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Modelling and Analysis of the Non-Linear Particle Movement and Fluid Flow in the Non-Inertial Complex Rotating Coordinate Systems

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Abstract: The differential equation array is derived in the non-inertial three-dimensional coordinate system rotating around the vertical axis. The equations are used in the Cartesian coordinates for inviscid liquid with account of the possible compressibility and effect of the mass forces (gravity and other ones, e.g. electromagnetic) as a starting point in the modeling and analysis. Transformation of the coordinates is performed to derive the Navier-Stokes equations in the non-inertial rotating coordinate system. Also it is considered particle movement in the rotating coordinate system, which is, in turn, rotating too as a whole around the other vertical z-axis of the first coordinate system being at the distance R_0 from the centre of z-axis. The Lagrangian of the free particle movement in the complex rotating coordinate system is written. Following the conventional technique for derivation of movement from the Lagrangian (derivative by time minus derivative by coordinate), the equation array for particle's movement is obtained. The both developed mathematical models are applied for computer simulation and analyzed for a number of physical situations.
Keywords: Modeling; Rotation of Coordinate System; Navier-Stokes Equations; Analysis; Centrifugal Forces; Particle Movement; Chaotic; Computer Simulation.

1 Introduction

Let us start with the coordinate system shown in Fig. 1 together with an example of the natural rotational flow. The one Cartesian coordinate system x_1, x_2, x_3 with the vertical axis x_3 is immovable, while the other coordinate system is rotating around the vertical axis, with the rotation speed ω. The rotating coordinate system can be done with the vertical axis coinciding with x_3 or shifted from the central axis ($x_1 = x_2 = 0$) on some distance R_0. It is known that diverse rotational (vortex) flows and particle motions, vortices and rotating flows were always wondering and sometimes scaring people: swirling flows at different scales in the nature - the spiral galaxies, the atmospheric hurricanes, sea and river vortices, and up to conventional in the everyday life [1-3].

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Interesting that normally vortices are with the counter clockwise rotations as in Fig. 1:

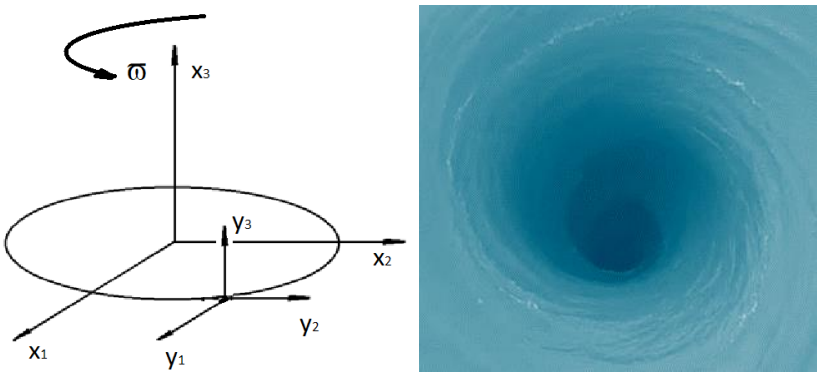


Fig. 1. Immovable and rotating around vertical axis Cartesian coordinate systems and example of natural vortex flow

The intensive rotational movement and mixing are fascinating phenomena and may be very effective in a number of engineering and technological applications [4-11]. Therefore many theoretical aspects have been studied for rotational flows in a number of different situations [12-19]. But the problem is still not enough studied in many theoretical and practical aspects. Let us start with the Navier-Stokes equation array for the inviscid liquid as follows:

$$\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{V}) = 0, \qquad \frac{\partial}{\partial t}\rho \boldsymbol{V} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho \boldsymbol{V} = -\boldsymbol{\nabla} P. \qquad (1)$$

The momentum equation can be written in more general form with mass forces:

$$\frac{\partial}{\partial t}\rho \boldsymbol{V} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho \boldsymbol{V} = -\boldsymbol{\nabla} P + \rho \boldsymbol{g} + \rho \boldsymbol{f} + \rho \nu \boldsymbol{\Delta V}. \qquad (2)$$

Here $\boldsymbol{\nabla}, \boldsymbol{\Delta}$ are the vector gradient and Laplace operator, ρ is density of liquid, *P*- its pressure, *t*- time, ν- kinematic viscosity coefficient, $\boldsymbol{V} = \{V_1, V_2, V_3\}$, $\boldsymbol{f} = \{f_1, f_2, f_3\}$, $\boldsymbol{g} = \{g_1, g_2, g_3\}$- vectors of the velocity, volumetric mass forces and gravity, respectively.

In the nature and technical systems sometimes fluid flow or just particle movement is going in a situation when the coordinate system rotates: river flow on Earth rotating itself and around the Sun, the flow of liquids in rotating parts of a car, and so on. In some cases such rotation is intensive and may affect the particle movement or fluid flow in the rotation system, sometimes a lot. This paper is devoted to some cases of those. First, derivation of the Navier-Stokes equation array for fluid flow and the equations for separate particle movement in the rotating coordinate system are considered. Then computer simulation and analysis is done. The focus is on the peculiarities in a fluid flow or particle movement, which are revealed and analyzed through the computer simulation based on the mathematical models derived for the examples of considered rotational systems.

The rotating coordinates y_1, y_2, y_3 are expressed through the outgoing immovable coordinates x_1, x_2, x_3 as follows

$$y_1 = cos(\omega t)x_1 - sin(\omega t)x_2,\ y_2 = sin(\omega t)x_1 + cos(\omega t)x_2,\ y_3 = x_3. \quad (3)$$

This is for rotation in the plane (x_1, x_2) with the constant speed of rotation ω.

2 Transformation of Coordinates and Derivatives due to Rotation

Transformation of the derivatives by the spatial coordinates from the immovable Cartesian coordinate system to the rotational one are performed as follows

$$\frac{\partial Q}{\partial x_1} = \cos(\omega t)\frac{\partial Q}{\partial y_1} + \sin(\omega t)\frac{\partial Q}{\partial y_2},\ \frac{\partial Q}{\partial x_2} = cos(\omega t)\frac{\partial Q}{\partial y_2} - \sin(\omega t)\frac{\partial Q}{\partial y_1}, \quad (4)$$

With account of (3), (4), the transformation of the coordinates and spatial derivatives depend on time and rotation speed. Only the vertical coordinate remains the same: $\frac{\partial Q}{\partial x_3} = \frac{\partial Q}{\partial y_3}$. The velocity components in the outgoing immovable Cartesian coordinates can be presented as

$$V_1 = \frac{dx_1}{dt} = \frac{d}{dt}(\ cos(\omega t)y_1 + \sin(\omega t)y_2), \quad (5)$$

$$V_2 = \frac{dx_2}{dt} = \frac{d}{dt}(-sin(\omega t)y_1 + cos(\omega t)y_2),\ \ V_3 = \frac{dx_3}{dt} = \frac{dy_3}{dt},$$

