Non-Shilnikov Chaos in Erbium doped Fiber Ring Laser

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Abstract: The Silnikov criteria requiring at least one fixed point to be of saddle focus type for chaos to be generated is tested here in Erbium doped fiber ring laser(EDFRL). The study of EDFRL linear stability analysis, nature and dynamics of fixed points is presented using theoretical and numerical approach. Two fixed points are calculated for EDFRL and their movement with change of key parameters is studied. The system converges to the same fixed points whenever we start the system with a slight deviation proving that the equilibrium points are not saddle focus type, yet the system is shown here and known previously to generate chaos violating Shilnikov criteria. This paper proves that EDFRL is a practical system which lacks the existence of a saddle focus type fixed point, hence generates Non- Silnikov chaos for some parameter range.

Keywords: Silnikov theorem, EDFRL, chaos, Fiber laser, linear stability, fixed points.

1 Introduction

Chaotic systems in the fields of physics, engineering, biology and medicine have been extensively studied since many decades. For a continuous dynamical system to produce chaos, at least three differential equations are required as per Poincare-Bendixon theorem [1]. Lasers are represented by their rate equations, meeting all the conditions for nonlinearity and subsequently possibility of chaos generation. The semiconductor laser [2–5], microchip laser [6–8] and fiber laser [9–13] have been extensively demonstrated both numerically and experimentally as successful chaos generators.
Erbium-doped fiber laser (EDFRL) is an important class of laser in the context of optical fiber communications. EDFRL can be used for generating very high output powers of magnitudes as high as tens of watts or even more [14]. Various experimental and numerical studies [9-16] have been conducted for this important class of lasers. Abarbanel et al for the first time developed a quantum level model of EDFRL and applied it to study bandwidth and frequency domain characteristics of chaos [15]. Luo et al developed mathematical model for pump

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modulated EDFRL to study optical bi-stability and bifurcation by varying modulating frequency [16]. Furthermore routes to chaos, chaos synchronization and master slave configurations were also explored and the experimental results were in very good agreement with theoretical study [16]. The mathematical model developed by Luo was simpler and at a higher level than that formulated by Abarbanel. Study of chaotic behavior using some form of forcing function and variation of different parameters remained the main focus for the analysis of EDFRL laser model. Dynamics of unforced system and linear stability analysis was not presented to the best of our knowledge, which is the main focus of this paper. The system is linearized at its fixed points using Jacobian. This analysis is quite important to explore the nature of fixed points, hence having an insight of rich dynamics of the system to relate the parameter ranges with stable, unstable or chaotic regimes. Another interesting observation as presented in this paper is that the EDFRL is different than the conventional chaotic systems i.e., chaos cannot be predicted using Silnikov theorem [17-18], which is used to prove the existence of chaos in continuous nonlinear dynamical systems.

As per Silnikov theorem a system must have at least one fixed point of saddle focus type. Furthermore three eigenvalues $\gamma, \sigma \pm i\omega$ of jacobian evaluated at that fixed point should satisfy $\gamma\sigma < 1, \omega \neq 0$ and with the assumptions that $|\gamma| > |\sigma| > 0$ and the existence of homoclinic orbit, chaos can be shown to exist. B. Chen et al [19] extended the Silnikov theorem to the case where one of the eigenvalues of the Jacobian evaluated at an equilibrium point is zero and the other two are complex. Modified Silnikov method is devised to cater for the degenerate case, where one of the eigenvalue is zero, for a system of particular form.

Until recently it was believed that conformity with Silnikov theorem is necessary for a system to exhibit chaotic trajectories. Xiong Wang, Guanrong Chen [20] reported recently that new generation of systems does exist which can be shown to violate Silnikov theorem and yet produce chaos. The system presented by Xion et al. has only one stable focus, so there is no question of homoclinic/heteroclinic trajectories. In the absence of any instability, it is expected that all trajectories will definitely converge to the stable focus. However the existence of chaotic trajectories was shown numerically using a range of different parameter values.

