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Asymptotic Behaviour of the Nonlinear Dynamical System Governing a Thermosyphon Model

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Abstract. Thermoyphons, in the engineering literature, is a device composed of a closed loop containing a fluid who motion is driven by several actions such as gravity and natural convection. Their dynamics are governing for a coupled differential nonlinear systems. In several previous work we show chaos in the fluid, even with a bynary fluid (Jiménez-Casas and Ovejero[6], Jiménez-Casas and Rodríguez-Bernal[5,8]). In this work I prove some result about the asymptotic behaviour for solutions of above system with binary fluids (water and antifreeze) when we consider a prescribed heat flux. These results are the generalizing of those obtained by Rodríguez-Bernal and Van Vleck[17] for a thermosyphon model with one-component fluid.
Keywords: Thermosyphon, Asymptotic behaviour, Inertial Manifold.

1 Introduction

In the engineering literature a thermosyphon is a device composed of a closed loop containing a fluid where some soluble substance has been dissolved. The motion of the fluid is driven by several actions such as gravity and natural convection. In particular, we will consider the convective movements caused by inner solute fluctuations generated by a temperature gradient; this fact is known as the Soret effect, and it has been studied experimentally by Hart, Hurle[2,4] between others. We study the evolution of the velocity of the fluid v, of the temperature T of the fluid and of the solute concentration S.

We assume that the section of the loop is constant and small compared with the dimensions of the physical device, so that the arc length coordinate along the loop (x) gives the position in the circuit. The velocity v of the fluid is assumed to be independent of the position in the circuit, i.e. it is assumed to be a scalar quantity depending only on time, $v = v(t)$. The other relevant quantities, namely temperature, $T(t, x)$, and concentration of the solute, $S(t, x)$, are assumed to depend on time and position along the loop.

We assume that the average circulation is generated by the net buoyancy torque exerted by both solute concentration and temperature, and is retarded

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by viscous drag at the wall, i.e. by friction forces. In addition we assume that the variations of temperature are independent on the solute concentration. We consider the distribution equation of solute into the loop as in Hart[2] and Keller[12], where has been used the conservation of mass for the solute and has been assumed that the fluid also transports the solute, and be generated by Soret diffusion and reduced by molecular diffusion.

The evolution of the above quantities is given by the following ODE/PDE system (cf. Jiménez-Casas and Ovejero[6], Jiménez-Casas and Rodríguez-Bernal [5,8], Jiménez-Casas[7],Velázquez[18] for further details)

$$\begin{cases} \dfrac{dv}{dt} + G(v)v = \oint (T-S).f, \quad v(0) = v_0 \\ \dfrac{\partial T}{\partial t} + v\dfrac{\partial T}{\partial x} = h(x) + \nu \frac{\partial^2 T}{\partial x^2}, \quad T(0,x) = T_0(x) \\ \dfrac{\partial S}{\partial t} + v\dfrac{\partial S}{\partial x} = c\dfrac{\partial^2 S}{\partial x^2} - b\dfrac{\partial^2 T}{\partial x^2}, \, S(0,x) = S_0(x) \end{cases} \tag{1}$$

It is important to note that all functions considered are 1-periodic respect to the spatial variable. The function f, describes the geometry of the loop and the distribution of gravitational forces, so note that $\oint f = 0$ where $\oint = \int_0^1 dx$ denotes integration along the closed path of the circuit. We consider the general geometries as in Velázquez[18]. In the sequel we assume that G and h, are given continuous functions, such that $G(v) \geq G_0 > 0$, and h is a prescribed heat flux as the heat transfer law across the loop wall. The functions $G(v)$, which specifies the friction law at the inner wall of the loop (Keller[12], Liñan[13], Rodríguez-Bernal and Van Vleck[16],Velázquez[18]), and h which prescribes the heat flux at the wall of the loop (cf.Liñan[13]), are given by different forms. The diffusion coefficients b, c are positive constants and we note that b is proportional to the Soret coefficient, therefore if we assume it to be zero, i.e. if we neglect the Soret effect, and we start with an homogeneous initial concentration of solute, then S remains constant in time and space in Eq. (1) and, since $\oint f = 0$, then Eq. (1) reduces exactly to the model in Rodríguez-Bernal and Van Vleck[16],Velázquez[18] and Rodríguez-Bernal and Van Vleck[17] when we consider a prescribed heat flux h as the heat transfer law across the loop wall instead of Newton's linear cooling law, i.e. $h = k(T_a - T)$ where k is a positive quantity, sometimes depending on the velocity, and T_a is the (given) ambient temperature distribution Welander[19].