For the velocity components in rotational coordinate system the following assignments are taken:

$$U_1 = \dot{y}_1,\ U_2 = \dot{y}_2,\ U_3 = \dot{y}_3.$$

where the dots assign the derivatives by time. Thus, from (5) results

$$V_1 = \omega(-\sin(\omega t)y_1 + cos(\omega t)y_2) + cos(\omega t)U_1 + \sin(\omega t)U_2, \quad (6)$$

$$V_2 = -\omega(cos(\omega t)y_1 + sin(\omega t)y_2) - sin(\omega t)U_1 + cos(\omega t)U_2,\ \ V_3 = U_3.$$

Then derivatives from the velocity components are transformed:

$$\frac{\partial}{\partial t}V_1 = -\omega^2(cos(\omega t)y_1 + sin(\omega t)y_2)$$

$$+2\omega(-\sin(\omega t)\dot{y}_1 + cos(\omega t)\dot{y}_2) + cos(\omega t)\ddot{y}_1 + \sin(\omega t)\ddot{y}_2, \quad (7)$$

$$\frac{\partial}{\partial t}V_2 = -\omega^2(-\sin(\omega t)y_1 + cos(\omega t)y_2) - 2\omega(cos(\omega t)\dot{y}_1 + sin(\omega \cdot t)\dot{y}_2)$$

$$-sin(\omega \cdot t)\ddot{y}_1 + cos(\omega \cdot t)\ddot{y}_2,\quad \frac{\partial}{\partial t}V_3 = \ddot{y}_3.$$

3 The Spatial and Temporal Derivatives in the Rotational Coordinate System and Derivation of Navier-Stokes Equations

According to the equations (4) the spatial derivatives can be written as follows

$$\frac{\partial(\rho V_1)}{\partial x_1} = (\omega(-sin(\omega t)y_1 + cos(\omega t)y_2) + cos(\omega t)U_1 + sin(\omega t)U_2)\cdot$$
$$\cdot\left(cos(\omega t)\frac{\partial}{\partial y_1} + sin(\omega t)\frac{\partial}{\partial y_2}\right)\rho +$$
$$+\rho\left(cos(\omega t)\frac{\partial}{\partial y_1} + sin(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(cos(\omega t)U_1 + sin(\omega t)U_2), \quad (8)$$
$$\frac{\partial(\rho V_1)}{\partial x_2} = (\omega(-sin(\omega t)y_1 + cos(\omega t)y_2) + cos(\omega t)U_1 + sin(\omega t)U_2)\cdot$$
$$\cdot\left(-sin(\omega t)\frac{\partial}{\partial y_1} + cos(\omega t)\frac{\partial}{\partial y_2}\right)\rho +$$
$$+\rho\left[\left(-sin(\omega t)\frac{\partial}{\partial y_1} + cos(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(cos(\omega t)U_1 + sin(\omega t)U_2) + \omega\right];$$
$$\frac{\partial(\rho V_2)}{\partial x_1} = (-\omega(cos(\omega t)y_1 + sin(\omega t)y_2) - sin(\omega t)U_1 + cos(\omega t)U_2)\cdot$$
$$\cdot\left(cos(\omega t)\frac{\partial}{\partial y_1} + sin(\omega t)\frac{\partial}{\partial y_2}\right)\rho +$$
$$+\rho\left[\left(cos(\omega t)\frac{\partial}{\partial y_1} + sin(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(-sin(\omega t)U_1 + cos(\omega t)U_2) - \omega\right],$$
$$\frac{\partial(\rho V_2)}{\partial x_2} = (-\omega(cos(\omega t)y_1 + sin(\omega t)y_2) - sin(\omega t)U_1 + cos(\omega t)U_2)\cdot$$
$$\cdot\left(-sin(\omega t)\frac{\partial}{\partial y_1} + cos(\omega t)\frac{\partial}{\partial y_2}\right)\rho$$
$$+\rho\left(-sin(\omega t)\frac{\partial}{\partial y_1} + cos(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(-sin(\omega t)U_1 + cos(\omega t)U_2);$$
$$\frac{\partial(\rho V_3)}{\partial x_1} = V_3\left(cos(\omega t)\frac{\partial\rho}{\partial y_1} + sin(\omega t)\frac{\partial\rho}{\partial y_2}\right) + \rho\left(cos(\omega t)\frac{\partial V_3}{\partial y_1} + sin(\omega t)\frac{\partial V_3}{\partial y_2}\right),$$
$$\frac{\partial(\rho V_3)}{\partial x_2} = V_3\left(cos(\omega t)\frac{\partial\rho}{\partial y_2} - sin(\omega t)\frac{\partial\rho}{\partial y_1}\right) + \rho\left(cos(\omega t)\frac{\partial V_3}{\partial y_2} - sin(\omega t)\frac{\partial V_3}{\partial y_1}\right);$$
$$\frac{\partial P}{\partial x_1} = cos(\omega t)\frac{\partial P}{\partial y_1} + sin(\omega t)\frac{\partial P}{\partial y_2}, \quad \frac{\partial P}{\partial x_2} = -sin(\omega t)\frac{\partial P}{\partial y_1} + cos(\omega t)\frac{\partial P}{\partial y_2}.$$

Here are: $\frac{\partial(\rho V_1)}{\partial x_3} = \frac{\partial(\rho V_1)}{\partial y_3}$; $\frac{\partial(\rho V_2)}{\partial x_3} = \frac{\partial(\rho V_2)}{\partial y_3}$; $\frac{\partial(\rho V_3)}{\partial x_3} = \frac{\partial(\rho V_3)}{\partial y_3}$; $\frac{\partial P}{\partial x_3} = \frac{\partial P}{\partial y_3}$.

For incompressible liquid, it is following from (8) that the flow gradient $(\rho V_1)/\partial x_1$ transforms to zero in the rotational coordinate system, while by the second coordinate x_2 it is just equal to $\omega\rho$. For the second velocity component, gradient $\partial(\rho V_2)/\partial x_2$ is, respectively -$\omega\rho$ and 0 by the coordinates x_1,x_2. The

vertical velocity component remains non-zero, it does not depend on rotation of the coordinate system. And the vertical gradients for all velocity components are not transformed in the new coordinate system.