In this context, EDFRL which is already shown to be a chaos generator [9-16], is presented here as a practical system which produces Non-Silnikov chaos. In this paper we have shown that EDFRL system has two equilibrium points, around which system is linearized and general expressions for equilibrium points and eigenvalues are found. It is shown that both of these equilibriums are not of saddle focus type, so one would not expect chaos to be generated as per Silnikov theorem. However the generation of chaos is further validated by numerical computation of some example in time series and phase space. Variation of fixed points for different parameter values and shape of converging trajectories around these fixed points is also shown. The general expressions of fixed points and eigenvalues are evaluated using linear stability analysis. Dependence of equilibrium points on system parameters, approaching trajectories and time series data is presented using numerical simulations.

2. Theoretical Model

The mathematical model representing the dynamics of Laser intensity I_{la} and population inversion D for erbium doped fiber is reported to be as follow

$$\dot{I}_{la} = -k(1 + \epsilon \cos\omega t)I_{la} + gI_{la}D \qquad (1)$$

$$\dot{D} = -\left(1 + I_p + I_{la}\right)D + I_p - 1 \qquad (2)$$

g and k are the cavity gain and loss parameters respectively, ϵ is modulation index and I_p is Laser threshold. Third degree of freedom is provided by time variable $\dot{T} = 1$. To find out the equilibrium or fixed points, time derivates will be set to zero and perturbation term is also ignored. So system takes the form

$$-kI_{la} + gI_{la}D = 0 \qquad (3)$$

$$-\left(1 + I_p + I_{la}\right)D + I_p - 1 = 0 \qquad (4)$$

3. Fixed Points and Shilnikov Theorem

Simultaneously solving for I_{la} and D, two fixed points are

$$\tilde{x}_1 = \begin{bmatrix} I_{la1} \\ D_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{I_p - 1}{I_p + 1} \end{bmatrix} \qquad (5)$$

$$\tilde{x}_2 = \begin{bmatrix} I_{la2} \\ D_2 \end{bmatrix} = \begin{bmatrix} I_p\left({}^{g}/_{k} - 1\right) - \left({}^{g}/_{k} + 1\right) \\ {}^{k}/_{g} \end{bmatrix} \qquad (6)$$

In order to analyze the nature of these fixed points, system is linearized at these points. Defining the functions *f* and *g* as

$$f = -kI_{la} + gI_{la}D \qquad (7)$$

$$g = -\left(1 + I_p + I_{la}\right)D + I_p - 1 \qquad (8)$$

The Jacobian of the system is defined as

$$J(x) = \begin{bmatrix} \frac{\partial f}{\partial I_{la}} & \frac{\partial f}{\partial D} \\ \frac{\partial g}{\partial I_{la}} & \frac{\partial g}{\partial D} \end{bmatrix} = \begin{bmatrix} -k + gD & gI_{la} \\ -D & -(1 + I_p + I_{la}) \end{bmatrix} \quad (9)$$

To linearize the system at first fixed point the Jacobian is evaluated at $\tilde{x}_1 = \begin{bmatrix} 0 \\ \frac{I_p - 1}{I_p + 1} \end{bmatrix}$

$$J(\tilde{x}_1) = \begin{bmatrix} -k + g\left(\frac{I_p - 1}{I_p + 1}\right) & 0 \\ -\left(\frac{I_p - 1}{I_p + 1}\right) & -(1 + I_p) \end{bmatrix} \quad (10)$$

The eigen values are given as

$$\lambda_1 = g\left(\frac{I_p - 1}{I_p + 1}\right) - k$$

$$\lambda_2 = -(I_p + 1)$$

For a homoclinic or hetroclinic orbit, stable and unstable manifolds should cross transversally, which imply that fixed point should be a saddle point. One of the given eigenvalues must be positive, for $\tilde{x}_1$ to be a saddle. Now λ_2 cannot be positive because $I_p > 0$ is necessary condition for lasing action. For λ_1 to be positive, following condition must be true

$$I_p > \frac{g+k}{g-k} \quad (11)$$

So $I_{th} = \frac{g+k}{g-k}$ is the laser threshold for EDFRL as already reported in literature. This is the minimum limit on I_p for the start of lasing action and also the lower bound on I_p for the equilibrium point to be a saddle. This does not guarantee the generation of chaos as Silnikov theorem requires additionally for saddle point to be a focus too. As all the eigenvalues are real and absence of imaginary part implies that there is no homoclinic orbit around fixed point $\tilde{x}_1$. These conditions show that Silnikov theorem is not applicable, and chaos is not likely to be generated. Now analyzing fixed point $\tilde{x}_2$.