The contribution in this paper (Section 3) is to prove that, under suitable conditions, any solution either converges to the rest state or the oscillations of velocity around $v = 0$ must be large enough. This result, generalizes the one proposed in Rodríguez-Bernal and Van Vleck[17] for a thermosyphon model including a two-component fluid.

2 Previous results about well posedness and global attractor

First, we note that in this section we consider the case in which all periodic functions in Eq. (1) have zero average, i.e. we work in $\mathcal{Y} = I\!R \times \dot{H}^1_{per}(0,1) \times$

$\dot{L}^2_{per}(0,1)$, where

$$\dot{L}^2_{per}(0,1) = \{u \in L^2_{loc}(\mathbb{R}),\ u(x+1) = u(x) a.e.,\ \oint u = 0\},$$

$$\dot{H}^m_{per}(0,1) = H^m_{loc}(\mathbb{R}) \cap \dot{L}^2_{per}(0,1).$$

In effect, we observe that, for $\nu > 0$, if we integrate the equation for the temperature along the loop, i.e. intregating the second equation of Eq. (1) with respect to x, we have that $\frac{d}{dt}(\oint T) = \oint h$ and then $\oint T(t) = \oint T_0 + t \oint h$. Therefore, the temperature is unbounded, as $t \to \infty$, unless $\oint h = 0$. However, taking $\theta = T - \oint T$ and $h^* = h - \oint h$ reduces to the case $\oint \theta = \oint \theta_0 = \oint h^* = 0$, since θ would satisfy:

$$\frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial x} = h(x) + \nu \frac{\partial^2 \theta}{\partial x^2}, \quad \theta(0) = \theta_0 = T_0 - \oint T_0 \tag{2}$$

Moreover, integrating with respect to x the third equation of Eq. (1) and taking into account the periodicity of T and S, we have $\oint \frac{\partial S}{\partial t} = -v \oint \frac{\partial S}{\partial x} - b\oint \frac{\partial^2 T}{\partial x^2} + c\oint \frac{\partial^2 S}{\partial x^2} = 0$. Therefore, $\frac{d}{dt}[\oint S dx] = 0$ and $\oint S$ is constant respect to t, i.e. $\oint S = \oint S_0 = m_0$.

Therefore, if we consider now $\theta = T - \oint T$ and $\sigma = S - \oint S_0$, then from the second and third equation of system Eq. (1), we obtain that θ and σ verify the equations (2) together with (3)

$$\frac{\partial \sigma}{\partial t} - c\frac{\partial^2 \sigma}{\partial x^2} = -v\frac{\partial \sigma}{\partial x} - b\frac{\partial^2 \tau}{\partial x^2}, \quad \sigma(0) = \sigma_0 = S_0 - \oint S_0. \tag{3}$$

Finally, since $\oint f = 0$, we have that $\oint (T-S) f = \oint (\theta - \sigma) f$, and the equation for v reads

$$\frac{dv}{dt} + G(v)v = \oint (\theta - \sigma).f, \quad v(0) = v_0. \tag{4}$$

Thus, from Eqs. (2), (3) and (4) we have (v, θ, σ) verifies system Eq.(1) with h^*, θ_0, σ_0 replacing h, T_0, S_0 respectively and now $\oint \theta = \oint \sigma = \oint h = \oint \sigma_0 = \oint \theta_0 = 0$.

We note that to obtain the original dynamics we put $v, T = \tau + \oint T, S = \sigma + m_0$, where $m_0 = \oint S_0$, which shows that the dynamics is essentially independent of m_0. Thus, using again variables v, T and S instead of v, θ and σ we consider the system Eq. (1) with $\oint T_0 = 0$, $\oint S_0 = 0, \oint h = 0$ and $\oint T(t) = \oint S(t) = 0$ for every $t \geq 0$.

Also, if $\nu > 0$ the operator $-\nu \frac{\partial^2}{\partial x^2}$, together with periodic boundary conditions, is an unbounded, self-adjoint operator with compact resolvent in $L^2_{per}(0,1)$ that is positive when resctricted to the space of zero-average functions $\dot{L}^2_{per}(0,1)$. Hence, the second equation in Eq. (1) is of parabolic type for $\nu > 0$, as is the third equation in Eq. (1). Thus, we can apply the result about the sectorial operator of Henry[3] to prove the existence of solutions of system Eq. (1), such that we get the following result on Jiménez-Casas and Ovejero[6].