4 Transformation of the Navier-Stokes Equation array

Now the equation array (1), (2), with account of (6), (8) can be written as:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho W_1)}{\partial y_1}+\frac{\partial(\rho W_2)}{\partial y_2}+\frac{\partial(\rho W_3)}{\partial y_3}=0, \tag{9}$$

$$\frac{\partial}{\partial t}(\rho W_1)+W_1\frac{\partial(\rho W_1)}{\partial y_1}+W_2\frac{\partial(\rho W_1)}{\partial y_2}+W_3\frac{\partial(\rho W_1)}{\partial y_3}=$$
$$=-\frac{\partial P}{\partial y_1}+\rho(cos(\omega t)g_1-sin(\omega t)g_2)+\rho(cos(\omega t)f_1-sin(\omega t)f_2),$$

$$\frac{\partial}{\partial t}(\rho W_2)+W_1\frac{\partial(\rho W_2)}{\partial y_1}+W_2\frac{\partial(\rho W_2)}{\partial y_2}+W_3\frac{\partial(\rho W_2)}{\partial y_3}=$$
$$-\frac{\partial P}{\partial y_2}+\rho(sin(\omega t)g_1+cos(\omega t)g_2)+\rho(sin(\omega t)f_1+cos(\omega t)f_2),$$

$$\frac{\partial}{\partial t}(\rho W_3)+W_1\frac{\partial(\rho W_3)}{\partial y_1}+W_2\frac{\partial(\rho W_3)}{\partial y_2}+W_3\frac{\partial(\rho W_3)}{\partial y_3}=-\frac{\partial P}{\partial y_3}+\rho g_3+\rho f_3,$$

where are: $W_1=U_1-\omega y_2$, $W_2=U_2+\omega y_1$, $W_3=U_3$.
Thus, the Navier-Stokes equation array in the new variables has similar to the above (1), (2) form represented in the immovable coordinate system. Therefore, in the appropriate variables the equations remain the same but the right parts contain the volumetric forces written in the new coordinate system (rotating one). The effect of rotation is hidden in the additive replace of the variable similar to [20, p. 60], when $\boldsymbol{W}=\boldsymbol{U}+[\boldsymbol{\omega}\mathrm{x}\boldsymbol{r}]$.
Despite the cumbersome view of the general case equations (9), they become much simpler for a number of specific conditions, e.g. for incompressible liquid:

$$\frac{\partial W_1}{\partial y_1}+\frac{\partial W_2}{\partial y_2}+\frac{\partial W_3}{\partial y_3}=0,\qquad \frac{\partial W_1}{\partial t}+W_1\frac{\partial W_1}{\partial y_1}+W_2\frac{\partial W_1}{\partial y_2}+W_3\frac{\partial W_1}{\partial y_3}=$$
$$=-\frac{1}{\rho}\frac{\partial P}{\partial y_1}+cos(\omega t)g_1-sin(\omega t)g_2+cos(\omega t)f_1-sin(\omega t)f_2, \tag{10}$$

$$\frac{\partial W_2}{\partial t}+W_1\frac{\partial W_2}{\partial y_1}+W_2\frac{\partial W_2}{\partial y_2}+W_3\frac{\partial W_2}{\partial y_3}=$$
$$=-\frac{1}{\rho}\frac{\partial P}{\partial y_2}+sin(\omega t)g_1+cos(\omega t)g_2+sin(\omega t)f_1+cos(\omega t)f_2,$$

$$\frac{\partial W_3}{\partial t}+W_1\frac{\partial W_3}{\partial y_1}+W_2\frac{\partial W_3}{\partial y_2}+W_3\frac{\partial W_3}{\partial y_3}=-\frac{1}{\rho}\frac{\partial P}{\partial y_3}+g_3+f_3,$$

or in the normal rotating coordinates:

$$\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}+\frac{\partial U_3}{\partial y_3}=0,\quad \frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial y_1}+U_2\frac{\partial U_1}{\partial y_2}+U_3\frac{\partial U_1}{\partial y_3}= \tag{11}$$

$$= \omega(y_2\frac{\partial U_1}{\partial y_1} - y_1\frac{\partial U_1}{\partial y_2}) + \omega(2U_2 + \omega y_1) - \frac{1}{\rho}\frac{\partial P}{\partial y_1} + cos(\omega t)g_1 - sin(\omega t)g_2 + cos(\omega t)f_1 - sin(\omega t)f_2$$

,

$$\frac{\partial U_2}{\partial t} + U_1\frac{\partial U_2}{\partial y_1} + U_2\frac{\partial U_2}{\partial y_2} + U_3\frac{\partial U_2}{\partial y_3} =$$

$$= \omega(y_2\frac{\partial U_2}{\partial y_1} - y_1\frac{\partial U_2}{\partial y_2}) + \omega(\omega y_2 - 2U_1) - \frac{1}{\rho}\frac{\partial P}{\partial y_2} + sin(\omega t)g_1 + cos(\omega t)g_2 + sin(\omega t)f_1 + cos(\omega t)f_2$$

,

$$\frac{\partial U_3}{\partial t} + U_1\frac{\partial U_3}{\partial y_1} + U_2\frac{\partial U_3}{\partial y_2} + U_3\frac{\partial U_3}{\partial y_3} = \omega(y_2\frac{\partial U_3}{\partial y_1} - y_1\frac{\partial U_3}{\partial y_2}) - \frac{1}{\rho}\frac{\partial P}{\partial y_3} + g_3 + f_3,$$

The first peculiarity of (11) is absence of the stationary solution if any of the gravitational components g_1, g_2 or any other mass force f_1, f_2 is acting in the plane y_1y_2. Only vertical components of these forces do not cause non-stationary flow regimes. The situation is very unusual: any (even constant!) external force in this plane makes the fluid flow regime principally non-stationary. The system (11) without external forces results in

$$\frac{\partial U_1}{\partial y_1} + \frac{\partial U_2}{\partial y_2} + \frac{\partial U_3}{\partial y_3} = 0, \quad \frac{\partial U_s}{\partial t} + U_1\frac{\partial U_s}{\partial y_1} + U_2\frac{\partial U_s}{\partial y_2} + U_3\frac{\partial U_s}{\partial y_3} =$$

$$= \omega\left(y_2\frac{\partial U_s}{\partial y_1} - y_1\frac{\partial U_s}{\partial y_2}\right) + \omega(2U_2 - 2U_1 + \omega(y_1 + y_2)) - \frac{1}{\rho}divP, \qquad (12)$$

$$\frac{\partial U_2}{\partial t} + U_1\frac{\partial U_2}{\partial y_1} + U_2\frac{\partial U_2}{\partial y_2} + U_3\frac{\partial U_2}{\partial y_3} =$$

$$= \omega(y_2\frac{\partial U_2}{\partial y_1} - y_1\frac{\partial U_2}{\partial y_2}) + \omega(\omega y_2 - 2U_1) - \frac{1}{\rho}\frac{\partial P}{\partial y_2},$$

$$\frac{\partial U_3}{\partial t} + U_1\frac{\partial U_3}{\partial y_1} + U_2\frac{\partial U_3}{\partial y_2} + U_3\frac{\partial U_3}{\partial y_3} = \omega(y_2\frac{\partial U_3}{\partial y_1} - y_1\frac{\partial U_3}{\partial y_2}) - \frac{1}{\rho}\frac{\partial P}{\partial y_3},$$

where $U_s = U_1 + U_2 + U_3$. In case of $U_s = const$ from the second equation (12) follows

$$divP = \omega\rho(2(U_2 - U_1) + \omega(y_1 + y_2)), \quad (13)$$

so that pressure in a flow is totally determined by the centrifugal and Coriolis forces independent of the value U_s. Only the position of the point in a flow $(y_1 + y_2)$ and the velocity difference $U_2 - U_1$ determine the pressure (but not their absolute values).

