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Simulation of Multidimensional Nonlinear Dynamics by One-Dimensional Maps with Many Parameters

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Abstract. We propose a concrete class of discrete dynamical systems as nonlinear matrix models to describe the multidimensional multiparameter nonlinear dynamics. In this article we simulate the system asymptotic behavior. A two-step algorithm for the computation of ω-limit sets of the dynamical systems is presented. In accordance with the qualitative theory which we develop for this class of systems, we allocate invariant subspaces of the system matrix containing cycles of rays on which ω-limit sets of the dynamical systems are situated and introduce the dynamical parameters by which the system behavior is described in the invariant subspaces. As the first step of the algorithm, a cycle of rays which contains the ω-limit set of the system trajectory, is allocated using system matrix. As the second step, the ω-limit set of the system trajectory is computed using the analytical form of one-dimensional nonlinear Poincare map dependent on the dynamical parameters. The proposed algorithm simplifies calculations of ω-limit sets and therefore reduces computing time. A graphic visualization of ω- limit sets of n- dimensional dynamical systems, $n > 3$ is shown.
Keywords: Computer simulation, Nonlinear dynamics, Discrete dynamical systems, Dynamical parameters.

1 Introduction

To understand and analyse nonlinear multidimensional dynamics simple one-dimensional semi-dynamical systems with complicated dynamics and fairly complete qualitative description are used. These are, first of all, one-dimensional discrete dynamical systems, i.e. iterations of real one-dimensional maps. The first systematic results on one-dimensional discrete dynamical systems appeared in the early 60's and are linked to A.N. Sharkovskii [1]. Many properties of the dynamical systems are the direct result of the theories developed by A.N. Sharkovskii [2] and M. Feigenbaum [3]. A representative of this class of systems is the dynamical system generated by the one-dimensional logistic map

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[4]. It was the first example of a complicated, chaotic behaviour of the system given by a simple nonlinear equation. Even though the properties of the one-dimensional logistic map are well studied, researchers continue referring to it as standard to check the many nonlinear phenomena [5]–[7]. However, up until now there is no well-developed qualitative theory available, which could be successfully applied in order to conduct a complete study of the multidimensional dynamical systems dependent on parameters. Therefore, it is appropriate to select concrete classes of the dynamical systems and to develop qualitative theories so as to be able to describe the properties and movements of the systems within these theories.

We focus our research on a concrete class of dynamical systems which represent a variant of generalization of one-dimensional discrete dynamical systems to the multidimensional multiparameter case. The systems are generated by a map in the form of the product of scalar and vector linear functions on compact sets of the real vector space. We propose the systems as nonlinear matrix models with limiting factors to describe the macro system dynamics, for example the dynamics of many group biological population in the presence of limited resources. In these models the scalar function plays a role of a limiting factor.

In recent years, the methods of computer simulation have become an essential tool in the study of the dynamical systems [8]–[10]. The modern computer capabilities make it possible to include in the system complicated nonlinear relationships between its variables and a large number of parameters. The presence of nonlinear relationships and multiparameter dependence reproduces in the model the phenomena which can be observed in actual experiments and which cannot be produced by splitting the system into separate components or reducing the number of parameters or variables. Thus, the improvement of current methods and the development of new ones for the dynamical system research are necessary and relevant [11,12]. In this case the quantitative research provides a theoretical basis for the algorithm constructions, and hence is particularly important.

