Chaotic Modeling and Simulation (CMSIM) 2: 91–103, 2021

A Statistical Ensemble Based Approach for Entropy in Cryptocurrencies Markets

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Abstract. In this paper, we point out an "affinity" between the system of agents trading in cryptocurrencies and statistical mechanics. In particular, we try to extend the concept of entropy in the sense of Boltzmann to a model in which the particles are replaced by N economic subjects (agents), that are completely described by their ability to buy and to sell a certain quantity of cryptocurrencies. In addition, by applying this model to the closing prices we show that entropy can be used as an indicator to forecast the price trend of cryptocurrencies.
Keywords: Cryptocurrency, Entropy, Prices Forecast, Boltzmann, Blockchain.

1 Introduction

The concept of entropy was first introduced by Clausius[23], whose definition was applied to a thermodynamic system that performs a transformation. Since the mid-19th century, entropy has been a key element linking mechanics to thermodynamics; however, this entropy suffered from a conceptual problem which, as demonstrated by Gibbs[28], was revealed in the case of identical gases (*Gibbs Paradox*). He solved this problem by changing the count of states. On the other hand, Boltzmann[16] presented his statistical interpretation of thermodynamic entropy, managing to link the macroscopic properties of a system with the microscopic ones. Based on Gibbs, in 1949 Shannon[8] developed a theory capable of evaluating the amount of information that is lost in receiving a message from a source to a recipient. This form of entropy was generalized by Rényi[3], Tsallis[7], Adler et al.[15] (in topology), redefined by Pincus[19] (*approximate entropy*) and - more recently - by Chen et al.[27] as a time series regularity measure.
The application of entropy in sectors such as economics or finance is linked to the work of Brissaud[4] that assimilated **entropy** to disorder, so as to make this tool that has always been applied the physics part of the economy. The forms of entropy most used in this case are Shannon entropy and the generalizations

Received: 6 July 2020 / Accepted: 12 April 2021

ISSN 2241-0503

by Rényi and Tsallis, who contributed to creating a new line of application for the management of financial portfolios. For example, these new types of entropy has been used by Philippatos and Wilson[6], Usta and Kantar[10], Jana et al.[22], Gulko[17], and Dionisio et al.[2].

What we want to demonstrate in this paper is that it is possible to assimilate a cryptocurrency system to a thermodynamic system. In this way, we are able to determine entropy in the sense of Boltzmann so that we can make price predictions related to the possibility that they move in a more or less wide range; unlike all the recent applications concerning theories based on Shannon entropy and its derivations. Innovation is linked to the reinterpretation of the monetary system of cryptocurrencies. In this sense, we can apply physical theories to a social science. Once the system has been described, our goal is to verify that entropy calculated in the physical sense also occurs in the economic context to allow us to make assumptions on how the process could move in the next future. This type of conjecture has been presented by Sergeev[20], Zakiras et al.[26] and Smith and Foley[9]. In particular McCauley[13], based on this previous theory, maintains that the illiquidity of the markets does not allow for the application of the concepts of statistical mechanics.

The paper structure is the following: in Section 2 we analyze cryptocurrencies and their key characteristics, focusing on the fact that they have a supply limit; in Section 3 we describe the evolution of a system of a particle in statistical thermodynamics and how to determine its entropy, subsequently applying these notions to our monetary system; in Section 4 we define the theoretical assumptions we can link to the system created previously to study the price evolution in these currency markets and we analytically describe the calculation of entropy using real data; finally in section 5 some conclusions are drawn.

2 Cryptocurrency

Cryptocurrencies represent a digital currency system with no guarantee institution and no transaction control. The main cryptocurrencies, by media coverage or by the possibility that some financial intermediaries offer to use them as a payment instrument, are: Bitcoin, Ethereum and Ripple. Unlike traditional financial assets, their value is not based on tangible assets such as the economy of a country or a company, but it is based on the security of an algorithm that tracks transactions. Their definition is controversial since by some entities [11] they are considered intangible assets (IFRS) while according to the German financial supervisory authority (BaFin) they are officially financial instruments [5]. All the cryptocurrencies have been based on the *Bitcoin*, a currency created by Nakamoto[24] who in 2009 released a software capable of implementing transactions. The currency itself is a unique alphanumeric string that represents a certain transaction, a transaction which will then be entered in a public register called *blockchain*.