Proposition 1. *If we assume that* $G^*(r) = rG(r)$ *is locally Lipschitz, and* $f, h \in \dot{L}^2_{per}$. *Then, given* $(v_0, T_0, S_0) \in \mathcal{Y} = I\!R \times \dot{H}^1_{per} \times \dot{L}^2_{per}$ *there exists a unique solution of Eq.* (1) *satisfying*

$$(v, T, S) \in C([0, \infty], \mathcal{Y}) \cap C(0, \infty, I\!R \times \dot{H}^3_{per}(0,1) \times \dot{H}^2_{per}(0,1)),$$

$$(\dot{v}, \frac{\partial T}{\partial t}, \frac{\partial S}{\partial t}) \in C(0, \infty, I\!R \times \dot{H}^{3-\delta}_{per}(0,1) \times \dot{H}^{2-\delta}_{per}(0,1))$$

for every $\delta > 0$. *In particular, Eq.* (1) *defines a nonlinear semigroup* $S^*(t)$ *in* $\mathcal{Y}$, *which is defined by* $S^*(t)(v_0, T_0, S_0) = (v(t), T(t, \cdot), S(t, \cdot))$.

Moreover, Eq. (1) *has a global compact and connected attractor,* $\mathcal{A}$, *in* $\mathcal{Y}$. *Also if* $h \in \times \dot{H}^m_{per}(0,1)$ *with* $m \geq 1$, *the global attractor* $\mathcal{A} \subset I\!R \times \dot{H}^{m+2}_{per} \times \dot{H}^{m+2}_{per}$ *and is compact in this space.*

Proof. This result has been proved in Theorem 2.1,Theorem 2.2 and Corollary 2.1 from Jiménez-Casas and Ovejero[6].

3 Asymptotic behaviour for solutions under orthogonality condition

In previous works, like Jiménez-Casas and Rodríguez-Bernal[8,5], Jiménez-Casas and Ovejero[6], the asymptotic behaviour of the system Eq.(1) for large enough time is studied.

In this sense the existence of a inertial manifold associated to the functions f (loop-geometry) and h (prescribed heat flux) have proved. The abstract operators theory (Henry[3], Foias et al. [1] and Rodríguez-Bernal[15,14]) has been used for this purpose.

In this section we prove in Proposition 2 the results which rise an important consequence: for large time the velocity reaches the equilibrium - null velocity -, or takes a value to make its integral diverge, which means that either it remains with a constant value without changing its sign or it will alternate an infinite number of times so the oscillations around zero become large enough to make the integral diverge.

3.1 Previous results and notations

In this section we assume also that $G^*(r) = rG(r)$ is locally Lipschitz, and $f, h \in \dot{L}^2_{per}$ are given by following Fourier expansions

$$T_a(x) = \sum_{k \in Z\!\!\!Z^*} b_k e^{2\pi kix}; \qquad f(x) = \sum_{k \in Z\!\!\!Z^*} c_k e^{2\pi kix}; \text{ where } Z\!\!\!Z^* = Z\!\!\!Z - \{0\}\,, \quad (5)$$

while $T_0 \in \dot{H}^2_{per}$ and $S_0 \in \dot{L}^2_{per}$ are given by

$$T_0(x) = \sum_{k \in Z\!\!\!Z^*} a_{k0} e^{2\pi kix}, S_0(x) = \sum_{k \in Z\!\!\!Z^*} d_{k0} e^{2\pi kix}.$$

Finally assume that $T(t,x) \in \dot{H}^2_{per}$ and $S(t,x) \in \dot{L}^2_{per}$ are given by

$$T(t,x) = \sum_{k\in \mathbb{Z}^*} a_k(t)e^{2\pi kix} \text{ and } S(t,x) = \sum_{k\in \mathbb{Z}^*} d_k(t)e^{2\pi kix} \quad \mathbb{Z}^* = \mathbb{Z} - \{0\} \quad (6)$$

We note that $\bar{a_k} = -a_k$ ($\bar{d_k} = -d_k$) since all functions consider are real and also $a_0 = d_0 = 0$ since they have zero average.