We develop a qualitative theory for the class of the dynamical systems considered (see e.g. [13] and references there). The systems possess the obvious properties which are determined by the linear vector function (the system matix) and which do not depend on the scalar function. In particular, in vector space we allocate invariant subspaces containing cycles of rays of the system matrix, on which ω-limit sets of dynamical systems are situated. On the other hand, the complicated nonlinear dynamics of the systems can occur due to the scalar function. We study the system dynamics in the invariant subspaces containing cycles of rays using one-dimensional nonlinear Poincare maps and introduce the dynamical parameters by which the system behavior is described in the invariant subspaces. In this article we show the results of the simulation of the system asymptotic behavior and present an algorithm for the computation of ω-limit sets of the class of the dynamical systems considered. The algorithm consists of two steps of calculations in accordance with the qualitative theory. As the first step, a cycle of rays which contains the ω-limit set of the system trajectory is allocated using system matrix. The period of the cycle of rays, the number and values of the dynamical parameters by which the system dynamics is described on the cycle of rays, are calculated as well. As the second step, the ω-limit set of the system trajectory is computed using one-dimensional nonlinear Poincare map dependent on the dynamical parameters. As a rule, these parameters differ from the system parameters and are unknown or not directly defined or computable [14]. The novelty of our research lies in the determination of the dynamical parameters and in the analytical form of one-dimensional nonlinear Poincare maps dependent on the dynamical parameters. We shall see below that the number of the dynamical parameters cannot be reduced without the loss of accuracy of the system behavior description, even when this number is greater than the number of the system parameters, i.e. entries of the system matrix.

2 Class of the dynamical systems

Let F be a map of the form [13]

$$F : \mathbb{R}^n \to \mathbb{R}^n, \ \ Fy = \Phi(y)Ay \tag{1}$$

where R^n is n- dimensional real vector-space, $\Phi(y)$ is a scalar function, A is a linear operator (a matrix of n-th order). Allocate set $\mathrm{X} \subseteq \mathbb{R}^n$ invariant under F i.e., $F : \mathrm{X} \to \mathrm{X}$. Map F in general is non invertible and generates in X a cyclic semi-group of maps $\{F^m\}$, $m \in Z_+$, which is called the dynamical system and is denoted by $\{F^m, \mathrm{X}, Z_+\}$. Set X is called phase space of the dynamical systems and specifies a set of valid states of the dynamical system, $Z_+ = \mathbb{N} \bigcup \{0\}$ is the set of nonnegative integers. Set $\{F^m y\}$ where y is fixed and m runs over Z_+, is called a trajectory of the point y. The dynamics of the system $\{F^m, \mathrm{X}, Z_+\}$ is understood as the process of transition from one state to another.

The dynamics of the system $\{F^m, \mathrm{X}, Z_+\}$ generally varies for different $\Phi(y)$. So, the systems $\{F^m, \mathrm{X}, Z_+\}$ are different too. But the systems possess similar properties which are determined by the linear operator A and do not depend on the function $\Phi(y)$. Therefore, the systems $\{F^m, \mathrm{X}, Z_+\}$ form one class of the dynamical systems.The elements of this class are, in particular, linear dynamical systems with $\Phi(y) = const$ and the dynamical system $\{f^m, \mathrm{X}, Z_+\}$ generated by the map f of the form [15]

$$f : \mathbb{R}^n \to \mathbb{R}^n, \ \ fy = (1 - \|y\|)Ay. \tag{2}$$

Here $\| \cdot \|$ is a vector norm in $\mathbb{R}^n$. If $n = 1$ then $A = \mu$ and we arrive at the well-known logistic map mentioned above

$$\psi_\mu : \mathbb{R}^1 \to \mathbb{R}^1, \quad \psi_\mu x = \mu(1 - x)x. \tag{3}$$

3 Mathematical models with limiting factors

We propose the class of the dynamical systems $\{F^m, \mathrm{X}, Z_+\}$ as mathematical models for describing the dynamics of model and real macro systems in the presence of limiting factors.

Let
n be a number of macro system's components,
$y \in \mathrm{X}$ be a vector of components' characteristics,
A be a matrix of components' interrelations and
$\Phi(y)$ be a limiting function (limiting factor).

Let X be a compact of the form

$$\mathrm{X} = \{y \in \mathbb{R}^n \mid y \geq 0, \ \ \|y\| \leq a\}, \quad a < \infty. \tag{4}$$

Here $y = (y_1, \ldots, y_n)' \geq 0$ means $y_i \geq 0$, $i = \overline{1,n}$ and is called a nonnegative vector. Note that X is invariant under F i.e., $F : \mathrm{X} \to \mathrm{X}$ if and only if [13]
1) $\Phi(y) \geq 0$ is continuous function on X,
2) $A = (a_{ij}) \geq 0$ ($a_{ij} \geq 0$, $i,j = \overline{1,n}$),
3) $\|A\| \leq aC^{-1}$ where $C = \max\limits_{y \in \mathrm{X}} \Phi(y)\|y\|$ and $\|A\|$ is a subordinate matrix norm for a matrix A based on the vector norm in $\mathbb{R}^n$.