The blockchain is the fulcrum of these systems and is essentially a register in which the data of the owners of the currency are entered, transactions occur in an encrypted manner. The blockchain is a data structure consisting of a list of transaction blocks linked together so that each refers to the previous

one in the chain. A block is a data structure that aggregates transactions to include them in the public register. The block is made of a header, containing metadata, followed by a long list of transactions. A complete block, with all transactions, is, thus, 1000 times larger than the block header [1]. The integrity of the blockchain network is guaranteed through consensus algorithms such as *Proof-of-Work* (PoW) and *Proof-of-Stake* (PoS), that solve the Byzantine Generals Problem[12] (problem of consent in the presence of errors). A consensus algorithm is a mechanism used by the network to reach consensus, i.e. ensuring that the protocol rules are followed and that transactions occur correctly so that coins can only be spent once.
The cryptocurrency generation process is called *mining*, which adds money to the supply. Cryptocurrencies are "minted" during the creation of each block at a fixed and decreasing rate [1]: each block generated on average every 10 minutes contains new currency. For example, if we consider Bitcoin, every 210000 blocks the currency issue rate decreases by 50% (the availability of new coins grows as a geometric series every 4 years). It is estimated that around the year 2140, the production of the last block will be reached (6930000) and the number of coins produced will tend to its upper limit of 21 million (precisely 20999999.97690000), value introduced by Nakamoto himself and contained in the variable "MAX_MONEY" as can be read in the source code present on GitHub. This value represents a sanity check, especially used to avoid bugs in which it is possible to generate currency from nothing and therefore moving towards a situation in which the blockchain diverges into different potential paths (called *fork*).

3 Methodology

The main assumption in this paper is that the prices of cryptocurrencies behave like a thermodynamic system, so it is possible to determine entropy by using the Boltzmann formula. In order to present the theoretical framework and the methodology, we need to briefly introduce the main physical results. In Statistical Mechanics a macroscopic system is made up of N molecules ($N \sim 10^{24}$ is the Avogadro's constant) whose mechanics provide the evolution of $6N$ dynamic variables describing completely the microscopic states of this system. Motion in the phase space can be studied using the $3N$ position components and the $3N$ momenta components, indicated with $\{\mathbf{q}_i\}$ and $\{\mathbf{p}_i\}$ whose evolution is driven by Hamilton's equations. Mechanics, therefore, provides a very detailed description of the system contrary to thermodynamics which studies the collective variations; for this reason, the mechanical point of view can be defined *microscopic* and the thermodynamic one *macroscopic*. The study of the system from a microscopic point of view concerns experimental observation on one or a few molecules.
Everything that happens from the microscopic side can be expressed in macroscopic terms through thermodynamics, defined in this case as a large amount of microscopic variables. We consider an isolated system of N particles described

Source: https://github.com/bitcoin/bitcoin/blob/master/src/amount.h

by the $3N$ coordinates and the $3N$ momenta in a $6N$-dimensional space at a certain time t. Particles are subject to the laws of classical mechanics and therefore $\mathbf{X}(t)$ evolves according to Hamilton's equations. Since the Hamiltonian $H(p,q)$ does not depend on time, the energy E is a conserved quantity during motion and develops on a fixed hypersurface. We want, for example, to measure an observable $A(\mathbf{X})$ (a function defined in the phase space) of the system in thermodynamic equilibrium, but since the scale of macroscopic times is much larger than the microscopic one, we can consider a datum as the result of a system that has gone through a large series of microscopic states; this implies that the observable must be compared with an average performed along with the evolution of the system calculated over very long times $\bar{A}$. The calculation of $\bar{A}$ would require knowledge of both the microscopic state at a certain moment and the determination of the corresponding trajectory in the phase space, which corresponds to a practically inexhaustible request. To determine the observable, the *ergodic* theory intervenes, according to which each energy surface is completely accessible to any motion with the given energy and the average residence time in a certain region is proportional to its volume. If these conditions are satisfied, the average $\bar{A}$ can be calculated as the average of $A(\mathbf{X})$ in which the states with the fixed energy contribute with equal weight. In applications it is convenient to consider on average all states with energy within a fixed range $[E, E + \Delta E]$; furthermore, we are only interested in some macroscopic properties such as particle number N and the volume V. There is an infinite number of systems that satisfy these conditions: these form the *Gibb's ensemble* which is represented by a set of points in the phase space characterized by a density function $\rho(p,q,t)$ defined so that $\rho(p,q,t)\, d^{3N}p\, d^{3N}q$ corresponds to the number of representative points of the system during the instant t contained in the infinitesimal volume of the phase space $d^{3N}p\, d^{3N}q$. Furthermore, since energy, volume and number of particles are constants of motion, the total number of systems in an ensemble is conservative.
We can thus introduce the *postulate of equal a priori probability* who claims that when a macroscopic system is in thermodynamic equilibrium its state can be with equal probability each of those which satisfies the macroscopic conditions of the system. This postulate implies that the system under consideration belongs to an ensemble called *microcanonic* with density function