In more general case, the equations (9) for the compressible liquid yield:

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial U_1}{\partial y_1} + \frac{\partial U_2}{\partial y_2} + \frac{\partial U_3}{\partial y_3}\right) + U_1\frac{\partial \rho}{\partial y_1} + U_2\frac{\partial \rho}{\partial y_2} + U_3\frac{\partial \rho}{\partial y_3} = \omega(y_2\frac{\partial \rho}{\partial y_1} - y_1\frac{\partial \rho}{\partial y_2}),$$

$$\rho(\frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial y_1}+U_2\frac{\partial U_1}{\partial y_2}+U_3\frac{\partial U_1}{\partial y_3})=(\omega y_2-U_1)\frac{\partial \rho}{\partial t}-(\omega y_2-U_1)^2\frac{\partial \rho}{\partial y_1}+(\omega y_2-U_1)(U_2+\omega y_1)\frac{\partial \rho}{\partial y_2}+(\omega y_2-U_1)U_3\frac{\partial \rho}{\partial y_3}+\omega\rho(y_2\frac{\partial U_1}{\partial y_1}-\frac{\partial U_1}{\partial y_2})+\omega\rho(2U_2+\omega y_1)-\frac{\partial P}{\partial y_1}+\rho(cos(\omega t)g_1-sin(\omega t)g_2+cos(\omega t)f_1-sin(\omega t)f_2)$$

$$,\quad (14)$$

$$\rho(\frac{\partial U_2}{\partial t}+U_1\frac{\partial U_2}{\partial y_1}+U_2\frac{\partial U_2}{\partial y_2}+U_3\frac{\partial U_2}{\partial y_3})=$$

$$=-(U_2+\omega y_1)\frac{\partial \rho}{\partial t}+(\omega y_2-U_1)(U_2+\omega y_1)\frac{\partial \rho}{\partial y_1}-(U_2+\omega y_1)^2\frac{\partial \rho}{\partial y_2}-(U_2+\omega y_1)U_3\frac{\partial \rho}{\partial y_3}+\omega\rho(y_2\frac{\partial U_2}{\partial y_1}-y_1\frac{\partial U_2}{\partial y_2})+\omega\rho(\omega y_2-2U_1)-\frac{\partial P}{\partial y_2}+\rho(sin(\omega t)g_1+cos(\omega t)g_2+sin(\omega t)f_1+cos(\omega t)f_2)$$

$$,$$

$$\rho\left(\frac{\partial U_3}{\partial t}+U_1\frac{\partial U_3}{\partial y_1}+U_2\frac{\partial U_3}{\partial y_2}+U_3\frac{\partial U_3}{\partial y_3}\right)=-U_3\frac{\partial \rho}{\partial t}+(\omega y_2-U_1)U_3\frac{\partial \rho}{\partial y_1}-(U_2+\omega y_1)U_3\frac{\partial \rho}{\partial y_2}-{U_3}^2\frac{\partial \rho}{\partial y_3}+\omega\rho(y_2\frac{\partial U_3}{\partial y_1}-y_1\frac{\partial U_3}{\partial y_2})-\frac{\partial P}{\partial y_3}+\rho(g_3+f_3)$$

.

An analysis of the obtained equation array (14), in contrast to (11), for the compressible liquid, shows the stationary flow regime impossible even in absence of any external mass forces because of the term $\frac{\partial \rho}{\partial t}$ present in all 3 momentum equations, except very specific case when this term is zero and there is only the spatial density distribution. The flow regime (13) is unique for the incompressible flow and is not available for compressible one as clearly seen from the (14). The first equation (14) contains the "density vortex" to the right, which shows that the higher is rotation speed, the more intensive is density variation by time and space. This, in turn, may cause further growing of the "density vortex", and so on. By the high rotation speed despite the small coordinate values the term $\omega(y_2\frac{\partial \rho}{\partial y_1}-y_1\frac{\partial \rho}{\partial y_2})$ can be big due to substantial density gradients, which may grow with time creating the singularity point in the centre of rotation. If we present the first equation in the form

$$\omega=\frac{\frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}+\frac{\partial U_3}{\partial y_3}\right)+U_1\frac{\partial \rho}{\partial y_1}+U_2\frac{\partial \rho}{\partial y_2}+U_3\frac{\partial \rho}{\partial y_3}}{(y_2\frac{\partial \rho}{\partial y_1}-y_1\frac{\partial \rho}{\partial y_2})},\quad (15)$$

we can consider (15) as the equation for the vortex formation in the compressible fluid flow due to the density spatial gradients. What is very interesting from this correlation that even in case of zero velocities and zero velocity gradients, the vortex formation can start because of the density gradients: $\frac{\partial \rho}{\partial t}$, $\frac{\partial \rho}{\partial y_1}$, $\frac{\partial \rho}{\partial y_2}$. The signs of the terms $\frac{\partial \rho}{\partial t}$ and $y_2\frac{\partial \rho}{\partial y_1}-y_1\frac{\partial \rho}{\partial y_2}$ determine the direction of the rotation.