Then the dynamical system $\{F^m, \mathrm{X}, Z_+\}$ describes the macro system's state changes over time m. For any nontrivial $\{F^m y\}$ we introduce a unit vector

$$e_m(y) = \|F^m y\|^{-1} F^m y \tag{5}$$

which is called a macro system structure and defines the ratio between components' characteristics at the time m. The state of macro system governed by the dynamical system $\{F^m, \mathrm{X}, Z_+\}$ (at the time m) we characterize by

$$S^m(y) = \{F^m y, e_m(y)\}. \tag{6}$$

The limiting factor concept was first coined in biology by Libig J. and generally, means a factor that restricts or constrains the dynamics of the system, process or phenomena. By using limiting factors, the state of the system is regulated.

On one hand, models given by the systems $\{F^m, \mathrm{X}, Z_+\}$ generalize in n- dimensional case many nonlinear one- and two-dimensional models widely used in practice. In particular, for describing the dynamics of n- group biological population with discrete generations in the presence of limited resources we propose the dynamical system generated by the map f of the form (2). In this representation y is a vector of densities of population age groups so, $\|y\| \leq 1$. If $n = 1$ then y is the total population density, $A \equiv \mu$ is the reproductive coefficient. The dynamical system $\{\psi_\mu^m, \mathrm{I}, Z_+\}$ in the interval $\mathrm{I}_1 = [0,1]$ describes a mechanism of self-regulation of one-species biological population with limited resources [2].

On the other hand, models given by the systems $\{F^m, \mathrm{X}, Z_+\}$ generalize many matrix models, in particular, Leslie models both linear and nonlinear [16,17]. The last ones contain matrices A of the special form (Leslie matrix and its generalizations) and concrete limiting functions $\Phi(y)$.

4 Qualitative theory

We develop a qualitative theory for the class of the dynamical systems $\{F^m, \mathrm{X}, Z_+\}$ and apply the results of the theory in computer simulation of their dynamics.

Denote by $\omega_F y$ ω- limit set of the trajectory $\{F^m y\}$ (the set which attracts $\{F^m y\}$ when $m \to +\infty$). A ray passing through $y \in \mathbb{R}^n$, $y \neq 0$ is the set $\mathrm{cone}(y) = \{\alpha y \mid \alpha \geq 0\}$. By a system of p elements we mean a sequence of these elements, $p \in \mathbb{N}$. Then the system of distinct rays $l_1, ..., l_p$ is called a cycle of rays of a linear operator A of period $p \in \mathbb{N}$ and is denoted by $\mathrm{L}_p = (l_1, ..., l_p)$ if

$$Al_k = l_{k+1}, \quad k = 1, \dots, p-1, \quad Al_p = l_1.$$

As easy to see, that invariant sets of the system $\{F^m, \mathrm{X}, Z_+\}$ are contained in invariant subspaces of A. Denote $\ker A = \{y \in \mathbb{R}^n \mid Ay = 0\}$ and let

$$\mathcal{P}(A; p, \mu) = A^p - \mu^p E, \quad \mu \in \mathbb{C}.$$

We call the intersection $l \cap \mathrm{X}$ as a segment of ray l (ray segment). Denote by ϕ_μ map F when $n = 1$,

$$\phi_\mu x = \mu \Phi(x) x \tag{7}$$

where $x \in \mathrm{I_a} = [0, a]$. According to the qualitative theory there exist p, $q \in \mathbb{N}$, $\mu \in \sigma(A)$ such that any nontrivial ($\neq \{0\}$) $\omega_F y$ is located in some invariant subspace