$$\rho(p,q) = \begin{cases} \rho^* & if\ E < H(p,q) < E + \Delta \\ 0 & otherwise \end{cases} \tag{1}$$

where ρ^* is constant and all members of the ensemble have the same number of particles and equal volume.
We can define $\Gamma(E)$ the volume occupied by the microcanonical ensemble in the phase space as:

$$\Gamma(E) \equiv \int_{E<H(p,q)<E+\Delta E} d^{3N}p\, d^{3N}q \tag{2}$$

and $\Sigma(E)$ the volume bounded by the energy surface E:

$$\Sigma(E) \equiv \int_{H(p,q)<E} d^{3N}p\, d^{3N}q \tag{3}$$

so that

$$\Gamma(E) = \Sigma(E + \Delta E) - \Sigma(E). \tag{4}$$

Entropy, then, can be defined as:

$$\begin{aligned} S_\Gamma &= \int_{E \leq H \leq E+\Delta E} d^{3N}p\, d^{3N}q\, \rho(-\kappa_B \ln \rho) \\ &= \int_{E \leq H \leq E+\Delta E} d^{3N}p\, d^{3N}q\, \frac{1}{\Gamma}\left(-\kappa_B \ln \frac{1}{\Gamma}\right) \\ &= \frac{1}{\Gamma}\kappa_B \ln \Gamma \int_{E \leq H \leq E+\Delta E} d^{3N}p\, d^{3N}q \\ &= \frac{1}{\Gamma}\kappa_B \ln \Gamma \cdot \Gamma = \kappa_B \ln \Gamma(E) \end{aligned} \tag{5}$$

where $\kappa_B \sim 1.3806 * 10^{-23}$ is the Boltzmann constant. To analytically calculate $\Gamma(E)$, which represents the number of states accessible to the system at temperature T, we must consider that a microcanonical ensemble is made up of J identical copies of the closed system, each of which is located in a microstate $(\mathbf{p}_i, \mathbf{q}_i)$ of the phase space. Being all on the same hypersurface E, we can divide it into cells of equal size, where in each there are j_i systems such that $J = \sum_i j_i$. To define the system it is necessary to find the most probable distribution of the j_i microstates, that is, to count the total number of ways in which we can obtain a certain macrostate. In the Boltzmann paradigm with an ideal gas consisting of identical particles under the same conditions, we can say that

$$\Gamma(E) = \frac{J!}{\prod_i j_i!} \tag{6}$$

The idea that entropy is connected to volumes in the phase space finds its origin in the *Helmholtz Theorem*, whose goal is to exactly bring thermodynamics down from mechanics.

Let us now try to translate this physical theory into a financial dress. Viaggiu et al.[25] have developed a representation of an economic model relating to money from a thermodynamic point of view. In their description the ensemble is made up of the N interacting economic subjects, entirely described by two variables $\{x_i, y_i\}$ which represent money and credit/debt capacity and which are not conjugated in the sense of mechanics Hamiltonian. The key characteristic is to consider a representative function of the total currency as a conservative law, to be able to exploit the ergodic hypothesis.

