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Informational Technologies for Quasilinear Research of Combined Stochastic and Chaotic Systems

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Abstract. This article is dedicated to the problems of modeling and simulation of combined stochastic and chaotic systems. Modern informational technologies for nonlinear stochastic systems are used to solve these problems. Discussed methods are based on the off-line information of stochastic systems. They use equivalent linearization and normal approximating techniques for solving equations for mathematical expectations and covariances. Two results for Duffing equation are given in this paper. This example contains the solution for harmonically and stochastically forced Duffing equation. Second example contains the solution for quasiharmonically forced Duffing equation.
Keywords: Combined stochastic and chaotic systems, Equivalent linearization techniques (ELT), Informational technologies, Analytical modeling, Duffing equation, Ito stochastic differential equation, Gaussian (normal) process, Wiener process, Poisson process, Mathematical expectation, Covariance matrix, Matrix of covariance functions, One-dimensional distribution, Two-dimensional distribution, MATLAB.

1 Introduction

It is known [5], [6] modern modeling (analytical, statistical and combined) and estimation (filtering, extrapolation, interpolation, parameters identification) informational technologies (IT) for nonlinear stochastic systems (StS) are based on off-line and on-line information. For engineering applications there are successfully used quasilinear methods based on equivalent linearization techniques (ELT) for solving equations (Eqs) for mathematical expectations and covariances. Experimental results received in [6],[7] for combined StS and chaotic systems (ChS) research IT show that these ELT may be used also.

The article is devoted to ELT for the combined stochastic and chaotic systems based on the off-line information. Problems of modeling and simulation based on the on-line information will be discussed in next articles.

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2 Eqs of combined stochastic and chaotic systems

For off-line analytical modeling of ChS with Gaussian perturbations based on Fokker-Plank- Kolmogorov Eq, Feller-Kolmogorov Eq for Poisson perturbations and Pugachev-Sinitsyn Eq for perturbations being the derivative of stochastic process with independent increments need to solve singular Eqs. The solution of these Eqs is very difficult for nowadays computers as for the on-line and the off-line information.

According to ELT one linearize the right hand of given stochastic differential Eq (in Ito sense) and implement known Pugachev-Dunkan ordinary deterministic Eqs for mathematical expectation, covariance matrix and matrix of covariance functions at corresponding initial conditions. The coefficients of these Eqs due to nonlinearity depend on mathematical expectation and covariance matrix. So these deterministic Eqs are interconnected and need joint solution.

For robust ELT it is quite enough to suppose the normal law for density of input signal for nonlinear element and use known formulae of statistical linearization for typical scalar and vector nonlinear functions [5],[6].

Following [5] let us consider differential combined StS and ChS described by the vector Ito Eq:

$$dY = a(Y,t)dt + b(Y,t)dW_0 + \int\limits_{R_0^q} c(Y,t,u)\, P^0(t,du), \quad Y(t_0) = Y_0. \tag{1}$$

Here $Y = [Y_1 \dots Y_p]^T$ is the state vector; $a = a(Y,t)$ and $b(Y,t)$ are known $(p\times 1)$ — dimensional and $(p\times m)$ — dimensional functions of Y, t; $W_0 = W_0(t)$ is the m — dimensional Wiener random process of the intensity $\nu_0 = \nu_0(t)$; $c = c(Y,t,u)$ is $(p \times 1)$ — dimensional function of Y, t and of the auxiliary $(q \times 1)$ — dimensional parameter u;

$$\mu = \int\limits_{\triangle} dP(t,A) - \int\limits_{\triangle} \nu_P(t,A)\, dt \tag{2}$$

is a centred Poisson measure, where the second integral term in (2) is a number of the jumps of a Poisson process $P(t,A)$ at time interval $\triangle$; $\nu_P = \nu_P(t,A)$ is the intensity of $P(t,A)$; A is the Borel set of the space R_0^q with the pricked origin of the coordinates.

The integral in Eq (1) is extended on R_0^q. The initial value Y_0 represents a random variable which does not depend on the increments of $W_0(t)$ and $P(t,A)$, $\triangle = (t_1,t_2]$ which follows t_0, $t_0 \le t_1 \le t_2$ for any set A.