$$\ker \mathcal{P}(A; p, \mu), \quad \mu^p > 0,$$

on a cycle of rays L_q where $\sigma(A)$ is a spectrum of A and q is a divisor of p, $1 \leq q \leq p$ [18]. More precisely, $\omega_F y \subseteq \mathrm{J_q} = \mathrm{L}_q \cap \mathrm{X} \subset \ker \mathcal{P}(A; p, \mu) \cap \mathrm{X}$ and $\mathrm{J_q}$ consists of q ray segments invariant under F^q. Without losing generality we agree $q = p$ and $\omega_F y \subseteq \mathrm{J}_p$. Then for the map F with $\Phi(\|y\|)$ map F^p represents in $\mathrm{J_p}$ as a superposition

$$F^p = \phi_{\mu_p} \circ \phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1} \tag{8}$$

with some numbers $\mu_1 > 0, \ldots, \mu_p > 0$.

If to consider the system $\{F^m, \ker \mathcal{P}(A; p, \mu) \cap \mathrm{X}, Z_+\}$, then $\mu_1, \ldots, \mu_p$ turns into parameters. We call them dynamical parameters in contrast to the system parameters i.e. entries of the matrix A. Thus, in the whole X, the system dynamics is defined by the trajectory behavior in the sets $\ker \mathcal{P}(A; p, \mu) \cap \mathrm{X}$. So, by the parameters $\mu_1, \ldots, \mu_p$ the system dynamics is described in the whole X. Every ray segment of $\mathrm{J_p}$ is the one-dimensional Poincare section for the trajectories located in $\mathrm{J_p}$ and F^p is the one-dimensional first return (Poincare) map for the map F in each ray segment of $\mathrm{J_p}$. For the special form $\Phi(\|y\|)$ map F^p has analytical representation (8).

Denote by $e_1, \ldots, e_p$ the unit vectors directed along the ray segments of $\mathrm{J_p}$. We define $e_1, \ldots, e_p$ and $\mu_1, \ldots, \mu_p$ by the recurrent formulas. Let $p = 1$. Then $e_1 \geq 0$ is an eigen vector of the matrix $A \geq 0$ and there exists an eigen value $\mu > 0$ such as $Ae_1 = \mu e_1$. So, (8) takes the form

$$F = \phi_\mu. \tag{9}$$

Let $p > 1$ then $e_1 \geq 0$ is not an eigen vector of A, $\|e_1\| = 1$ and $e_2, \ldots, e_p$ are defined by the sequence

$$e_j = \|Ae_{j-1}\|^{-1} Ae_{j-1}, \quad j = \overline{2, p}. \tag{10}$$

Denote

$$\mu_j = \|Ae_j\|, \quad j = \overline{1,p}. \tag{11}$$

For the map F with different $\Phi(\|y\|)$, parameters $\mu_1, \ldots, \mu_p$ and vectors e_1, $\ldots, e_p$ are the same and their computation by the formulas (10)-(11) does not cause difficulties.

It should be noted that the dynamical parameters, their number and values depend on the location of the sets $\ker \mathcal{P}(A; p, \mu) \cap \mathrm{X}$ in X and J_p in $\ker \mathcal{P}(A; p, \mu) \cap \mathrm{X}$ and vary, as a rule, at the fixed entries of the matrix A. So, the dynamical parameters differ from the system parameters and identify the regions with different dynamics. Their number is less than or equal to p and may be very large, in particular, when $p > n^2$ at $n \geq 19$ [18]. According to (8) all parameters are involved in the representation of the map F^p so, their number cannot be reduced.

5 Computer simulation

We present computer simulation of multidimensional dynamics by the numerical realization of the models for the dynamics of biological population governed by the system $\{f^m, \mathrm{X}, Z_+\}$.

In the population model:
f is a map of the form (2): $fy = (1 - \|y\|)Ay$,
n is a number of population age groups,
y is a vector of densities of the age groups, $y \in \mathrm{X}$,
X is of the form (4) if $a = 1$ i.e.,

$$\mathrm{X} = \{y \in \mathbb{R}^n \mid y \geq 0, \ \|y\| \leq 1\},$$

A is a matrix of intergroup relations,
$\Phi(\|y\|) = 1 - \|y\|$ is a population size limiting function corresponding to the assumption of limited resources or available living space.