Our idea is to go back to their hypothesis by applying it to the case of cryptocurrencies. We consider a model in which the particles are replaced by N economic subjects (agents) who intend to trade in cryptocurrencies (compared only to a reference currency, such as the USD). These agents are completely described by 2 variables, which we can, however, identify as $\{x_i, y_i\}$, where x_i and y_i indicate, respectively, the ability to buy and to sell a certain quantity of cryptocurrencies (both expressed in monetary terms). The latter hypothesis is possible according to the fact that the market to which we refer is influenced only by the supply and demand leverage. As for [25], even if the complete Hamiltonian formalism is not respected, we can consider as a conserved quantity the total number of cryptocurrencies in circulation which by their definition is constant over a suitable time interval through the function $M(x_i, y_i)$ (as in the particular case of Bitcoins for which the supply limit is fixed at 21 million). However, since the supply limit has not yet been reached by any cryptocurrency we consider this quantity constant concerning the currency in circulation in a precise time t, therefore:

$$M = \sum_{i=1}^{N} x_i + y_i. \tag{7}$$

In this sense, the sum of the ability to sell and buy of the N agents fully describes the cryptocurrencies in circulation. The ergodic hypothesis allows us, given a certain function $f(x_i, y_i)$, to express its average with respect to the time in terms of an average over the ensemble at fixed M:

$$\bar{f} = \int_{M=const} f(x, y)\rho(x, y)\, dx\, dy \tag{8}$$

where $\rho(x, y)$ denotes the probability distribution of the ensemble. Through these assumptions we can verify the economic transformations through thermodynamics; in particular, as in statistical mechanics, we can calculate the *volume in the phase space* [25] . If we integrate over all the available volume of the configuration space spanned by $\{x, y\}$ with $\bar{M} = m$ (where $\bar{M}$ denotes the average over the whole configuration space) we have $\int_{\bar{M}=m} d^N x\, d^N y = 0$. So introducing a thick shell Δ where $\Delta \ll m$ we can define:

$$\Gamma(m) = \int_{m<M<m+\Delta M} \frac{d^N x\, d^N y}{k^{2N}} \tag{9}$$

where $d^N x\, d^N y$ is understood as the phase space and k is a normalization factor such that Γ is dimensionless. This functional represents the number of microscopic realizations of the system under examination and allows us to calculate the entropy S as described in the equation (5).

4 The model

In this section, however, we try to define, through a new type of approach, how it is possible to calculate entropy considering essentially the prices obtainable from the currency markets (FOREX).

First, we know that cryptocurrencies are used by an approximate number of economic entities equal to 44 million (based on the number of blockchain portfolios[21]) for which $N \gg 1$. We also know that every subject in our system is fully described by its ability to buy and sell ($\{x_i, y_i\}$). Let us consider that these two variables are summarized in the *last prices* of the cryptocurrency on the currency markets, a type of price used to keep track of changes in the value of an asset throughout a session. In this sense, the latest prices allow us to understand whether, compared to the previous session, the ability to buy or sell prevailed. We can summarize this price capability in the sentence *"prices describe the strength with which agents position themselves in the phase space"*. The key point is that we can use the function M (described above) because in a certain time t the quantity of cryptocurrencies is constant and quantifiable, in this way we can go back to the previous economic model and determine Γ as described in the equation (9). Analytically, we do not consider the number of economic subjects present in the market but indirectly deduce their "position" in the phase space from the difference between the closing prices. In particular, first we cluster the closing price series based on a certain reference interval (5 days); as for each cluster there is a maximum and a minimum price, we calculate the difference in terms of necessary steps to pass from one to the other obtaining a certain value of **gap G** (this assumption is based on the idea that the distance between maximum and minimum is a measure of the dispersion of agents in our phase space); to calculate the "volume" occupied by the disposition of the agents we use combinatorial analysis, therefore:

$$\Gamma = G^5 \tag{10}$$

Once the value of Γ is determined, entropy can be calculated by using the Boltzmann formula:

$$S = \kappa_B \ln \Gamma. \tag{11}$$

Finally, precisely because Boltzmann's constant is of the order of Avogadro's number, we can *"rationalize"* this entropy value obtained by multiplying it by 10^{23}. Our data analysis shows that in situations where entropy is drastically reduced, in the following phase it must grow in an "almost obligatory" way; this in terms of cryptocurrency prices indicates that in situations in which the gap between the maximum and the minimum is drastically reduced in the transition from one cluster to another "almost compulsorily" follows a situation in which it is certainly wider than the previous one. This type of price-based entropy defines how agents move in the phase space, so it allows us to understand if there is more movement towards one area rather than another.

