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Chaotic Time Series by Time-Discretization of Periodic Functions and Its Application to Engineering

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Abstract. It is shown firstly that chaotic time series are generated by time-discretizing continuous periodic functions, and the time-dependent chaos functions, which have non-periodicity and sensitivity on initial values, are obtained from the Chebyshev differential equation and the pendulum model equation with the Jacobi elliptic functions. Then, the proposed numerical method for nonlinear time series expansion is represented for analysing chaotic time series and for generating 1/f noise on the basis of the time-dependent chaos functions. Finally, noise analyser, chaos function generator and chaos controller are briefly discussed as an application of the chaotic time series and the nonlinear expansion method to engineering.
Keywords: Chaotic time series, Chaos function, Chebyshev polynomials, Jacobi elliptic functions, Nonlinear time series expansion, Noise, Fluctuation, Application to engineering.

1 Introduction

Over the last fifty years, a large number of papers and books on nonlinear physics have appeared, for soliton, chaos, fractals and so on [1, 2]. In order to describe the nonlinear dynamics, physics and mathematics explain intricate patterns and the repeated application of dynamic procedures, and the fundamental rules underlying the variety of physical phenomena have led to searching and defining them in scientific terms. For example, after the proposal of a theory for shallow water waves, soliton has been a self-reinforcing solitary wave that maintains the shape during the travel with constant speed, and arises as the solution to a widespread class of weekly nonlinear dispersive partial differential equations representing physical systems [3-6]. In the last two decades, the field of soliton and nonlinear optics has grown steadily for technological applications, and presents research problems from a fundamental and an applied point of view [7-9]. In the meantime, it has been shown that the first-order nonlinear difference equations arise in the biological, economic and social sciences, and possess a rich spectrum of dynamical behavior as chaos in many respects [10-12]. The population growth of insects is modeled by the simplest nonlinear difference equation called the logistic map. After many attempts for chaos, as an electrical analogue, a piecewise-linear circuit has been proposed to generate chaos, and has been accepted as a powerful paradigm for

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learning chaos [13]. Furthermore, various chaotic sequences have been proposed for pseudo-random numbers [14] and the application to cryptosystems [15-17]. At the same time, a family of shapes and many other irregular patterns called fractals have been proposed for the geometric representation [18], as an irregular set consisting of parts similar to the whole. Therefore, the concept of fractals is useful for describing various natural objects, such as clouds, coasts, rivers and road networks [19, 20]. For the application of fractals to engineering, fractal compression has been proposed as a method to compress images using fractals [21]. On the other hand, a soliton wave generator using nonlinear diodes has been proposed [22]. In addition, for an application of chaos, an algorithm of exact long time chaotic series has been constructed without the accumulation of round-off error caused by numerical iterations [23], and has been applied to the generation of pseudo-random numbers and to cryptosystems [24]. Recently, a nonlinear time series expansion of the logistic chaos has been proposed [25], and high dimensional chaotic maps and fractal sets with physical analogues have been presented [26].

In this paper, it is shown firstly that chaotic time series are constructed by time-discretizing continuous periodic functions in Section 2. Next, the time-dependent chaos functions, which have non-periodicity and sensitivity on initial values, are obtained on the basis of the Chebyshev polynomials and the Jacobi elliptic functions in Sections 3 and 4, respectively. Then, numerical calculation steps for constructing the proposed nonlinear time series expansion are represented in Section 5, and finally an application of the chaotic time series and the nonlinear expansion to engineering is briefly discussed in Section 6. The last Section is for conclusions.

2 Time-Discretization of Periodic Functions

Firstly, we introduce an exact chaos solution

$$x_n = \cos(C2^n), \quad n = 0,1,2,\ldots, \tag{1}$$

with a real coefficient

$$C \neq \pm m\pi / 2^l, \tag{2}$$

and finite positive integers $\{l, m\}$, to the logistic map $x_{n+1} = 2x_n^2 - 1$. For the solution (1) with (2), we can regard it as a time-dependent function;

$$x_n(t) = \cos(2^n t), \tag{3}$$

with a condition

$$t \neq \pm m\pi / 2^l, \tag{4}$$

where the function (3) with (4) is known to have a fractal curve as $n \to \infty$ [27]. For example, chaotic time series of (3) are calculated and illustrated in Fig. 1, and the initial value is given by $x_n(t_0 = 0) = \cos(2^n \varepsilon)$ from the function;