Let $\|y\| = \sum\limits_1^n y_i$ then the condition 3) for the invariance of X is as follows: $\|A\| = \max\limits_j \sum\limits_{i=1}^n a_{ij} \leq 4$. For any nontrivial $\{f^m y\}$ a unit vector $e_m(y) = \|f^m y\|^{-1} f^m y$ is an age structure of many-group population and defines the ratio between densities of age groups in total population density (at the time m). The state of the population governed by the dynamical system $\{f^m, \mathrm{X}, Z_+\}$ (at the time m) is $S^m(y) = \{f^m y, e_m(y)\}$.

According to the section 4, for any nonzero initial state $S^0(y)$, the structure of many-group population is asymptotically stabilized as p- periodic and is characterized by p vectors $e_1, \ldots, e_p$ defined by (10).

As to the population dynamics, we get that the many-group population model given by the dynamical system $\{f^m, \mathrm{X}, Z_+\}$ asymptotically has the same behavior as a family of one-species population models given by the one-dimensional systems $\{(\psi_{\mu_p} \circ \psi_{\mu_{p-1}} \circ \ldots \circ \psi_{\mu_1})^m, \mathrm{I}, Z_+\}$ where $\psi_{\mu_p} \circ \psi_{\mu_{p-1}} \circ \ldots \circ \psi_{\mu_1}$ is a superposition (8) with the map ψ_μ of the form (3) and $\mu_1, \ldots, \mu_p$ defined by (11).

Therefore, the population governed by the system $\{f^m, \mathrm{X}, Z_+\}$, has stabilized p- periodic structure at its final state, $p < \infty$ and densities of its age groups that change periodically or not. The same asymptotic behavior has the macro system governed by the system $\{F^m, \mathrm{X}, Z_+\}$ i.e., exactly p- periodic structure, $p < \infty$ and periodic or nonperiodic changes of its components' characteristics.

6 Method of one-dimensional superpositions

For correct determining cyclic ω-limit sets of large periods or chaotic ω-limit sets of the system $\{F^m, \mathrm{X}, Z_+\}$, we propose a computer method which we call as a method of one-dimensional superpositions. Let F be the map with a function $\Phi(\|y\|)$. The method implies calculations in two steps.

As the first step, a stable set J_p is determined for any nonzero $y \in \mathrm{X}$ using n- dimensional linear dynamical system $\{A^m, \mathrm{R}^n, Z_+\}$. At this step, period p is obtained and the unit vectors $\{e_1, \ldots, e_p\}$ along the rays of a cycle of rays L_p in which $\omega_F y$ is located, are computed by the matrix A. The number $1 \leq t \leq p$ and values of the dynamical parameters by which the trajectory dynamics in J_p is described, are computed as well. Here t is a divisor of p.

As the second step, set $\omega_F y$ is determined using the one-dimensional dynamical system $\{(\phi_{\mu_p} \circ \phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1})^m, \mathrm{I}, Z_+\}$. At this step, the norm x of the projection of the vector y in the set J_p is obtained and a one-dimensional ω-limit set of the trajectory $\{(\phi_{\mu_p} \circ \phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1})^m x\}$ is computed by the one-dimensional nonlinear Poincare map F^p with t parameters $\mu_1 > 0, \ldots, \mu_t > 0$. The points of this ω- limit set are coordinates of vectors which compose the part of $\omega_F y$ along the vector e_1.

The parts of $\omega_F y$ along the other vectors $e_2, \ldots, e_p$ are of the same type and structure and the vectors which compose these parts, are computed as well.

The method proposed simplifies calculations for large n and p for instance, $p > n^2$, $p > n^3$ and so on. Indeed, at first we detect the stable cyclic set J_p and later on we describe the trajectory dynamics in it. Using the method we compute any nontrivial set $\omega_F y$, in particular, we obtain the final state of many-group population for any nonzero initial state $S^0(y)$. The method also provides graphic visualization of ω- limit sets of n- dimensional dynamical systems $\{F^m, \mathrm{X}, Z_+\}$ at $n > 3$ and for large p.