Now we observe the dynamics of each Fourier mode and from Eq. (1), we get the following system for the new unknowns, v and the coefficients $a_k(t)$ and $d_k(t)$.

$$\left\{ \begin{array}{c} \frac{dv}{dt} + G(v)v = \sum_{k\in \mathbb{Z}^*} (a_k(t) - d_k(t))c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + 4\nu\pi^2k^2\right] a_k(t) = b_k \\ \dot{d}_k(t) + \left[2\pi kiv(t) + 4c\pi^2k^2\right] d_k(t) = 4b\pi^2k^2 a_k(t) \end{array} \right. \quad (7)$$

- Assume that the prescribed heat flux $h \in \dot{H}^m_{per}$, are given by

  $$h(x) = \sum_{k\in K} b_k e^{2\pi kix},$$

  and $b_k \neq 0$ for every $k \in K \subset \mathbb{Z}$ with $0 \neq K$, since $\oint h = 0$. We denote by V_m the clousure of the subspace of $\dot{H}^m_{per}$ generated by $\{e^{2\pi kix}, k \in K\}$. Then we have from Theorem 2.3 in Jiménez-Casas and Ovejero[6] the set $\mathcal{M} = \mathbb{R} \times V_m \times V_{m-1}$ is an **inertial manifold** for the flow of $S^*(t)(v_0, T_0, S_0) = (v(t), T(t), S(t))$ in the space $\mathcal{Y} = \mathbb{R} \times \dot{H}^m_{per} \times \dot{H}^{m-1}_{per}$. By this, the dynamics of the flow is given by the flow in $\mathcal{M}$ associated to the prescribed heat flux h. This is

  $$\left\{ \begin{array}{c} \frac{dv}{dt} + G(v)v = \sum_{k\in K} (a_k(t) - d_k(t))c_{-k} \\ \dot{a}_k(t) + \left[2\pi kiv(t) + 4\nu\pi^2k^2\right] a_k(t) = b_k,\ k \in K \\ \dot{d}_k(t) + \left[2\pi kiv(t) + 4c\pi^2k^2\right] d_k(t) = 4b\pi^2k^2 a_k(t),\ k \in K \end{array} \right. \quad (8)$$

- Moreover, we assume that the function associated to the geometry of the loop f, are given by

  $$f(x) = \sum_{k\in J} c_k e^{2\pi kix}$$

  and $c_k \neq 0$ for every $k \in J \subset \mathbb{Z}$ with $0 \neq K$, since $\oint f = 0$.
  We note also that on the inertial manifold
  $\oint (T - S)f = \sum_{k\in K\cap J} a_k c_{-k} - \sum_{k\in K\cap J} d_k c_{-k}$. Thus, the dynamics of the system depends only on the coefficients in $K \cap J$.
- Herafter, we consider de functions h and f are given by following Fourier expansions

  $$h(x) = \sum_{k\in K} b_k e^{2\pi kix}; \qquad f(x) = \sum_{k\in J} c_k e^{2\pi kix}; \quad (9)$$

  where

  $$K = \{k \in \mathbb{Z}^*/b_k \neq 0\}, J = \{k \in \mathbb{Z}^*/c_k \neq 0\} \text{ with } \mathbb{Z}^* = \mathbb{Z} - \{0\},$$

First, from the equations Eq. (7) we can observe the velocity of the fluid is independent of the coefficients for temperature $a_k(t)$ and the salinity $d_k(t)$ for every $k \in \mathbb{Z}^* - (K \cap J)$. That is, the **relevant coefficients** for the evolution of the velocity are only $a_k(t)$ and $d_k(t)$ with k belonging to the set $K \cap J$. This important result about the asymptotic behaviour has been proved in Corollary 2.2 from Jiménez-Casas and Ovejero[6].

We also note that $0 \notin K \cap J$ and since $K = -K$ and $J = -J$ then the set $K \cap J$ has an even number of elements, wihch we denote by $2n_0$. Therefore the number of the positive elements of $K \cap J$, $(K \cap J)_+$ is n_0. Moreover the equations for a_{-k} and d_{-k} are conjugates of the equations for a_k and d_k, and therefore we have $\sum_{k \in K \cap J} a_k c_{-k} = 2Re(\sum_{k \in (K \cap J)_+} a_k c_{-k})$ and analogously $\sum_{k \in K \cap J} d_k c_{-k} = 2Re(\sum_{k \in (K \cap J)_+} d_k c_{-k})$. Thus

$$\oint (T - S) f = 2Re(\sum_{k \in (K \cap J)_+} a_k c_{-k}) - 2Re(\sum_{k \in (K \cap J)_+} d_k c_{-k}). \tag{10}$$

The aim is to prove the Proposition 2 which generalize the result of thermosyphon model without solute of Rodríguez-Bernal and Van Vleck[17] in the case of a prescribed heat flux, i.e. $h = h(x)$. To do so we examine which are these steady-state solutions, also called *equilibrium points.*

We have to make the difference between equilibrium points (constants respect to the time) null velocity, called *rest equilibrium*, and equilibrium points with non-vanishing constant velocity.