$$x_n(t) = \cos(2^n(t+\varepsilon)) \tag{5}$$

with a real parameter ε >0, in order to show the chaotic properties numerically. It is found in Fig. 1 that the difference of time series of two cases $\varepsilon = 0$ (—×—) and $\varepsilon = 0.0001$ (—o—) is small at (a) n=0 and (b) n=1, respectively. However, for (c) n=10 and (d) n=20, the sensitivity on initial values appears clearly. That is, the time-dependent chaos function (3) generates chaotic time series without a period. The algorithm for long time chaotic series to avoid the accumulation of round-off error caused by the numerical iteration of (3) with (4) is given by;

$$\begin{aligned} & n = 0,\ 1,\ 2,\ 3,\cdots, \\ & t_i = i(\Delta t) = 0,\ (\Delta t),\ 2(\Delta t),\ 3(\Delta t),\cdots,\ (i = 0,\ 1,\ 2,\ 3,\cdots) \\ & \Delta t = t_{i+1} - t_i = (l_0 / p_r)\pi, \end{aligned} \tag{6}$$

$$\begin{aligned} x_n(t_i) &= \cos(2^n t_i) \\ &= \cos(2^n i(\Delta t)) \\ &= \cos(2^n l_0 (i / p_r)\pi) \\ &\equiv \cos(l_n (i / p_r)\pi) \end{aligned} \tag{7}$$

with
$$l_{n+1} i \equiv 2 l_n i \pmod{2p_r}, \tag{8}$$

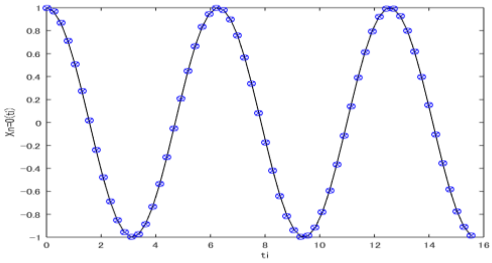
where $\{l_n / p_r\}$ are rational numbers, l_0 is the initial integer of l_n, and p_r is a large prime number [23-25].
Similarly, from a chaos solution;

$$y_n = \sin(C2^n) \tag{9}$$

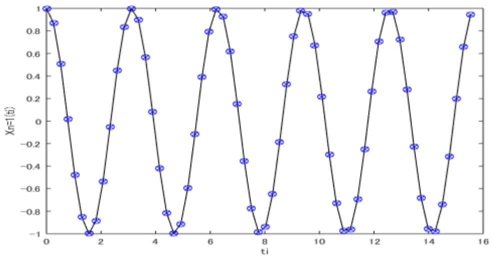
with the condition (2) and satisfying the delayed solvable chaos map $y_{n+1} = 2y_n(1-2y_{n-1}^2)$ [26], we regard it as a time-dependent function;

$$y_n(t) = \sin(2^n t) \tag{10}$$

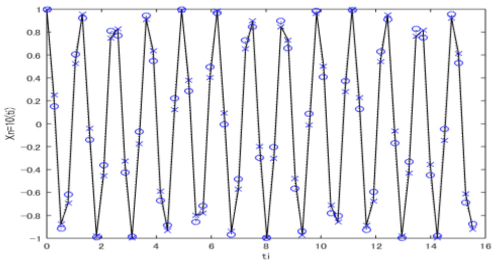
with the condition (4), where (10) gives chaotic time series, and (2) and (10) are applied to the nonlinear time series expansion in Section 5.
In this paper, we call the functions (1) and (9) 'chaos function,' and the functions (3) and (10) time-dependent 'chaos function.' In Sections 3 and 4, the chaos functions are obtained from the Chebyshev polynomials and the Jacobi elliptic functions, respectively.



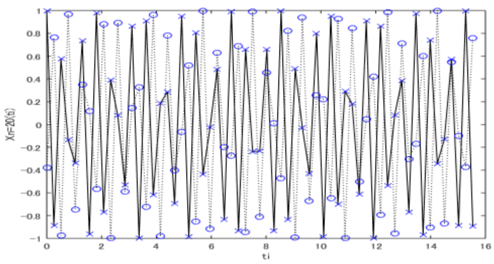
(a) n=0



(b) n=1



(c) n=10



(d) n=20

Fig. 1. Time series of $x_n(t_i)=\cos(2^n t_i)$ (3) with (4), $x_n(t_0=0)=\cos(2^n\varepsilon)$, ε=0 (—×—), ε=0.0001 (⋯⋯o⋯⋯), l_0=100 and p_r=1213 in (7).