At practice when the integrand $c(Y,t,u)$ admits presentation: $c(Y,t,u) = b(Y,t)c'(u)$ we need to assume the formulae for stochastic process with independent increments:

$$W(t) = W_0(t) + \int\limits_{R_0^q} c'(u)\, P^0(t,du). \tag{3}$$

In this case we get instead of Eq (1) the following Eq:

$$\dot{Y} = a(Y,t) + b(Y,t)V, \quad Y(t_0) = Y_0. \tag{4}$$

Here $V = \dot{W}$ is the white noise with one-dimensional characteristic function $h_1 = h_1(\rho;t)$ and its logarithmic time derivative

$$\begin{aligned}\Psi(\rho;t) = \frac{1}{h_1(\rho;t)}\frac{\partial h_1(\rho;t)}{\partial t} &= -\frac{1}{2}\rho^T\nu_0(t)\rho + \\ &\int\limits_{R_0^q}\left[e^{i\rho^T c'(u)} - 1 - i\rho^T c'(u)\right]\nu_P(t,u)\,du,\end{aligned} \tag{5}$$

where $\nu_0(t)$ id the intensity of $W_0(t)$ and $\nu_P(t,u)\,du$ is the intensity of the Poisson stream of the jumps equal to $c'(u)$.

3 Basic Eqs for ELT

Approximating the one-dimensional distribution Y (characteristic function and probability density) by normal (Gaussian) one we shall have

$$\begin{aligned} g_1(\lambda;t) &\approx \exp\{i\lambda^T m_y - \frac{1}{2}\lambda^T K_y\lambda\}, \\ f_1(\lambda;t) &\approx \left[(2\pi)^p|K_y|\right]^{-1/2}\exp\{-\frac{1}{2}(y-m_y)^T K_y^{-1}(y-m_y)\} \end{aligned} \tag{6}$$

In Eqs (6) m_y and K_y are the expectation and covariance matrix of the state vector defined by the ordinary differential Eqs:

$$\dot{m}_y = \varphi_1(m_y,K_y,t), \quad m_y(t_0) = m_0, \tag{7}$$

$$\dot{K}_y = \varphi_2(m_y,K_y,t), \quad K_y(t_0) = K_0. \tag{8}$$

Here

$$\begin{aligned} \varphi_1(m_y,K_y,t) &= E_N a(Y,t), \\ \varphi_2(m_y,K_y,t) &= \varphi_{21}(m_y,K_y,t) + \varphi_{21}^T(m_y,K_y,t) + \varphi_{22}(m_y,K_y,t), \\ \varphi_{21}(m_y,K_y,t) &= E_N\left\{a(Y,t)(Y-m_y)^T\right\}, \\ \varphi_{22}(m_y,K_y,t) &= E_N\overline{\sigma}(Y,t)^T, \end{aligned} \tag{9}$$

$$\begin{aligned} \overline{\sigma}(Y,t) &= \overline{\sigma_0}(Y,t) + \int\limits_{R_0^q} c'(u)b(Y,t)b(Y,t)^T {c'}^T(u)\,\nu_P(t,du), \\ \overline{\sigma_0}(Y,t) &= E_N b(Y,t)\nu_0(t)b(Y,t)^T \end{aligned} \tag{10}$$

and the subscript N denotes that the expectation E_N is calculated for normal distribution $N(m_y, K_y)$. The number of Eqs for normal approximation method (NAM) is equal to $Q_{NAM} = p(p+3)/2$.

NAM for approximate determination of stationary m_y^* and K_y^* gives Eqs:

$$\varphi_1(m_y^*, K_y^*) = 0, \quad \varphi_2(m_y^*, K_y^*) = 0. \tag{11}$$

For two-dimensional distributions in addition to Eqs (7)–(11) we have

$$\begin{gathered} g_{t_1 t_2}(\lambda_1, \lambda_2) \approx \exp\{i\overline{\lambda}^T \overline{m}_2 - \frac{1}{2}\overline{\lambda}^T \overline{K}_n \overline{\lambda}\}, \\ f_2(y_1, y_2; t_1, t_2) \approx \left[2\pi^{2p}|\overline{K}_2|\right]^{-1/2} \exp\{-\frac{1}{2}(\overline{y_2} - \overline{m}_2)^T \overline{K}_n^{-1}(\overline{y_2} - \overline{m}_2)\}, \end{gathered} \tag{12}$$

and p^2 ordinary differential Eqs:

$$\begin{gathered} \frac{\partial K_y(t_1, t_2)}{\partial t_2} = K_y(t_1, t_2) K_y(t_2)^{-1} \varphi_{21}(m_y(t_2), K_y(t_2), t_2)^{-1}, \\ K_y(t_1, t_2) = K_y(t_2, t_1)^T, K_y(t_1, t_1) = K_y(t_1) \text{ at } t_1 = t_2, \end{gathered} \tag{13}$$

$$\begin{gathered} \frac{dk_y(\tau)}{d\tau} = \varphi_{21}(m_y^*, K_y^*) K_y^{*-1} k_y(\tau), \\ k_y(\tau) = K_y(t_1, t_1 + \tau) = k_y(-\tau)^T, k(0) = K_y^*, \end{gathered} \tag{14}$$

where

$$\lambda = [\lambda_1^T \, \lambda_2^T]^T, \quad \overline{m}_2 = \left[m_y(t_1)^T \, m_y(t_2)^T\right]^T,$$

$$\overline{K}_2 = \begin{bmatrix} K_y(t_1, t_1) & K_y(t_1, t_2) \\ K_y(t_2, t_1) & K_y(t_2, t_2) \end{bmatrix}, \quad \overline{y}_2 = [y_1^T \, y_2^T]^T.$$

For stationary systems in Eqs (11), (14) one use the spectral density instead of covariance matrix and amplitude-frequency characteristic.

ELT Eqs (6), (7), (8), (12), (13) for the combined StS and ChS described by Eq (4) at $b(Y, t) = b_0(t)$

$$\dot{Y} = a(Y, t) + b_0(t)V, \quad Y(t_0) = Y_0 \tag{15}$$

after statistical linearization by known formulae [5]:

$$a(Y, t) = a_0(m_y, K_y) + a_1(m_y, K_y)Y^0, \quad (Y^0 = (Y - m)) \tag{16}$$

may be presented in the form:

$$\dot{m}_y = a_0(m_y, K_y, t) = E_N a(Y, t), \quad m_y(t_0) = m_{y_0}, \tag{17}$$

$$\dot{K}_y = a_1(m_y, K_y, t)K_y + K_y a_1(m_y, K_y, t)^T + b_0(t)\nu(t)b_0(t)^T, \quad K_y(t_0) = K_{y_0}, \quad (18)$$

$$\frac{\partial K_y(t_1,t_2)}{\partial t_2} = K_y(t_1,t_2)a_1(t_1,t_2)^T \text{ at } t_2 > t_1$$
$$K_y(t_1,t_2) = K_y(t_2,t_1)^T \text{ at } t_2 < t_1. \quad (19)$$

Putting $\dot{m}_y = 0$, $\dot{K}_y = 0$, $K_y(t_1, t_1 + \tau) = k_y(\tau)$ we get Eqs for stationary processes. Eqs (17)–(19) are the basic Eqs for ELT analytical modeling of Combined StS and ChS.

It is evident that in combined StS and ChS chaotic properties will prevail when corresponding amplitude-frequency characteristic for Eq (17) parametrically dependent on m_y and K_y is not simple.

When the stochastic input signal contains the harmonic component one use combined harmonic and statistical linearization [6].

For input signal given by canonical expansion (CE) we develop corresponding ELT based on [6] and symbolic calculations [7], [8]. IT based on CE ELT are effective as for analytical, statistical and combined modeling (simulation).

For the on-line information optimal nonlinear estimation in ChS with small stochastic perturbations is based on Kushner-Stratonovich Eq for conditional density or corresponding Eq for conditional characteristic function. These Eqs are very difficult for solution on nowadays computers even for simple models [5], [6].

According to robust ELT based on normal density one may use Kalman, Kalman-Bucy or Pugachev Eqs with coefficients depending on mathematical expectation and covariance matrix.

The discussed ELT was realized by modern software tools developed in MATLAB.

Applications: statistical and chaotical vibromechanics, fluctuations of the Earth Pole and irregular rotation of the Earth.