Equilibrium conditions.

i) The system Eq. (7) presents the *rest equilibrium* $v = 0$, $a_k = \frac{b_k}{4\nu\pi^2 k^2}$ and $d_k = \frac{b}{c} a_k = \frac{b}{c} \frac{b_k}{4\nu\pi^2 k^2}$ $\forall k \in K \cap J$ under the assumption of the following orthogonality condition:

$$I_0 = Re(\sum_{k \in (K \cap J)_+} \frac{b_k c_{-k}}{k^2}) = 0. \tag{11}$$

ii) Any other equilibrium position will have a non-vanishing velocity and the equilibrium is given by:

$$\begin{cases} G(v)v = 2Re(\sum_{k \in (K \cap J)_+} a_k c_{-k}) - 2Re(\sum_{k \in (K \cap J)_+} d_k c_{-k}) \\ a_k = \frac{b_k}{4c\pi^2 k^2 + 2\pi k i v} \\ d_k = \frac{4b\pi^2 k^2}{4c\pi^2 k^2 + 2\pi k i v} \frac{b_k}{4c\pi^2 k^2 + 2\pi k i v} \end{cases} \tag{12}$$

3.2 Asymptotic behaviour

Lemma 1. *If we assume that a solution of Eq. (7) satisfies* $\int_0^\infty |v(s)| ds < \infty$, *then for every* $\eta > 0$ *there exists* t_0 *such that*

$$\int_{t_0}^t e^{-4\nu\pi^2 k^2 (t-r)} (e^{-\int_r^t 2\pi i k v} - 1) dr \le \eta \quad \text{with } t \ge t_0. \tag{13}$$

Moreover

$$limsup_{t\to\infty}|\int_{t_0}^{t}(a_k(r)e^{-\int_r^t 2\pi kiv}-b_k^*)e^{-4c\pi^2k^2(t-r)}dr|\leq$$

$$\leq limsup_{t\to\infty}|a_k(t)-b_k^*|+\eta|b_k^*| \quad with\ b_k^*=\frac{b_k}{4\nu\pi^2k^2}\ and\ t\geq t_0 \qquad (14)$$

Proof: If $\int_0^\infty |v(s)|ds<\infty$, then for all δ there exits $t_0>0$ such that for every $t_0\leq r\leq t$ we have $|\int_r^t v|\leq\delta$. Then, for any $\eta>0$ we can take t_0 large enough such that

$$|e^{-\int_r^t 2\pi ikv}-1|\leq\eta \text{ for all } t_0\leq r\leq t. \qquad (15)$$

Therefore, we get

$$\int_{t_0}^{t}e^{-4\nu\pi^2k^2(t-r)}(e^{-\int_r^t 2\pi ikv}-1)dr\leq\frac{\eta}{4\nu\pi^2k^2}(1-e^{-4\nu\pi^2k^2(t-t_0)})\leq\frac{\eta}{4\nu\pi^2}\leq\eta$$

with $t\geq t_0$ and taking into account that $\eta\to 0$ for $t\to\infty$ and $\nu>0$, we get Eq. (13).

To prove Eq. (14), we write

$$\int_{t_0}^{t}(a_k(r)e^{-\int_r^t 2\pi kiv}-b_k^*)e^{-4c\pi^2k^2(t-r)}dr=$$

$$=\int_{t_0}^{t}(a_k(r)-b_k^*)e^{-\int_r^t 2\pi ikv}e^{-4c\pi^2k^2(t-r)}dr+\int_{t_0}^{t}b_k^*(e^{-\int_r^t 2\pi ikv}-1)e^{-4c\pi^2k^2(t-r)}dr$$

and taking modulus in this expression the first term in the right member remains

$$|\int_{t_0}^{t}(a_k(r)-b_k^*)e^{-\int_r^t 2\pi ikv}e^{-4c\pi^2k^2(t-r)}dr|\leq$$

$$\leq\int_{t_0}^{t}|a_k(r)-b_k^*|e^{-4c\pi^2k^2(t-r)}dr\leq limsup_{t\to\infty}(|a_k(t)-b_k^*|\frac{(1-e^{-4c\pi^2k^2(t-t_0)})}{4c\pi^2k^2}.$$