3 The Chebyshev Polynomials

As well known, the Chebyshev differential equation is given by

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+k^2y=0,\ \ k=0,1,2,3,\ldots, \tag{11}$$

with a parameter k, and by introducing a variable transformation $x\equiv\cos\theta$, we have the general solution

$$\begin{aligned} y(x) &= A\cos(k\cos^{-1}x)+B\sin(k\cos^{-1}x) \\ &\equiv AT_k(x)+BU_k(x) \end{aligned} \tag{12}$$

with $|x|<1$ and integration constants $\{A, B\}$, where $T_k(x)$ and $U_k(x)$ are defined as the Chebyshev polynomials of the first and the second kind of degree k, respectively [28]. For the polynomials of $T_k(x)$, we have from (12) as

$$k=1; T_1=\cos(\theta)=x, \tag{13}$$
$$k=2; T_2=\cos(2\theta)=2x^2-1, \tag{14}$$
$$k=3; T_3=\cos(3\theta)=4x^3-3x, \tag{15}$$
$$\ldots$$

and for $U_k(x)$;

$$k=1; U_1=\sin(\theta)\equiv X, \tag{16}$$
$$k=2; U_2=\sin(2\theta)=2xX, \tag{17}$$
$$k=3; U_3=\sin(3\theta)=3X-4X^3, \tag{18}$$
$$\ldots$$

Then, we find the Chebyshev maps in a general form;

$$x_{n+1}=T_k(x_n),\ \ X_{n+1}=U_k(X_n), \tag{19}$$

which map the interval [-1, 1] k times onto itself. Especially, it has been considered that the maps with $k\geq 2$ are ergodic and strongly mixing [29], and have statistical properties [30]. Here, it is interesting to note that from the maps (19) we have the following chaos maps and the chaos solutions [26] as

$$\begin{aligned} &k=2; x_{n+1}=2x_n^2-1, x_n=\cos(C2^n), \\ &\qquad X_{n+1}=2X_n(1-2X_{n-1}^2), X_n=\sin(C2^n), \end{aligned} \tag{20}$$
$$\begin{aligned} &k=3; x_{n+1}=4x_n^2-3x_n, x_n=\cos(C3^n), \\ &\qquad X_{n+1}=3X_n-4X_n^3, X_n=\sin(C3^n), \end{aligned} \tag{21}$$
$$\ldots$$

where $C \neq \pm m\pi / p^l$ with finite positive integers {l, m} for the general chaos solution $x_n = \cos(Cp^n)$. Then, we obtain the time-dependent chaos function; $x_n(t) = \cos(p^n t),\ t \neq \pm m\pi / p^l,\ p = 2,3,4,\cdots,$ which give chaotic time series.

4 The Jacobi Elliptic Functions

The mathematical model of pendulum is well discussed and described by the second-order nonlinear differential equation;

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \tag{22}$$

where θ is the pendulum angle of deviation, g is the acceleration due to gravity, and l is the thread length. In order to find the solution to (22), the elliptic integral is defined by

$$u \equiv \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, (0 < k < 1) \tag{23}$$

with a modulus k, $x \to \sin u$ as $k \to 0$, and $x \to \tanh u$ as $k \to 1$. Then, we find the inverse function of (23), and it is called the Jacobi elliptic function of the first kind;

$$x = sn(u,k), \tag{24}$$

$$cn(u,k) = \sqrt{1 - sn^2(u,k)}, \tag{25}$$

$$dn(u,k) = \sqrt{1 - k^2 sn^2(u,k)}. \tag{26}$$

Then, the duplication formulas [28] are derived as

$$sn(2u,k) = \frac{2sn(u,k)cn(u,k)dn(u,k)}{1 - k^2 sn^4(u,k)}, \tag{27}$$

$$cn(2u,k) = \frac{1 - 2sn^2(u,k) + k^2 sn^4(u,k)}{1 - k^2 sn^4(u,k)}. \tag{28}$$

Therefore, we get the following map, for example, from (28);

$$\begin{aligned} X_{n+1} &\equiv cn(2u,k) \\ &\equiv f(X_n) \\ &= \frac{1 - 2(1 - X_n^2) + k^2(1 - X_n^2)^2}{1 - k^2(1 - X_n^2)^2} \end{aligned} \tag{29}$$

with

$$X_n \equiv cn(C2^n, k), \tag{30}$$

which has been considered as the Katsura-Fukuda map [31], and has been discussed in the class of exactly solvable chaos map [32].
Furthermore, we find the logistic map $X_{n+1} = 2X_n^2 - 1$ and the chaos solution $X_n = \cos(C2^n)$ as $k \to 0$ from (29) and (30), respectively. Thus, we have the generalized time-dependent forms;

$$X_n(t) = cn(p^n t, k), \tag{31}$$

$$Y_n(t) = sn(p^n t, k) \tag{32}$$

with (4). Here, we call functions (31) and (32) time-dependent 'elliptic chaos function,' which give chaotic time series.