It must be underlined that zero value of the term $(y_2\frac{\partial\rho}{\partial y_1} - y_1\frac{\partial\rho}{\partial y_2})$ in the equation (15) excludes influence of the rotation in the first equation (14), while any small values of this term in (15) may create the reason for an abrupt grow of the rotation (vortex) in a compressible fluid, which may start a vortex flow in the immovable liquid or create a vortex in the fluid flow. For example, the vortex birth in a volume of liquid or gas being initially in the rest can appear as follows. From (15) it is got

$$\omega = \frac{\frac{\partial\rho}{\partial t}}{(y_2\frac{\partial\rho}{\partial y_1} - y_1\frac{\partial\rho}{\partial y_2})}, \qquad (16)$$

which shows that the vortex cannot be created in case of symmetrical density gradients. Naturally such conditions may happen due to abrupt local heating causing the remarkable variation of the liquid or gas density in time and space e.g. from a solar radiation concentrated in a local region. Assuming that gravity is acting in the vertical direction and substituting (16) into (14) yields:

$$\rho(\frac{\partial U_1}{\partial t} + U_1\frac{\partial U_1}{\partial y_1} + U_2\frac{\partial U_1}{\partial y_2} + U_3\frac{\partial U_1}{\partial y_3}) = (\omega y_2 - U_1)\frac{\partial\rho}{\partial t} - (\omega y_2 - U_1)^2\frac{\partial\rho}{\partial y_1} + (\omega y_2 - U_1)(U_2 + \omega y_1)\frac{\partial\rho}{\partial y_2} + (\omega y_2 - U_1)U_3\frac{\partial\rho}{\partial y_3} + \omega\rho(y_2\frac{\partial U_1}{\partial y_1} - y_1\frac{\partial U_1}{\partial y_2}) + \omega\rho(2U_2 + \omega y_1) - \frac{\partial P}{\partial y_1},$$

$$\frac{\partial\rho}{\partial t} + \rho\left(\frac{\partial U_1}{\partial y_1} + \frac{\partial U_2}{\partial y_2} + \frac{\partial U_3}{\partial y_3}\right) + U_1\frac{\partial\rho}{\partial y_1} + U_2\frac{\partial\rho}{\partial y_2} + U_3\frac{\partial\rho}{\partial y_3} = \omega(y_2\frac{\partial\rho}{\partial y_1} - y_1\frac{\partial\rho}{\partial y_2}), \qquad (17)$$

$$\rho(\frac{\partial U_2}{\partial t} + U_1\frac{\partial U_2}{\partial y_1} + U_2\frac{\partial U_2}{\partial y_2} + U_3\frac{\partial U_2}{\partial y_3}) = -(U_2 + \omega y_1)\frac{\partial\rho}{\partial t} + (\omega y_2 - U_1)(U_2 + \omega y_1)\frac{\partial\rho}{\partial y_1} - (U_2 + \omega y_1)^2\frac{\partial\rho}{\partial y_2} - (U_2 + \omega y_1)U_3\frac{\partial\rho}{\partial y_3} + \omega\rho(y_2\frac{\partial U_2}{\partial y_1} - y_1\frac{\partial U_2}{\partial y_2}) + \omega\rho(\omega y_2 - 2U_1) - \frac{\partial P}{\partial y_2},$$

$$\rho\left(\frac{\partial U_3}{\partial t} + U_1\frac{\partial U_3}{\partial y_1} + U_2\frac{\partial U_3}{\partial y_2} + U_3\frac{\partial U_3}{\partial y_2}\right) = -U_3\frac{\partial\rho}{\partial t} + (\omega y_2 - U_1)U_3\frac{\partial\rho}{\partial y_1} - (U_2 + \omega y_1)U_3\frac{\partial\rho}{\partial y_2} - {U_3}^2\frac{\partial\rho}{\partial y_2} + \omega\rho(y_2\frac{\partial U_3}{\partial y_1} - y_1\frac{\partial U_3}{\partial y_2}) - \frac{\partial P}{\partial y_2} + \rho g_3.$$

Thus, the initial vortex birth is available to the stated initial density gradients in accordance with (16). Then the equation array (17) is solved for the flow field and density distribution. For the gas flow it is added the equation of state to close the system (17), e.g.: $P = \rho RT$, R=286 J/(kg*K)- the universal gas constant for the air. The (16) can be considered as the origin for the vortex flow formation by different density variation at the initial time. Solving the equation array (17) under simplification that the vortex formation is starting as the two-dimensional process at the initial moment can be done. From (17) follows:

$$\frac{\partial\rho}{\partial t}+\rho\left(\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}\right)+U_1\frac{\partial\rho}{\partial y_1}+U_2\frac{\partial\rho}{\partial y_2}=\omega(y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2}),$$

$$\rho(\frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial y_1}+U_2\frac{\partial U_1}{\partial y_2})=(\omega y_2-U_1)\frac{\partial\rho}{\partial t}-(\omega y_2-U_1)^2\frac{\partial\rho}{\partial y_1}+(\omega y_2-U_1)(U_2+\omega y_1)\frac{\partial\rho}{\partial y_2}+\omega\rho(y_2\frac{\partial U_1}{\partial y_1}-y_1\frac{\partial U_1}{\partial y_2})+\omega\rho(2U_2+\omega y_1)-\frac{\partial P}{\partial y_1}$$

, (18)

$$\rho(\frac{\partial U_2}{\partial t}+U_1\frac{\partial U_2}{\partial y_1}+U_2\frac{\partial U_2}{\partial y_2})=$$

$$=-(U_2+\omega y_1)\frac{\partial\rho}{\partial t}+(\omega y_2-U_1)(U_2+\omega y_1)\frac{\partial\rho}{\partial y_1}-(U_2+\omega y_1)^2\frac{\partial\rho}{\partial y_2}+\omega\rho(y_2\frac{\partial U_2}{\partial y_1}-y_1\frac{\partial U_2}{\partial y_2})+\omega\rho(\omega y_2-2U_1)-\frac{\partial P}{\partial y_2}$$

.

Numerical solution of the boundary problem (16), (18) was performed in the circle of radius a with the following initial and boundary conditions:

$$t=0,\ \frac{\partial\rho}{\partial t}=d\rho 0,\ \frac{\partial\rho}{\partial y_1}=d\rho 1,\ \frac{\partial\rho}{\partial y_2}=d\rho 2,\ U_i=0,\ \omega=\omega_0=\frac{d\rho 0}{a(d\rho 1-d\rho 2)};$$

(19)

$$y_2=y_1=a,\ U_1=U_2=0. \qquad (20)$$

5 Computer simulation of the vortex flow

The numerical solution of the boundary problem (18)-(20) presented in Figs 2-5 shows the peculiarities of the process. The density and pressure fields do not change during the initial moment from zero to t=1, as well as up to t=60, as clearly shown from the computer experiment. The velocities are growing since starting point but at t=60 they are decreasing. It was revealed high sensitivity to the fidelity of the initial data. Remarkably the density and pressure are not prone to any changes during all this process.