Now, for the second term in the right, considering the previous result together with Eq. (15), we have

$$|\int_{t_0}^{t}b_k^*(e^{-\int_r^t 2\pi ikv}-1)e^{-4c\pi^2k^2(t-r)}dr|\leq\eta|b_k^*|\frac{(1-e^{-4c\pi^2k^2(t-t_0)})}{4c\pi^2k^2}$$

and taking into account that $\frac{(1-e^{-4c\pi^2k^2(t-t_0)})}{4c\pi^2k^2}\leq 1$, we get Eq. (14).

□

Proposition 2. *i) We assume that* $I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{k^2}) = 0$, *with* $K\cap J$ *finite set, and that a solution of Eq.* (7) *satisfies* $\int_0^\infty |v(s)|ds < \infty$. *Then the system reaches the rest stationary solution, that:*

$$\begin{cases} v(t) \to 0, \ \ as\ t\to\infty \\ a_k(t) \to \frac{b_k}{4\nu\pi^2k^2}, \ \ as\ t\to\infty \\ d_k(t) \to \frac{b}{c}\frac{b_k}{4\nu\pi^2k^2}, \ \ as\ t\to\infty \end{cases}$$

ii) Conversely, if $I_0 = Re(\sum_{k\in(K\cap J)_+} \frac{b_k c_{-k}}{k^2}) \neq 0$ *then for every solution* $\int_0^\infty |v(s)|ds = \infty$, *and* $v(t)$ *does not converge to zero.*

Proof: i) First, we study the behaviour for large time.

The distance between the coefficients that represents the solution of the system, $a_k(t)$ and $d_k(t)$ to the values of those coefficients in the equilibrium, $\frac{b_k}{4\nu\pi^2k^2}$ and $\frac{b}{c}\frac{b_k}{4\nu\pi^2k^2}$ are computed.

For t_0 enough large, we known that for every $t > t_0$ we have

$$a_k(t) = a_k(t_0)e^{-\int_{t_0}^t 2\pi ikv+4\nu\pi^2k^2} + b_k\int_{t_0}^t e^{-\int_r^t 2\pi ikv+4\nu\pi^2k^2}dr \tag{16}$$

and using $\int_{t_0}^t e^{-\int_r^t 4\nu\pi^2k^2} = \frac{1}{4\nu\pi^2k^2}(1-e^{-4\nu\pi^2k^2(t-t_0)})$ we have that

$$a_k(t) - (1-e^{-4\nu\pi^2k^2(t-t_0)})\frac{b_k}{4\nu\pi^2k^2} = a_k(t_0)e^{-\int_{t_0}^t 2\pi ikv+4\nu\pi^2k^2} +$$

$$+b_k\int_{t_0}^t e^{-\int_r^t 4\nu\pi^2k^2}(e^{-\int_r^t 2\pi ikv}-1)dr.$$

Taking limits when $t\to\infty$, we get

$a_k(t) - (1-e^{-4\nu\pi^2k^2(t-t_0)})\frac{b_k}{4\nu\pi^2k^2} \to 0$, since $a_k(t_0)e^{-\int_0^t 2\pi ikv+4\nu\pi^2k^2} \to 0$ and from Eq. (13) we have that $b_k\int_{t_0}^t e^{-\int_r^t 4\nu\pi^2k^2}(e^{-\int_r^t 2\pi kiv}-1) \to 0$. Now taking into account that $(1-e^{-4\nu\pi^2k^2(t-t_0)})\frac{b_k}{4\nu\pi^2k^2}$ converges to $\frac{b_k}{4\nu\pi^2k^2}$ for large time we conclude that:

$$\begin{cases} a_k(t) \to \frac{b_k}{4\nu\pi^2k^2} \\ I_1(t) = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}) \to 2Re(\sum_{k\in(K\cap J)_+}\frac{b_kc_{-k}}{4\nu\pi^2k^2}) = I \end{cases} \tag{17}$$