4 Examples

4.1 Harmonically and stochastically forced Duffing Eq

Let us consider harmonically and stochastically forced Duffing Eq:

$$\ddot{X} + \alpha\dot{X} + \beta X + \gamma X^3 = A_0 + A\cos(\omega t) + V, \; X(t_0) = X_0, \dot{X}(t_0) = \dot{X}_0, \quad (20)$$

where α, β, γ, A_0, A, ω being constant values; $V = \dot{W}$ is the white noise of the intensity $\nu = \nu(t)$; W being the process with independent increments. After putting $X_1 = X$, $X_2 = \dot{X}$ and X^3 statistical linearization performing by the formulae [5]:

$$X_1^3 \approx K_0(m_1, D_1)m_1 + K_1(m_1, D_1)X_1^0,$$
$$K_0(m_1, D_1) = (m_1^2 + 3D_1), \quad K_1(m_1, D_1) = 3(m_1^2 + D_1) \tag{21}$$

we get Eqs for mathematical expectations:

$$\dot m_1 = m_2,\ \dot m_2 = -\alpha m_2 - \beta m_1 - \gamma(m_1^2 + 3D_1)m_1 + A_0 + A\cos(\omega t),$$
$$m_1(t_0) = m_{1_0}, \quad m_2(t_0) = m_{2_0} \tag{22}$$

and centered variables:

$$\dot X_1^0 = X_2^0,\ \dot X_2^0 = -\alpha X_2^0 - \beta X_1^0 - 3\gamma(m_1^2 + D_1)X_1^0 + V,$$
$$X_1^0(t_0) = X_{1_0}^0, \quad X_2^0(t_0) = X_{2_0}^0. \tag{23}$$

Ordinary differential Eqs for $K_{11}(t) = D_1 = MX_2^{02}$, $K_{12}(t) = K_{X\dot X}$, $K_{22}(t) = D_2 = MX_2^{02}$ may be presented in the form

$$\begin{cases} \dot K_{11} = 2K_{12},\ \dot K_{22} = -2\alpha K_{22} - 2\big[\beta + 3\gamma(m_1^2 + D_1)\big]K_{12} + \nu, \\ \dot K_{12} = K_{22} - \alpha K_{12} - \big[\beta + 3\gamma(m_1^2 + D_1)\big]K_{11} \end{cases} \tag{24}$$

at corresponding initial conditions. Eqs (22) and (24) are interconnected due to dependence of coefficients at m_i, D_i $(i = 1, 2)$ on m_1 and D_1.

Eqs (22) and (24) permit to detect the following effects.

1. At $A_0 = 0$ and given D_1 the chaos structure described by Eqs (22) coincides with the structure of chaos in Eqs (22) at $V = 0$ for $\beta_\xi = \beta + 3\gamma D_1$ [4].
2. At $A_0 = 0$ and given $\nu = const$ the chaos structure described by Eqs (22) coincides with the structure of chaos in Eqs (22) at $V = 0$ for m_1 and D_1 described by joint set of Eqs (22) and (24) for $\beta = \beta + \triangle\beta(\nu)$, $\triangle\beta(\nu) = 3\gamma D_1(t, \nu)$.
3. At given periodical intensity $\nu = \nu(\Omega t)$ we find the bias detecting effect due to parametric term $\langle D_1 m_1\rangle$ at $n_1\omega + n_2\Omega = 0$. Such structure of chaos corresponds to bias chaos structure [4] at $A_{\xi 0} = A_0 - 3\gamma\langle m_1 D_1\rangle$.

4.2 Quasiharmonically forced Duffing Eq

Let us consider quasiharmonically forced Duffing Eq:

$$\ddot X + \alpha\dot X + \beta X + \gamma X^3 = A_0 + A\cos(\psi),$$
$$X(t_0) = X_0, \quad \dot X(t_0) = \dot X_0, \tag{25}$$

$$\dot\psi = \omega + Y, \tag{26}$$

$$\dot{Y} = -\alpha_1 Y + V, \quad Y(t_0) = Y_0 \tag{27}$$

where α, β,γ, A_0, A, ω, being constants; $\nu = \nu(t)$ being the intensity of the white noise V. Putting

$$\begin{gathered} X_1 = X,\ X_2 = \dot{X}_1,\ X_3 = Y, \\ X_1^3 \approx (m_1^2 + 3D_1)m_1 + 3(m_1^2 + D_1)X_1^0, \\ \cos(\psi) \approx e^{-D_3/2}\cos(\omega t + m_3) + \left[-e^{-D_3/2}\sin(\omega t + m_3)\right]X_3^0, \end{gathered} \tag{28}$$

we get the joint set of Eqs for m_i, $K_{ii} = D_i$ and K_{ij} $(i, j = 1, 2, 3)$:

$$\begin{cases} \dot{m}_1 = m_2,\ \dot{m}_2 = -\alpha m_2 - \beta m_1 - \gamma(m_1^2 + 3D_1)m_1 + \\ \qquad A_0 + Ae^{-D_3/2}\cos(\omega t + m_3), \\ \dot{m}_3 = -\alpha_1 m_3, \end{cases} \tag{29}$$