The calculation algorithm for the computation of the set $\omega_F y$ and the final macro system state by the method of one-dimensional superpositions is as follows:

1. enter initial vector $y \geq 0$ and matrix $A \geq 0$ ($\|y\| < 1$, $\|A\| \leq 4$);

1'. calculate eigen values and eigen vectors of matrix A;

2. calculate period p, vectors $e_1, \ldots, e_p$ of the set J_p and t distinct parameters $\mu_1, \ldots, \mu_t$ of the set $\{\mu_1, \ldots, \mu_p\}$ using (10), (11);

3. determine projection y' of vector y in the set $\mathrm{J_p}$ and calculate its norm $x = \|y'\|$;

4. obtain ω- limit set of the trajectory $\{\phi_{\mu_p} \circ \phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1})^m x\}$ as some trajectory i.e.,

$$\omega_{\phi_{\mu_p} \circ \phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1}} x = \{x_i^*\}_{i \geq 0}$$

where $x_i^* = (\phi_{\mu_p} \circ \phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1})^i x^*$. After these calculations the iteration process stops;
5. The set $\omega_F y$ is a result of calculations done in step 4 and is the following set

$$\omega_F y = \{x^* e_1,\, (\phi_{\mu_1} x^*) e_2,\, (\phi_{\mu_2} \circ \phi_{\mu_1} x^*) e_3,\, (\phi_{\mu_3} \circ \phi_{\mu_2} \circ \phi_{\mu_1} x^*) e_4,\, \ldots,$$

$$(\phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1} x^*) e_p,\, x_1^* e_1,\, \ldots\}.$$

So, vectors of $\omega_F y$ are of the form ue_i where $u \in \omega_{\phi_{\mu_p} \circ \phi_{\mu_{p-1}} \circ \ldots \circ \phi_{\mu_1}} x$, $i = \overline{1,p}$.

The final macro system state is a pair

$$S^*(y) = \{\omega_f y, E\} \quad \text{where } E = \{e_1, \ldots, e_p\}.$$

7 Examples

7.1 The dynamics of the Northern Spotted Owl

The algorithm for computing of the population dynamics is implemented in Matlab as a function with the following input data: n- dimensional initial vector $y \geq 0$ and matrix $A \geq 0$ of n order. The final state of the population is given as an output data in the form of two arrays of vectors.

Let us demonstrate this algorithm by simulating the dynamics of the Northern Spotted Owl. As an input data we use the real (3×3)- matrix A from article of Lamberson R., McKelvey R., at al. [19]. We would like to take into account limited resources for the population. For this purpose, in contrast to the linear model considered in [19], we propose nonlinear models given by the dynamical system $\{f^m, \mathrm{X}, Z_+\}$. In the models a proportional coefficient c is introduced as an input data to make the dynamics nontrivial.

Example 1. 1. enter a) $y = (0.1, 0.1, 0.1)'$,

b) $A = c \cdot \begin{pmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{pmatrix}$.

The elements in the top row of matrix A are fertility rates; the sub-diagonal elements are survival rates; nonzero diagonal element a_{ii} is the probability that females in stage i remain in the same stage next year;

c) $c = 3.1$ (almost maximum available value of c to fulfil $\|A\| \leq 4$);

2. vectors e_j of the set E with accuracy $\epsilon = 10^{-5}$ and parameters μ_j, $j = \overline{1,p}$, are computed. As a result, after 8 iterations, a convergence of the sequence of vectors $e_i(y)$ to E is obtained. The ultimate result is $p = 1$, $(n \times p)$- array $E = \{e\}$ and array $U = \{\mu\}$ where $e = (0.2402, 0.0440, 0.7159)'$, $\mu = 3.0491$;

3. given $x = 0.8$;

4. given accuracy $\epsilon = 10^{-5}$ for the trajectory $\{\psi_\mu^m x\}$ obtain $\omega_{\psi_\mu} x$ as a cycle of period 2 per 56 iterations,

$$\omega_{\psi_\mu} x = \{x^*, \psi_\mu x^*\} = \{0.5909, 0.7371\};$$

5. for the trajectory $\{f^m y\}$

$$\omega_f y = \{0.5909e, 0.7371e\} =$$

$$= \{(0.1419, 0.0260, 0.4230)', (0.1770, 0.0324, 0.5277)'\}.$$

The final population state is

$$S^*(y) = \{\omega_f y, E\} \quad \text{where } E = \{e\}$$

and is shown in Figure 1a.