Integrating the equation for $d_k(t)$ we have that

$$d_k(t) = d_k(t_0)e^{-\int_{t_0}^t(2\pi ikv+4c\pi^2k^2)} + \int_{t_0}^t (4b\pi^2k^2)a_k(r)e^{-\int_r^t(2\pi ikv+4c\pi^2k^2)}dr \tag{18}$$

from Eq. (18) we have that

$$\begin{array}{c} d_k(t) - (1-e^{-4c\pi^2k^2(t-t_0)})\frac{b}{c}\frac{b_k}{4\nu\pi^2k^2} = d_k(t_0)e^{-\int_{t_0}^t(2\pi ikv+4c\pi^2k^2)} + \\ +\int_{t_0}^t(4c\pi^2k^2)e^{-4c\pi^2k^2(t-r)}\frac{b}{c}(a_k(r)e^{-\int_r^t 2\pi ikv} - \frac{b_k}{4\nu\pi^2k^2})dr \end{array} \tag{19}$$

working as in the a_k case, we prove that Eq. (19) tends to zero and then we obtain the result about d_k.

In fact, taking limits when $t \to \infty$, we obtain

$$d_k(t) - (1 - e^{-4c\pi^2 k^2 (t-t_0)})\frac{b}{c}\frac{b_k}{4\nu\pi^2 k^2} \to 0,$$

since $d_k(t_0)e^{-\int_{t_0}^t (2\pi k v i + 4c\pi^2 k^2)} \to 0$ and taking into account Eq. (14) from Lemma 1 together Eq. (17), we have that

$$\int_{t_0}^t (4c\pi^2 k^2)e^{-4c\pi^2 k^2 (t-r)}\frac{b}{c}(a_k(r)e^{-\int_r^t 2\pi k v i} - \frac{b_k}{4\nu\pi^2 k^2})dr \to 0.$$

So that we get

$$\begin{cases} d_k(t) \to \frac{b}{c}\frac{b_k}{4\nu\pi^2 k^2} \\ I_2(t) = 2Re(\sum_{k\in(K\cap J)_+} d_k(t)c_{-k}) \to \frac{b}{c}2Re(\sum_{k\in(K\cap J)_+}\frac{b_k c_{-k}}{4\nu\pi^2 k^2}) = \frac{b}{c}I \end{cases} \quad (20)$$

To conclude, we study now when the velocity $v(t)$ goes to zero. From (10) we can reading the equation for v, the first equation of system Eq. (7), as

$$\frac{dv}{dt} + G(v)v = (I_1(t) - I) + (I_2(t) - \frac{b}{c}I) + (1 + \frac{b}{c})I$$

we have that

$$\begin{aligned} v(t) = v_{t_0}e^{-\int_{t_0}^t G(v)} + \int_{t_0}^t (I_1(t) - I)e^{-\int_r^t G(v)}dr + \int_{t_0}^t Ie^{-\int_r^t G(v)}dr + \\ + \int_{t_0}^t (I_2(t) - \tfrac{b}{c}I)e^{-\int_r^t G(v)}dr + \tfrac{b}{c}\int_{t_0}^t Ie^{-\int_r^t G(v)}dr \end{aligned} \quad (21)$$

where

$$\begin{cases} I_1(t) = 2Re(\sum_{k\in(K\cap J)_+} a_k(t)c_{-k}) \\ I_2(t) = 2Re(\sum_{k\in(K\cap J)_+} d_k(t)c_{-k}) \end{cases}$$

Now from Eqs.(17) and (20) for every $\delta > 0$ there exists t_0 such that $|I_1(s) - I| \leq \delta$ and $|I_2(s) - \frac{b}{c}I| \leq \delta$ for every $t_0 \leq s \leq t < \infty$.

Let $F(t) = \int_{t_0}^t e^{-\int_r^t G}dr$, with

$$F(t) = \int_{t_0}^t e^{-\int_r^t G}dr = \frac{\int_{t_0}^t e^{-\int_{t_0}^r G}dr}{e^{-\int_{t_0}^t Gdr}} \quad (22)$$

and then using L'Hopital's Lemma from [12] for the function F, we have

$$limsup_{t\to\infty}F(t) \leq \frac{e^{\int_{t_0}^t Gdr}}{Ge^{\int_{t_0}^t Gdr}} \leq limsup_{t\to\infty}\frac{1}{G(v)} \quad (23)$$

Hence, we find that

$$limsup_{t\to\infty}|v(t)-(1+\frac{b}{c})IF(t)| \leq 2\delta limsup_{t\to\infty}\frac{1}{G(v)},\ \forall\delta>0 \tag{24}$$

i.e., since δ is arbitrary, we get

$$v(t)-(1+\frac{b}{c})IF(t)\to 0. \tag{25}$$

We note that all above result, are valid for every I always we have the conditions $\int_0^\infty |v(s)|ds<\infty$.