$$\tilde{x}_2 = \begin{bmatrix} I_{la2} \\ D_2 \end{bmatrix} = \begin{bmatrix} I_p\left({}^{g}/_{k} - 1\right) - \left({}^{g}/_{k} + 1\right) \\ {}^{k}/_{g} \end{bmatrix} \tag{12}$$

Evaluating the Jacobian at $\tilde{x}_2$

$$J(\tilde{x}_2) = \begin{bmatrix} -k + g\left(\frac{k}{g}\right) & g\left(I_p\left({}^{g}/_{k} - 1\right) - \left({}^{g}/_{k} + 1\right)\right) \\ \left(\frac{k}{g}\right) & -\left(1 + I_p + I_p\left({}^{g}/_{k} - 1\right) - \left({}^{g}/_{k} + 1\right)\right) \end{bmatrix} \tag{13}$$

Finding eigenvalues

$$\lambda_{1,2} = -\frac{1}{2}\left[\frac{g}{k}(I_p - 1)\right] \pm \frac{1}{2}\left[\sqrt{\left[\frac{g}{k}(I_p - 1)\right]^2 - 4\left[I_p(g-k) - (g+k)\right]}\right] \tag{14}$$

For all practical purposes cavity gain parameter g remains greater than cavity loss parameter k and I_p should be greater than laser threshold, so these eigenvalues will always have negative real part. So a stable fixed point attracts all the trajectories around neighborhood and there is no saddle point unless the third eigenvalue is positive. From the third equation of model, the third eigenvalue $\lambda_3 = 0$, violates Silnikov theorem conditions $\gamma\sigma < 1, \omega \neq 0$ and $|\gamma| > |\sigma| > 0$.

Modified Silnikov theorem may be applied when there is a degenerate case of zero eigenvalue, but it is applicable for the following general normal form of the system

$$\dot{x} = \rho x - \omega y + axz + byz + o(3)$$

$$\dot{y} = \omega x + \rho y + ayz + bxz + o(3)$$

$$\dot{z} = cx^2 + o(3)$$

Then for chaos to occur the modified Silnikov theorem is based on two assumptions; (1) $c\rho < 0,\ c\omega > 0;$ (2) there exists a homoclinic orbit connected at (0,0,0).

Here we see that the system model of fiber laser is not in conformity with the above model and assumptions presented are not true for this system, so the modified Silnikov theorem is also not applicable. It is evident that Silnikov criterion is not being fulfilled by EDFRL system, so one would expect that there is no possibility for chaos generation. However, fiber laser is well known to exhibit chaos as reported our earlier work[9-10]. In the following section some sample phase space and time series data is presented to show the typical shape of chaotic waveforms.

4. Simulations and Results

The solution for the mathematical model of fiber laser is computed numerically using Fourth order Runge-Kutta solver. At first simulation is run without perturbation term. Trajectories are expected to converge to a fixed point as calculated in section II. Using k=3.3e7; g=2*k; I_p=5; two fixed points are calculated as $\tilde{x}_1$= [0, 0.6667] and $\tilde{x}_2$ = [2, 0.500000]. Trajectories are converging to a particular fixed point, when started from different initial points in the neighborhood. When started from some neighborhood the system converges to fixed point $\tilde{x}_1$= [0, 0.6667] as in Fig. 1. This is in total agreement with fixed point location as found theoretically. First fixed point $\tilde{x}_1$ shows the rest condition of laser when I = 0 and lasing action has not started.

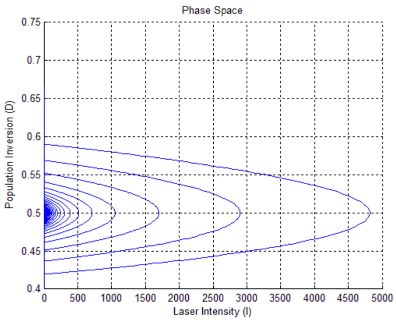


Fig. 1. System convergence to fixed point $\tilde{x}_1$= [0, 0.6667]

Table-1 depicts some instances for the change of location of fixed points in phase space with respect to change of parameters. The first fixed point always has lasing intensity zero which means lasing is not started yet but population inversion is building up. However, the second fixed point has nonzero value of lasing intensity as well which means a steady lasing condition.

