle either the stiffness, due to different scales, or the discontinuity of the vector field, we have considered numerical method which are semi-implicit and of low order of accuracy. We tested our numerical methods on examples known in literature.

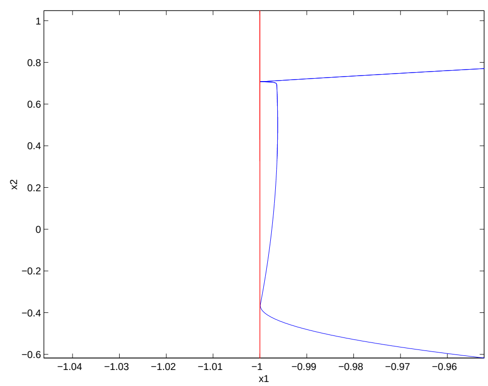

Fig. 6. Example 2. Case $\theta = 0.5$. Zoom of the solution.

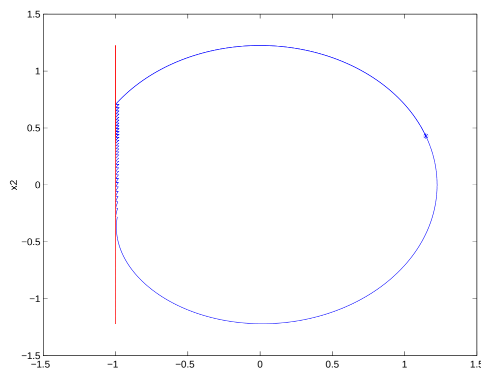

Fig. 7. Example 2. Case $\theta = -0.5$. Periodic solution.

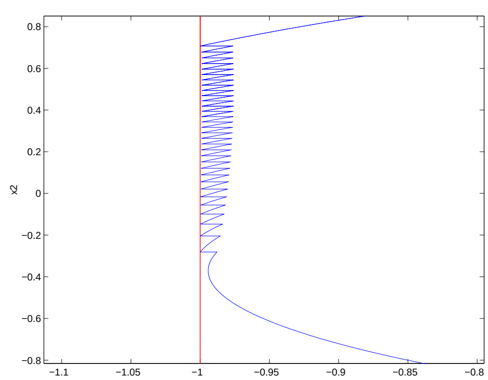

Fig. 8. Example 2. Case $\theta = -0.5$. Zoom of the solution.

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