Now, we note that $I_0 = Re(\sum_{k\in(K\cap J)_+}\frac{b_k(t)c_{-k}}{k^2}) = 0$ is equivalent to $I = \frac{1}{2\nu\pi^2}I_0 = 0$.

Therefore, if $I_0 = 0$ then $I = 0$ and we get from Eq. (25) that $v(t)\to 0$.

ii) If $I_0 = Re(\sum_{k\in(K\cap J)_+}\frac{b_k(t)c_{-k}}{k^2}) \neq 0$ this is $I \neq 0$ and we assume that $\int_0^\infty |v(s)|ds<\infty$. Then using again L'Hopital's Lemma from of Rodríguez-Bernal and Van Vleck[17], for F, with Eq. (22) we have that

$$liminf_{t\to\infty}F(t) \geq liminf_{t\to\infty}\frac{e^{\int_{t_0}^t Gdr}}{Ge^{\int_{t_0}^t Gdr}} \geq liminf_{t\to\infty}\frac{1}{G(v)} > 0,$$

and therefore from this together with Eq. (25) we conclude $liminf_{t\to\infty}|v(t)| > 0$, which implies that $\int_0^\infty |v(s)|ds=\infty$. This result is in contradiction with the initial condition $\int_0^\infty |v(s)|ds<\infty$, what implies that it is not a valid hypothesis.

□

3.3 Concluding remarks

Recalling that functions associated to circuit geometry, f, and to prescribed heat flux, h, are given by $f(x)=\sum_{k\in J}c_k e^{2\pi kix}$ and $h(x)=\sum_{k\in K}b_k e^{2\pi kix}$, respectively. In Jiménez-Casas and Ovejero[6] Corollary 2.2, using the operator abstract theory, it is proved that if $K\cap J=\emptyset$, then the global attractor for system Eq. (1) in $I\!R\times\dot{H}^1_{per}\times\dot{L}^2_{per}$ is reduced to a point $\{(0,\theta_\infty,\frac{b}{c}\theta_\infty)\}$, where θ_∞ is the unique solution in $\dot{H}^2_{per}(0,1)$ of

$$-\nu\frac{\partial^2\theta}{\partial x^2}=h(x).$$

In this sense the Proposition 2 offers the possibility to obtain the same asymptotic behaviour for the dynamics, i.e., the attractor is also reduced to a point taking functions f and h without this condition, that is with $K\cap J\neq\emptyset$, its enough that the set $(K\cap J)\neq\emptyset$, but $Re(\sum_{k\in(K\cap J)_+}\frac{b_k c_{-k}}{k^2})=0$.

We note, the result about the inertial manifold (Jiménez-Casas and Ovejero[6]) reduces the asymptotic behaviour of the initial system Eq. (1) to the dynamics of the reduced explicit system Eq. (7) with $k\in K\cap J$.

We observe also that from the analysis above, it is possible to design the geometry of circuit, f, and/or heat flux, h, so that the resulting system has an arbitrary number of equations of the form $N = 4n_0 + 1$ where n_0 is the number of elements of $(K \cap J)_+$ and we consider the real an imaginary parts of relevant coeffiicents for the temperature $a_k(t)$ and solute concentration $d_k(t)$ with $k \in (K \cap J)_+$.

Note that it may be the case that K and J are infinite sets, but their intersection is finite. Also, for a circular circuit we have $f(x) \sim asen(x) + bcos(x)$, i.e. $J = \{\pm 1\}$ and then $K \cap J$ is either $\{\pm 1\}$ or the empty set.

Recently, we have considered a thermosyphon model containing a viscoelastic fluid and we have shown chaos in some closed-loop thermosyphon model with one-component viscoelastic fluid (Yasappan and Jiménez-Casas et al. [10]), and even in some cases with a viscoelastic binary fluid (Yasappan and Jiménez-Casas et al. [11] and Jiménez-Casas and Castro [9]).

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