Example 2. 1. enter a) $y = (0.1, 0.1, 0.1)'$,

b) $A = c \cdot \begin{pmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0 \end{pmatrix}$. In this model we suppose that there are no females remaining in the same stage next year;

c) $c = 5$ (almost maximum available value of c to fulfil $\|A\| \leq 4$);

2. vectors e_j with accuracy $\epsilon = 10^{-5}$ and parameters μ_j, $j = \overline{1,p}$, are computed. As a result, after 3 iterations, a convergence of sequence of vectors e_j to set E is obtained. The ultimate result is $p = 3$, $(n \times p)$- array $E = \{e_1, e_2, e_3\}$ and array $U = \{\mu_1, \mu_2, \mu_3\}$ where $e_1 = (0.3333, 0.3333, 0.3333)'$, $e_2 = (0.2705, 0.1475, 0.5820)'$, $e_3 = (0.5559, 0.1409, 0.3032)'$, $\mu_1 = 2.0333$, $\mu_2 = 1.7275$, $\mu_3 = 1.5009$;

3. as $A^3 = \lambda^3 I$ then y, $\{f^m y\}$ and $\omega_f y$ are located in the same set J_3. Here $\lambda = 1.7404$ is the maximum eigenvalue of $A \geq 0$ and I is identity matrix. So, calculate $x = \sum_1^3 y_i = 0.3$;

4. given accuracy $\epsilon = 10^{-5}$ for the trajectory $\{(\psi_{\mu_3} \circ \psi_{\mu_2} \circ \psi_{\mu_1})^m x\}$ obtain $\omega_{\psi_{\mu_3} \circ \psi_{\mu_2} \circ \psi_{\mu_1}} x$ as a fixed point per 8 iterations,

$$\omega_{\psi_{\mu_3} \circ \psi_{\mu_2} \circ \psi_{\mu_1}} x = \{x^*\} \quad \text{where } x^* = 0.368;$$

5. for the trajectory $\{f^m y\}$

$$\omega_f y = \{x^* e_1,\ (\psi_{\mu_1} x^*) e_2,\ (\psi_{\mu_2} \circ \psi_{\mu_1} x^*) e_3\} = \{(0.1227; 0.1227; 0.1227)',$$

$$(0.1279; 0.0698; 0.2752)', (0.2394; 0.0607; 0.1306)'\}.$$

The final population state is

$$S^*(y) = \{\omega_f y, E\} \quad \text{where } E = \{e_1, e_2, e_3\}$$

and is shown in Figure 1b.

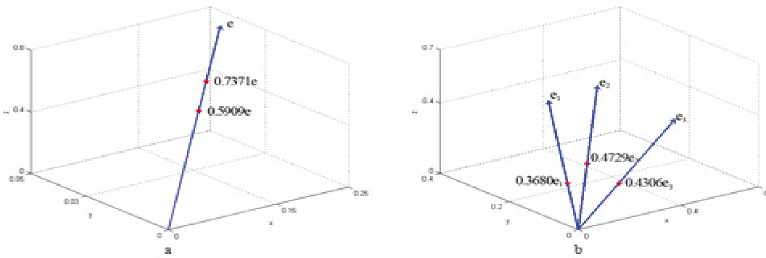

Fig. 1. Final states $S^*(y)$ of three-group populations with two different matrices $A \geq 0$ and the same initial vectors y

7.2 The dynamics of macro system composed of a large number of components

In the next two examples we demonstrate the advantages of the method of one-dimensional superposition in graphic visualization of the final macro system state at $n > 3$ and $p > n$. We briefly summarize the results obtained by the method. Assume that the macro system dynamics is described by the dynamical system $\{f^m, \mathrm{X}, Z_+\}$.