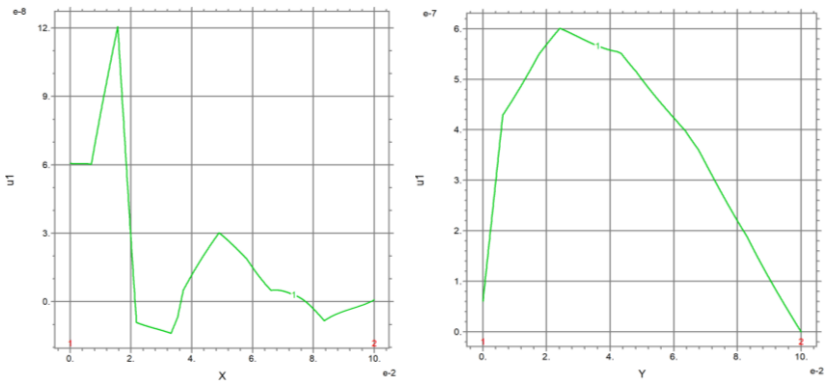


Fig. 2 Velocity field u_1 by y_1 y_2 (X,Y) at t=1

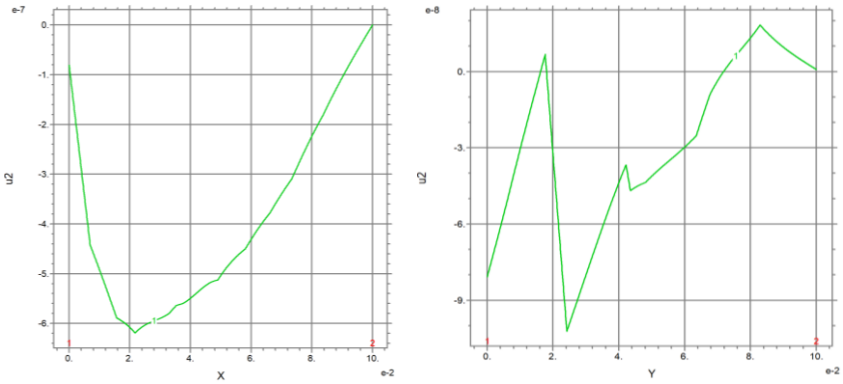


Fig. 3 Velocity field u_2 by y_1 y_2 (X,Y) at t=1

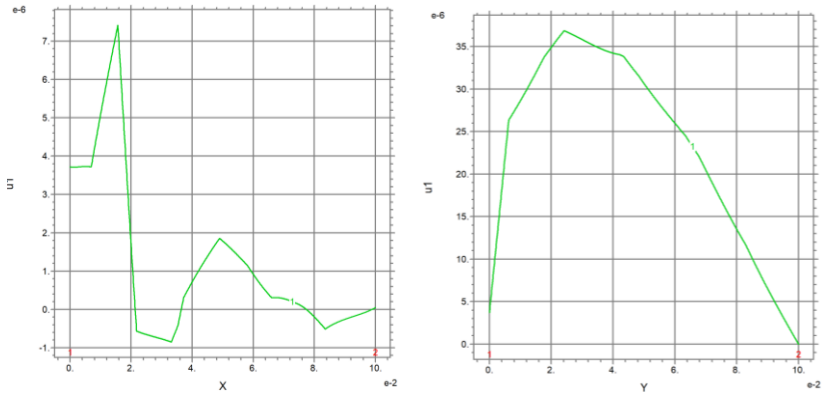


Fig. 4 Velocity field u_1 by y_1 y_2 (X,Y) at t=60

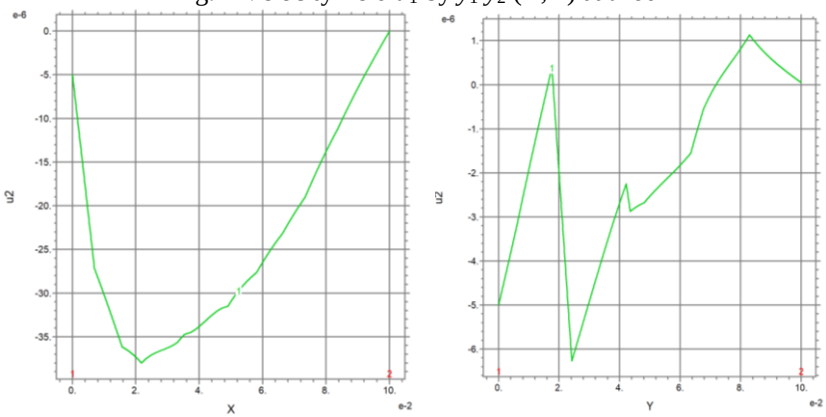


Fig. 5 Velocity field u_2 by y_1 y_2 (X,Y) at t=60

More information about the computer simulation of this process in detail can be found in [21].

6 Movement of particle in complex rotations in different directions

Now it is considered particle movement due to two perpendicular rotations as shown in Fig. 6:

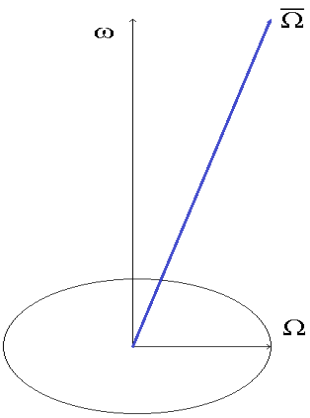


Fig. 6. To the adding of the rotation vectors in two perpendicular directions

Introducing the following assignments for the particle coordinates and velocities:

$$\vec{R} = \{r_1(t)e_1, r_2(t)e_2, r_3(t)e_3\}, \quad \vec{V} = \{\dot{r}_1(t), \dot{r}_2(t), \dot{r}_3(t)\}, \quad \dot{r}_i(t) = \frac{dr_i}{dt}, \qquad (21)$$

where r_i and e_i are the coordinates and normal vectors of the Cartesian coordinate system. The double rotation is assumed around the first z-axis with the frequency **Ω** and the other rotation is going with the frequency ω around the axis coinciding with the tangential direction of the circle of first rotation around the z-axis, at the distance R_0 from the centre of rotation around the z-axis. The Lagrangian of the free particle movement in the above-described double rotating reference coordinate system is L=0.5m($V_1^2 + V_2^2 + V_3^2$), where m is a mass of the particle. Following the conventional technique of the derivation of movement from the Lagrangian (derivative by time minus derivative by coordinate), the equation array of the particle's movement is as follows

$$\frac{d}{dt}\frac{\partial L}{\partial V_i} = \frac{\partial L}{\partial r_i}. \qquad (22)$$

Transformation of basic vectors in rotating (around z) reference frame

$$e_1 = f_1 \cos \Omega t - f_2 \sin \Omega t\,, \quad e_2 = f_1 \sin \Omega t + f_2 \cos \Omega t\,, \quad e_3 = f_3\,. \qquad (23)$$