5 Application to Engineering

In this Section, it is presented that chaotic time series obtained in Sections 2-4 could be applied to the nonlinear time series expansion method [25] and to engineering.
Usually, the Fourier series expansion decomposes periodic functions or periodic signals in terms of an infinite sum of simple oscillating functions, and has been applied to finding an approximation for original problems as harmonic analysis [33]. The well-known expansion for a given continuous periodic function $f(t)$ has been represented by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t)), \tag{33}$$

where ω is the angular frequency. Recently, a nonlinear time series expansion has been proposed for a continuous periodic function $g(t)$ with a 2π-period as

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(p^n t) + b_n \sin(p^n t)) \tag{34}$$

with a positive integer $p>1$ and the coefficients;

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} g(t) dt, \tag{35}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} g(t) \cos(p^n t) dt, \tag{36}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} g(t) \sin(p^n t) dt. \tag{37}$$

Here, it is found that the coefficient $n\omega$ in (33) is linear, and the coefficient p^n

in (34) is nonlinear with respect to *n*. Steps for the construction of (34) are represented as follows:

Step 1

The number N_0 of discrete-time data X_i is determined as $i = 0,1,2,\ldots,N_0$. Then, we find a maximum prime number $n \le N_0$. For example, the case of $N_0 = 200$ gives $n = 199$, and we introduce the following correction function for the data X_i as a pretreatment;

$$y_i \equiv X_i - ai, a \equiv (X_n - X_0)/N, N \equiv n+1, \quad (38)$$

with i=0,1,2,…,N to have a periodicity, that is, a 2π-period under the condition $y_0 = y_N = 0.0$.

Step 2

Next, we evaluate $\{a_0/2, a_n, b_n\}$ by setting $g(t_i) \equiv y_i$ for the numerical integration of (35)-(37) with the discrete-time t_i by dividing a 2π-period evenly into N intervals.

Step 3

Then, we calculate the discrete-time form of (34), which is given by

$$g(t_i) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(p^n t_i) + b_n \sin(p^n t_i)), \quad (39)$$

and define the following error function by

$$\varepsilon \equiv \sqrt{\sum_{i=1}^{N}(y_i - g(t_i))^2 / N}, \quad (40)$$

where y_i and $g(t_i)$ are the revised data (38) and the calculated data (39), respectively. Finally, we find the optimal parameter $\{p, l_0\}$ with the initial value l_0 of l_n in the numerical iteration to minimize the error function (40) as an optimization problem of (38)-(40). Here, for the calculation of functions $\cos(p^n t_i)$ and $\sin(p^n t_i)$, the proposed algorithm [23, 24] to avoid the accumulation of round-off error plays an important role, where the details are shown in the proposed expansion method [25].

Thus, we obtain numerical results of the chaotic time series expansion and the power spectrum. For example, a resultant expansion is given as

$$g(t_i) = \frac{a_0}{2} + \{a_1 \cos(54 t_i) + \cdots + a_{100} \cos(54^{100} t_i) + b_1 \sin(54 t_i) + \cdots + b_{100} \sin(54^{100} t_i)\}, \quad (41)$$

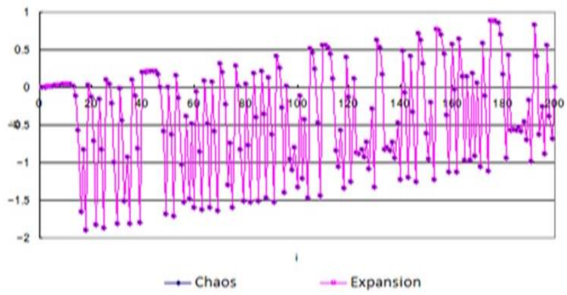


Fig. 2. The data y_i (38) and the expansion $g(t_i)$ (39) with parameters (p, l_0) = (54, 70) and $\varepsilon = 6.08\times10^{-16}$ obtained by the numerical iteration [25].