Table-1 Change in two fixed points with parameter changes

I_p	k	g	Fixed point $\tilde{x}_1$	Fixed point $\tilde{x}_2$
5	3.3 x 10^7	2k	[0,0.6667]	[2,0.500000]
10	3.3 x 10^7	2k	[0, 0.8182]	[7, 0.500000]
20	3.3 x 10^7	2k	[0, 0.9048]	[17, 0.500000]
100	3.3 x 10^7	2k	[0, 0.9802]	[97, 0.500000]
10	3.3 x 10^7	4k	[0, 0.8182]	[25,0.2500]
10	3.3 x 10^7	10k	[0, 0.8182]	[79,0.1000]

All the simulated results show that system is dissipative and converging to equilibrium points from all directions for a variety of parameter ranges. Fig. 2

shows the path of changing second fixed point when I_p=10 and g is changed from 2k to 30k.

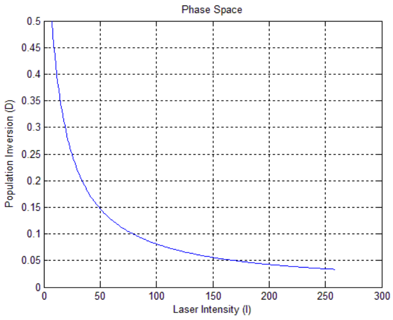
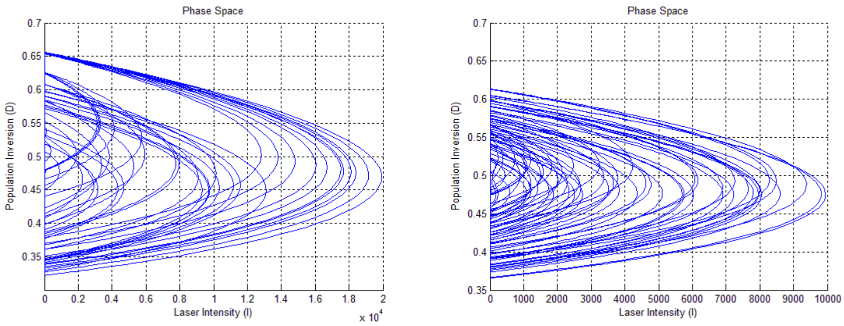


Fig.2 Trajectory of second fixed point for change of g_a from $2k_{a0}$ to $30k_{a0}$

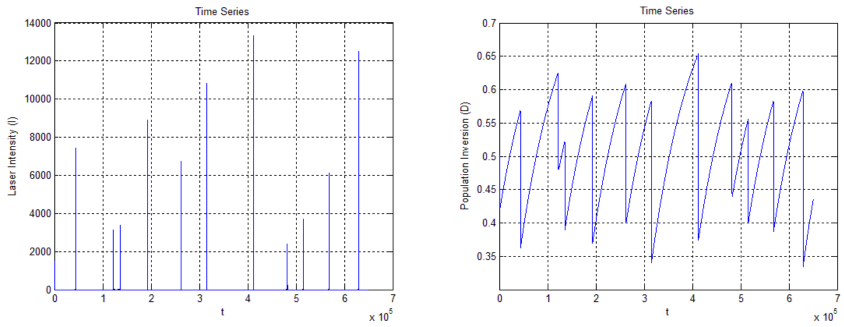
Next $\cos\omega t$ is used as perturbation term or forcing function to generate chaotic dynamics. Fig. 3(a) and (b) show typical phase space chaotic trajectories when modulation index is taken as 0.3 and 0.05 respectively. Fig. 4(a) and (b) show chaotic time variation of lasing intensity I_{la} and population inversion D.



(a) (b)

Fig. 2 Phase space trajectories for different modulation index

(a) m_a=0.05 (b)m_a=0.3



(a) (b)

Fig. 4. Time variation (a) Lasing Intensity I_{la} (b) Population inversion D

5. Conclusions

This work shows that practical systems like EDFRL can produce Non- Silnikov chaos which is evident from linear stability analysis and the nature of fixed points. Eigenvalues at both the fixed points are either real or having negative real part hence suggesting a stable system. Linear stability analysis and general form of fixed points is presented theoretically, while simulation results show that system actually converges to theoretical fixed point locations in phase space. This study suggests that EDFRL is an exception from conventional chaos generators who follow Shilnikov criteria.

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