New coordinates of radius-vector in rotating around z reference frame

$$\vec{R} = \left\{ \left(r_1(t) \cos \Omega t + r_2(t) \sin \Omega t \right) f_1, \left(r_2(t) \cos \Omega t - r_1(t) \sin \Omega t \right) f_2, r_3(t)\, f_3 \right\}. \qquad (24)$$

Shift of radius vector in x direction orthogonal to z leads to the following

$$\vec{R}_0 = \left(r_1(t) \cos \Omega t + r_2(t) \sin \Omega t + R_0 \right) f_1 + \left(r_2(t) \cos \Omega t - r_1(t) \sin \Omega t \right) f_2 + r_3(t)\, f_3\,. \qquad (25)$$

Then transformation of the basic vectors in the rotating around x frame yields

$$f_1 = g_1(t), \quad f_2 = g_2(t) \cos \omega t - g_3(t) \sin \omega t\,, \quad f_3 = g_2(t) \sin \omega t + g_3(t) \cos \omega t\,. \qquad (26)$$

The new coordinates of radius-vector in complex rotating reference frame are

$$\vec{R}_0 = \left(r_1(t) \cos \Omega t + r_2(t) \sin \Omega t + R_0 \right) g_1(t) + \left[\left(r_2(t) \cos \Omega t - r_1(t) \sin \Omega t \right) \cos \omega t + r_3(t) \sin \omega t \right] g_2(t) +$$
$$+ \left[r_3(t) \cos \omega t - \left(r_2(t) \cos \Omega t - r_1(t) \sin \Omega t \right) \sin \omega t \right] g_3(t), \qquad (27)$$
$$R_2 = r_1(t) \cos \Omega t + r_2(t) \sin \Omega t + R_0\,, \quad R_2 = \left[r_2(t) \cos \Omega t - r_1(t) \sin \Omega t \right] \cos \omega t + r_3(t) \sin \omega t\,,$$
$$R_3 = r_3(t) \cos \omega t + \left[r_1(t) \sin \Omega t - r_2(t) \cos \Omega t \right] \sin \omega t\,.$$

The old coordinates of the radius-vector in such reference frame (inverse transformation) are:

$$r_1(t) = \sin \Omega t \left[R_3(t) \sin \omega t - R_2(t) \cos \omega t \right] + \left[R_1(t) - R_0 \right] \cos \Omega t\,, \qquad (28)$$
$$r_2(t) = \cos \Omega t \left[R_2(t) \cos \omega t - R_3(t) \sin \omega t \right] + \left[R_1(t) - R_0 \right] \sin \Omega t\,, \quad r_3(t) = R_2(t) \sin \omega t + R_3(t) \cos \omega t\,.$$

The corresponding velocity components are:

$$v_1(t) = \cos \Omega t \left[U_1(t) + \Omega R_3(t) \sin \omega t \right] + \sin \Omega t \left[U_3(t) \sin \omega t + \omega R_3(t) \cos \omega t \right] +$$
$$+ \Omega \left[R_0 - R_1(t) \right] \sin \Omega t + R_2(t) \left[\omega \sin \omega t \sin \Omega t - \Omega \cos \omega t \cos \Omega t \right] - U_2(t) \sin \Omega t \cos \omega t, \qquad (29)$$
$$v_2(t) = \cos \Omega t \left[\left(R_1(t) - R_0 \right) - U_3(t) \sin \omega t \right] + \sin \Omega t \left[U_1(t) + \Omega R_3(t) \sin \omega t \right] +$$
$$+ \cos \Omega t \cos \omega t \left[U_2 - \omega R_3(t) \right] - R_2(t) \left[\cos \omega t \sin \Omega t + \Omega \sin \omega t \cos \Omega t \right],$$
$$v_3(t) = \cos \omega t \left[\omega R_2(t) + U_3(t) \right] + \sin \omega t \left[U_2(t) - \omega R_3(t) \right],$$

where are: $U_1 = \dfrac{dR_1(t)}{dt} = \dfrac{dS_1(t)}{dt}, U_2 = \dfrac{dR_2(t)}{dt}, U_3 = \dfrac{dR_3(t)}{dt}, S_1 = R_1 - R_0$. Then from the equations (27) follows:

$$U_1(t) = \cos \omega t \left[v_1(t) + \Omega R_2(t) \right] + v_2(t) \sin \Omega t - \Omega R_3(t) \sin \omega t\,,$$
$$U_2(t) = \cos \omega t \left[v_2(t) \cos \Omega t - v_1(t) \sin \Omega t - \Omega S_1(t) \right] + R_3(t)\, \omega\,, \qquad (30)$$
$$U_3(t) = \sin \omega t \left[v_1(t) \sin \Omega t - v_2(t) \cos \Omega t + \Omega S_1(t) \right] - R_2(t)\, \omega\,.$$

Now the following simple movement of the particle is considered with $\Omega = k\omega$:

$$r_1 = v_1 t + r1, r_2 = v_2 t + r2, r_3 = v_3 t + r3,$$

where from with account (27) follows

$$S_1 = (tv_1 + r1)\cos(k\omega t) + (tv_2 + r2)\sin(k\omega t),$$

$$R_2 = (tv_3 + r3)\sin\omega t + \left[(tv_2 + r2)\cos(k\omega t) - (tv_1 + r1)\sin(k\omega t)\right]\cos\omega t,$$

$$R_3 = (tv_3 + r3)\cos\omega t + \left[(tv_1 + r1)\sin(k\omega t) - (tv_2 + r2)\cos(k\omega t)\right]\sin\omega t.$$

The results of simulation in cylindrical coordinates attached to the horizontal plane rotating with frequency Ω around the axis z ($S_1 = \rho\cos\varphi, R_2 \sin\varphi, R_3 = Vt$) are given in Figs 7-9:

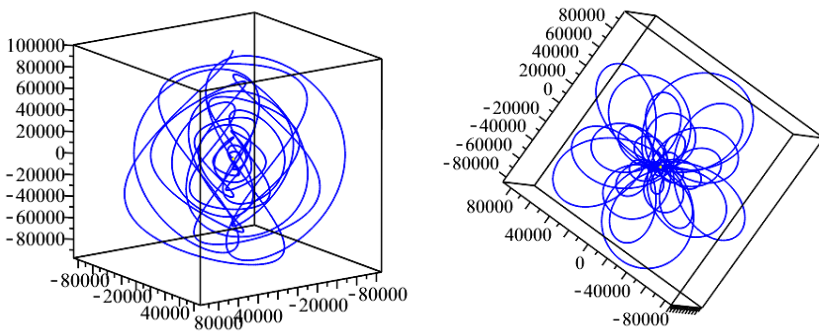


Fig. 7. k=2/($\sqrt{5}+1$),

$$Radius := 30, L := 27, A := 6, \delta := 300, V := 1000, \Omega := k\omega, \omega := 1, \phi := \frac{1}{4}\pi,$$

$$P := Radius + A\sin\left(\frac{2\pi V t}{L}\right), Rho\_ := Radius + \delta + A\sin\left(\frac{2\pi V t}{L}\right)$$