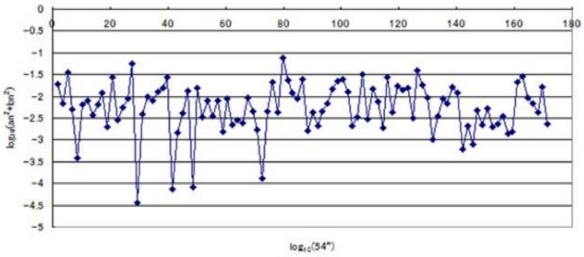


Fig. 3. The power spectrum of $g(t_i)$ (41).

and the revised original data y_i and the calculated data $g(t_i)$ are illustrated in Fig. 2. In this case, the resultant power spectrum is shown in Fig. 3, which has a flat average value with a property like white noise.

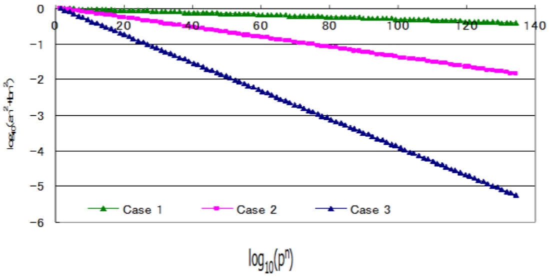
Similarly, we can construct another nonlinear time series expansion, by introducing chaotic time series (31) and (32) derived from the Jacobi elliptic functions in Section 4, as

$$g(t_i) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cn(p^n t_i, k) + b_n sn(p^n t_i, k)), \tag{42}$$

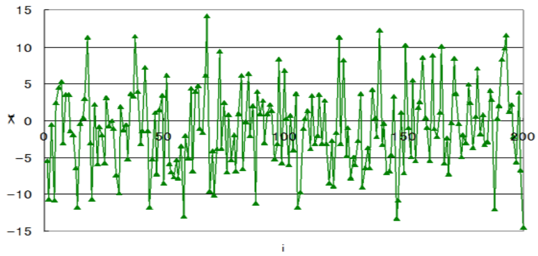
where k is the modulus for elliptic functions.

In addition, on the basis of the given discrete-time data X_i, we can construct other chaotic time series expansions, which generate 1/f fluctuation and noise by setting arbitrarily the coefficients $\{a_0, a_n, b_n\}$ of (41) as shown in Fig. 4.

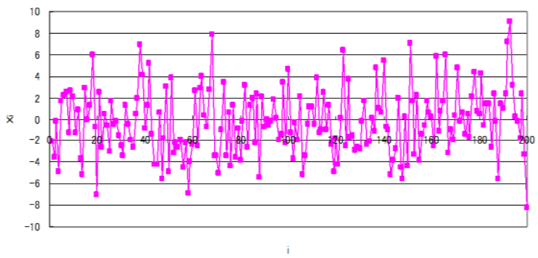
Therefore, the proposed time-dependent chaos functions and the nonlinear expansion could be applied to analyzing the data of noise, to constructing the chaotic time series expansion and to generating 1/f noise, as noise analyzer, chaos function generator and 1/f noise generator, respectively, and to input voltage of chaotic circuit and system, natural illumination, natural sound, natural vibration in the fields of engineering and technology.



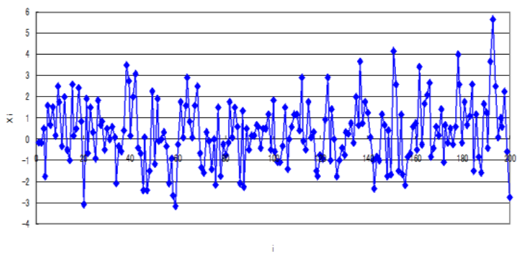
(a) Power spectra



(b) Case 1 in (a)



(c) Case 2 in (a)



(d) Case 3 in (a)

Fig. 4. Power spectra and 1/ f noise obtained by (41) with setting the coefficients $\{a_0, a_n, b_n\}$ arbitrarily [25].

Conclusions

We have shown in this paper firstly that the chaos function (1) with (2) and the time-dependent chaos function (3) with (4) have the chaotic properties of non-periodicity and sensitivity on initial values. Then, the chaotic time series are derived from the Chebyshev polynomials, and the generalized chaotic time series (31) and (32) are obtained on the basis of the Jacobi elliptic functions. Finally, numerical calculation steps for the chaotic time series expansion are represented, and an application of the chaotic time series and the nonlinear expansion is briefly discussed for the fields of engineering and technology.

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