$$\begin{cases} \dot{X}_1^0 = X_2^0 \\ \dot{X}_2^0 = -\alpha X_2^0 - \beta X_1^0 - 3\gamma(m_1^2 + D_1)X_1^0 - Ae^{-D_3/2}\sin(\omega t + m_3)X_3^0, \\ \dot{X}_3^0 = -\alpha_1 X_3^0 + V; \end{cases} \tag{30}$$

$$\begin{aligned} &\dot{K}_{11} = 2K_{12},\ \dot{K}_{33} = \nu - 2\alpha_1 K_{33}, \\ &\dot{K}_{22} = -2\left[\beta + 3\gamma(m_1^2 + D_1)\right]K_{12} + 2\alpha K_{22} - 2Ae^{-D_3/2}\sin(\omega t + m_3)K_{23}, \\ &\dot{K}_{12} = K_{22} - \left[\beta + 3\gamma(m_1^2 + D_1)\right]K_{11} - \alpha K_{12} + Ae^{-D_3/2}\sin(\omega t + m_3)K_{13}, \\ &\dot{K}_{13} = K_{23} - \alpha_1 K_{13}, \\ &\dot{K}_{23} = -\left[\beta + 3\gamma(m_1^2 + D_1)\right]K_{13} + Ae^{-D_3/2}\sin(\omega t + m_3)K_{33} - (\alpha + \alpha_1)K_{23}. \end{aligned}$$

5 Conclusions

1. At given $m_3^* = 0$, $D_1 = D_1^*$, $D_3 = D_3^*$, Eqs (25) have chaos structure for Eq (25) at $Y \equiv 0$ [4] if we put $\beta_\xi = \beta + 3\gamma D_1^*$, $A_\xi = Ae^{-D_3^*/2}$.
2. At given $\nu = const$ the chaos structure for Eq (25) is described by known formulae [4] at $\beta_\xi = \beta + \triangle\beta(\nu)$, $\triangle\beta(\nu) = 3\gamma D_1(t, \nu)$, $A_\xi = Ae^{-D_3(t,\nu)/2}$. More then that due to the dependence of $D_1(t)$ on $\sin(n\omega t)$ and $\cos(n\omega t)$ $(n = 2, 3, \dots)$ we find the parametric effect of pulsation.
3. At given periodical intensity $\nu(\Omega t)$ we may find bias effect on frequencies $n\Omega$ and $n\omega$ in chaos structure.

The derived results of modeling of the chaos structure coincides with the results from [4], [3], [1], [2].

In Figure 1 presented the secant plane $t = [mod 2\pi/\omega]$ for the case of rigid elastic characteristic: $\alpha = 0.2$, $\beta = 1$, $\gamma = 1$, $A_0 = 0$, $A = 50$, $\omega = 1.9$.

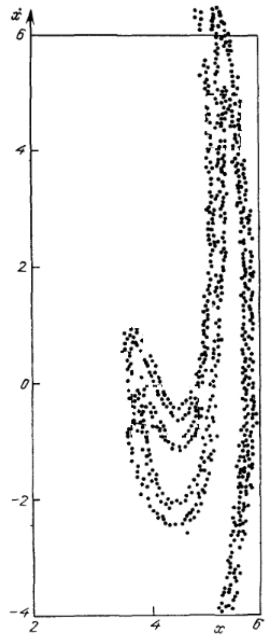

Fig. 1. Chaos structure for rigid elastic characteristic.

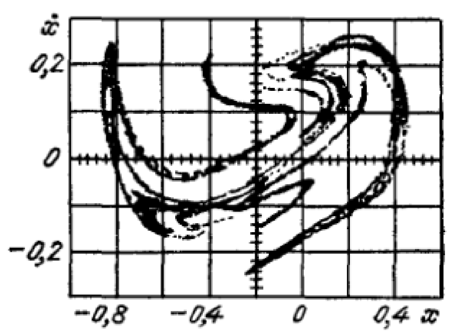

Fig. 2. Chaos structure for asymmetrical external influence.

In Figure 2 presented the secant plane $t = [mod 2\pi/\omega]$ for the case of asymmetrical external influence: $\alpha = 0.05$, $\beta = 0$, $\gamma = 1$, $A_0 = 0.04$, $A = 0.14$, $\omega = 1$.

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