It is interesting that it looks chaotic due to complex rotation but it is not indeed. There is a plane, in which a process is regular as shown in Fig. 8 and Fig. 9:

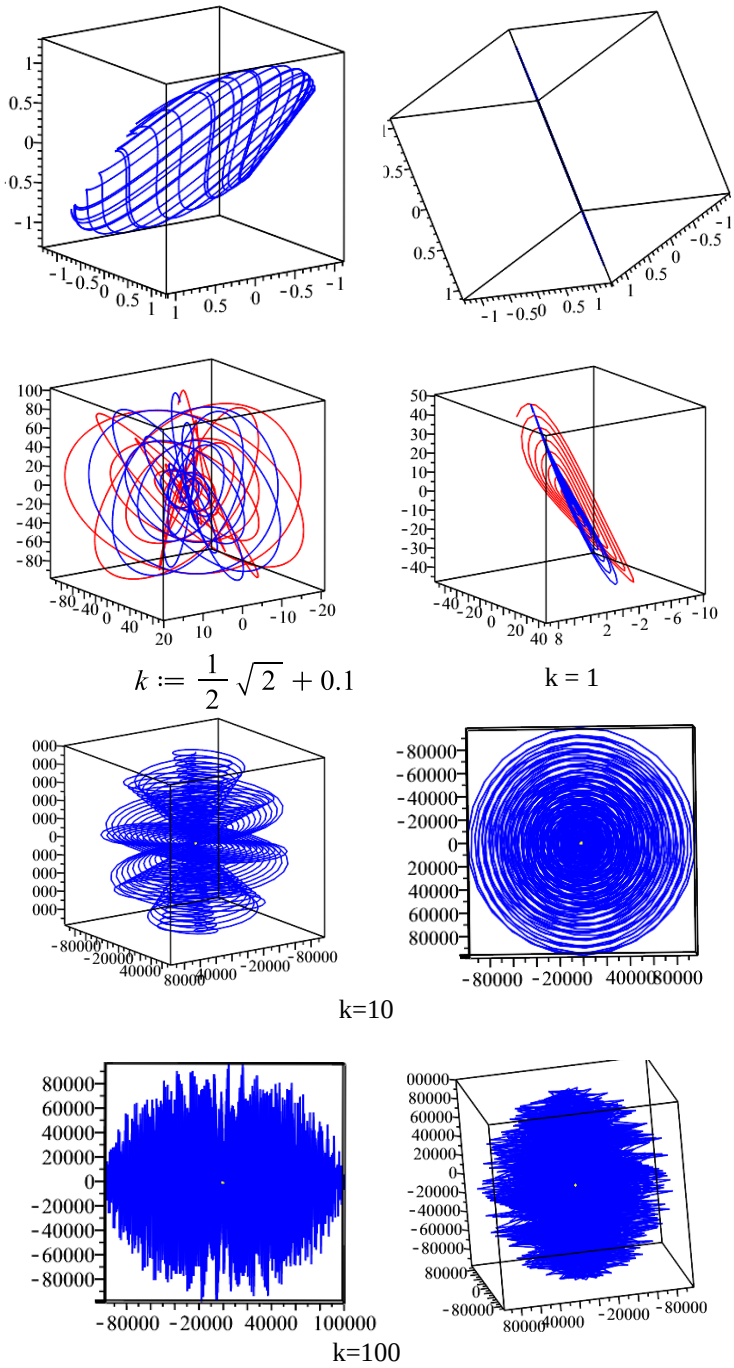
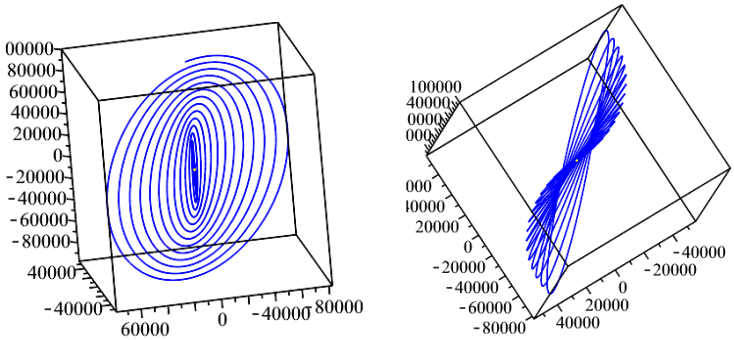
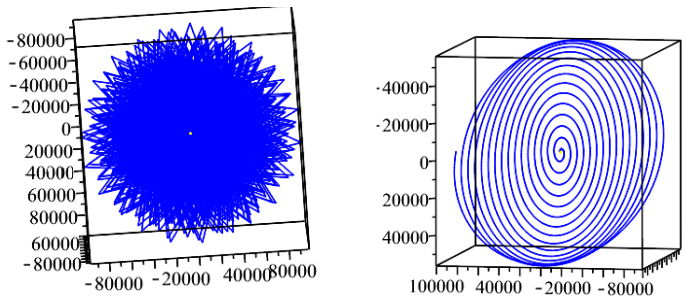


Fig. 8. Resulting plane of complex rotation where chaotic process is regular



k=0,01



k=100

Fig. 9. The chaotic and regular view of the same trajectories of particle in its complex rotation

By V=100 picture is kept the same (k=100, 0.01). And similarly by V=10, and V=1, V=0.1. Thus, there is intensive mixing of the particles inside in the described complex rotations around two axes, which is mainly depending on the frequencies of rotations but does not depend on the velocity of movement along the axis. The higher is frequency of the main rotation, the stronger is mixing.

7 The Conclusions

Mathematical models obtained allowed studying the interesting features of the complex rotational flows and particle movement. The non-inertial rotational coordinate system was applied and the Navier-Stokes equations were analyzed. The computer simulation for the air rotational flow created by the initial density gradients has been done using the gas equation of state by the temperature 300K. It revealed starting the slow flow in a circle region of 10 sm radius under the initial density gradients by time 0.01K/s, and 0.25K/m, 0.05K/m by coordinates, which is growing up to the flow velocities about 0.25-0.35 mm/s and then is gradually fading after *t*=600 s. This is the first attempts to reveal the

features of the rotational flows using the non-inertial coordinates systems, which will be continued in the future study. Also some peculiarities of the particle movement in complex two simultaneous rotations in perpendicular directions have been revealed, e.g. that chaotic movement of particle is not chaotic indeed as far as it is looked at from the resulting plane, where all the trajectories are located like in attractor.

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