4.1 Dataset

The empirical analysis has been applied to the closing prices of three cryptocurrencies, all related to the US dollar (USD), that are:

- Tether, whose price with 4 decimal places requires a step equal to 0.0001;
- Bitcoin Cash, whose price with 2 decimal places requires a step equal to 0.01;
- Litecoin, whose price with 3 decimal places requires a step equal to 0.001.

Source: Investing.com

Prices are considered with a daily time frame over 1 year, from 1/1/2019 to 31/12/2019 and they are clustered in 5 days. To make the figures more clear, the 1-year interval has been divided into 4 trimesters. Furthermore, to better test the idea, the same test was carried out also on daily prices at 1 minute of 1/4/2020 recorded from 10:56 to 11:52, instead of clustered in 5 minutes. The difference from the daily case is that these prices were collected, always from the same source, but observed on different currency markets.

4.2 Numerical examples

We can start the analysis from the annual case. The first cryptocurrency analyzed is Tether (USDT/USD), whose price moves in a neighborhood of 1 and consists of 4 decimal places; distinguish the trend of entropy compared to prices in the 4 ranges previously defined.

(a) Closing prices and entropy 1/1 - 31/3

(b) Closing prices and entropy 1/4 - 29/6

(c) Closing prices and entropy 30/6 - 27/9

(d) Closing prices and entropy 28/9 - 31/12

Fig. 1: Prices (blue) and entropy (orange) Tether in the period 1/1 - 31/12

As can be seen graphically, when entropy reaches a point of relative minimum falling below a certain threshold (it therefore undergoes a sharp reduction) it is forced in the next cluster to grow, almost as if to rebalance itself. In terms of prices, this implies that in the cluster in which the entropy descent occurred there was a very small gap and, in the subsequent cluster, since entropy increases the gap also increases. In this case, the range of variation of prices is very "narrow" and every movement is important. It is possible, however, to notice for example looking at the figure 1 (d) what is the gap value and therefore the entropy threshold that, if "under"-passed, will cause an immediate growth

in the next future.
The next cryptocurrency analyzed is Bitcoin Cash (BCH/USD):

(a) Closing prices and entropy 1/1 - 31/3

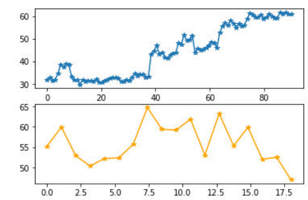
(b) Closing prices and entropy 1/4 - 29/6

(c) Closing prices and entropy 30/6 - 27/9

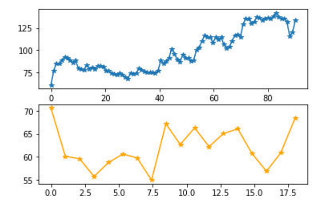
(d) Closing prices and entropy 28/9 - 31/12

Fig. 2: Prices (blue) and entropy (orange) Bitcoin Cash in the period 1/1 - 31/12

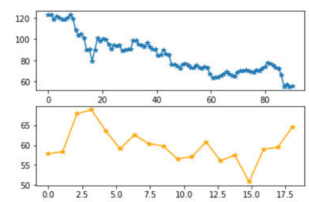
In this case the figure 2 (d) shows how the gap threshold below which a sharp drop in entropy occurs can also be quite high (especially in currencies where high volatility allows it to to move many points from one price to another).
The last cryptocurrency we have considered is Litecoin (LTC/USD):



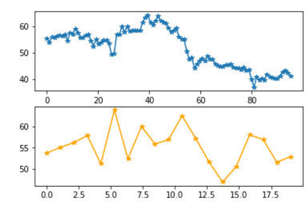
(a) Closing prices and entropy 1/1 - 31/3



(b) Closing prices and entropy 1/4 - 29/6



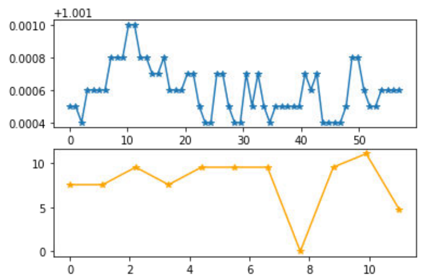
(c) Closing prices and entropy 30/6 - 27/9



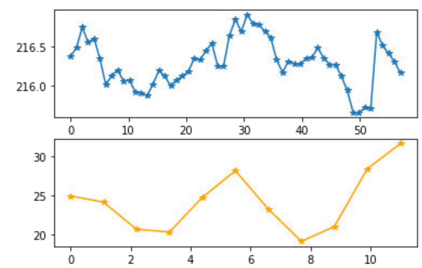
(d) Closing prices and entropy 28/9 - 31/12

Fig. 3: Prices (blue) and entropy (orange) Litecoin in the period 1/1 - 31/12

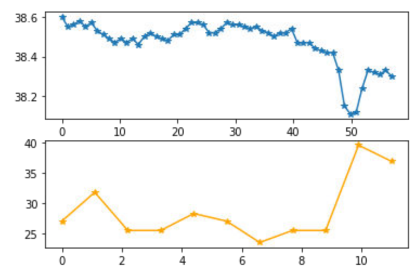
Also in this cryptocurrency all the situations defined above occur, in particular from the figure 3(d) it can be seen how, following the fact that the first 4 clusters are growing despite the gap value being quite low, the gap threshold to define the drastic descent of entropy is quite low. As for the case of 1-minute prices, we can summarize the trend of the different cryptocurrencies together as shown in figure 4 which shows how all the assumptions made in the previous case are also respected for prices of this type



(a) Tether



(b) Bitcoin Cash



(c) Litecoin

Fig. 4: Prices (blue) and entropy (orange) of cryptocurrencies based on 1 minute

In particular, the hypotheses made previously are very evident in the case of the Tether (figure 4 (a)).

4.3 Possible effects on prediction

Thanks to previous results we can use entropy as an indicator to make predictions on the price trend of cryptocurrencies in the currency markets. For example, we can suppose that we are in a certain cluster X where entropy has declined sharply. As previously defined, we expect entropy to grow in the next cluster and this leads to an increase in the price gap. The hypothesis we can make is that the value of the gap in the cluster $X+1$ is at least one unit higher than the value in the cluster X: we can use this information to understand what the future price range will be. In this case, knowing the value of the gap in the cluster X, we can create a bifurcation that represents the possible evolution of the price in the event of a bullish or bearish trend. Assuming, moreover, that the first cryptocurrency price close enough to the last price of the previous cluster what we can expect is such a situation: if the second closing price of the cluster $X+1$ is **higher** than the previous price in the same cluster and assuming an upward trend we can assume that the series of prices continues in an area that we have defined as Gap^{-}; while if the second closing price of the cluster $X+1$ is **lower** than the previous price in the same cluster and assuming a bearish trend we can assume that the price series continues in an area that we have defined as Gap^{+}. Such information can be fundamental for example for an investor who intends to choose the ideal moment to enter (or exit) the market or balance any price limits.

5 Conclusions

In this paper, we have defined a similarity between a thermodynamic system and a currency system. Thanks to this assumption, we have shown how it's possible to apply Boltzmann's entropy to cryptocurrencies. This system is characterized by the presence of N subjects interested in buying (or selling) this type of currency. Assuming that the quantity of money at a certain moment t is fixed and determinable, it is possible to hypothesize that the position of each economic entity is summarized by the last price of the cryptocurrency itself in the currency markets, as an indicator characterized by the ability to buy and sell. With this hypothesis, it was possible to determine the entropy using the Boltzmann formula, dividing the time interval into clusters and calculating the gap between the different prices. This analysis has shown that when entropy falls sharply then it must necessarily grow shortly; which in terms of price corresponds to a situation in which the gap between maximum and minimum is wider than the previous one.

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