Example 3. Let $n = 10$ and A be (10×10)- matrix of a quasidiagonal form $\{A_1, A_2, A_3\}$ with matrices A_j on the main diagonal,

$$A_1 = \begin{pmatrix} 0 & 3.2 \\ 3.2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2.56 & 0 \\ 0 & 0 & 3.2 \\ 4 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 2.56 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3.2 \\ 2.56 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(Matrix A is not a real matrix of the subsystems' relations, just some model matrix). Matrix A has 10 eigenvalues in modulus 3.2.

Enter $y = (0, 0.1, 0.1, 0.1, 0.1, 0.05, 0.05, 0, 0.3, 0.05)'$ and A as an input data.

As an ultimate result we get $p = 15$ and (10×15)- array E consisting of 15 vectors.

Given accuracy $\epsilon = 10^{-10}$ for the trajectory $\{(\psi_{\mu_{15}} \circ \psi_{\mu_{14}} \circ \ldots \circ \psi_{\mu_1})^m x\}$ its ω- limit set is a cycle of period 4.

For the trajectory $\{f^m y\}$ its ω- limit set $\omega_f y$ is a cycle of period $60 = 4 \cdot 15$.

The final macro system state is $S^*(y) = \{\omega_f y, E\}$.

We present graphic visualization of the part of $\omega_f y$ located in $\mathrm{J_p}$ along the vector e_1 i.e., four vectors with coordinates x^*, $(\psi_{\mu_p} \circ \ldots \circ \psi_{\mu_1})x^*$, $(\psi_{\mu_p} \circ \ldots \circ \psi_{\mu_1})^2 x^*$, $(\psi_{\mu_p} \circ \ldots \circ \psi_{\mu_1})^3 x^*$. In XY coordinate system four vectors with the same coordinates along the unit vector of the bisector of the first coordinate angle are drawn and their graphic image is shown in Figure 2a. The parts of $\omega_f y$ located in $\mathrm{J_p}$ along the vectors $e_2, \ldots, e_p$, are of the same type i.e., each of them consists of four vectors.

Example 4. Change the initial vector to $y = (0.1, 0.1, 0, 0.3, 0.1, 0, 0, 0, 0.3, 0.05)'$.

The ultimate result is $p = 30$ (the maximum possible value at $n = 10$), (10×30)- array E now consists of 30 vectors.

Given accuracy $\epsilon = 10^{-10}$ for the trajectory $\{(\psi_{\mu_{30}} \circ \psi_{\mu_{29}} \circ \ldots \circ \psi_{\mu_1})^m x\}$ we get non-stop iterative process when calculating its ω- limit set. It means that ω- limit set is irregular or a cycle of a very large period. In this case we agreed to accept the last 200 iterations when calculating the trajectory $(\psi_{\mu_{30}} \circ \psi_{\mu_{29}} \circ \ldots \circ \psi_{\mu_1})^m x$ as its ω- limit set.

Graphic visualization of 200 vectors which are the part of $\omega_f y$ located in $\mathrm{J_p}$ along the vector e_1, is presented in XY coordinate system by 200 vectors with the same coordinates, along the unit vector of the bisector of the first coordinate angle. The graphic image of these vectors is shown in Figure 2b.

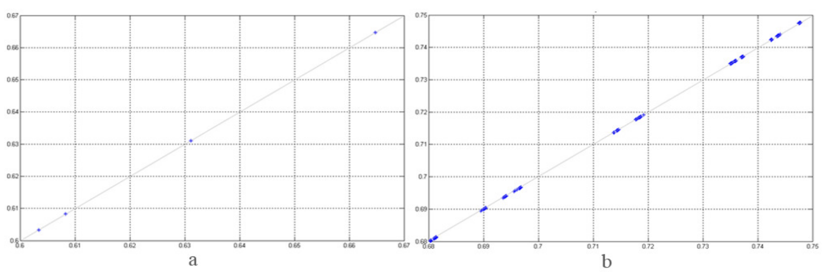

Fig. 2. Graphic visualization of the parts of $\omega_f y$ located along the vectors e_1 for two different initial vectors y

7.3 Outcomes of examples

The examples 1-2 show that in the first model the population structure is asymptotically stabilized and does not vary any more and population size, as well as age group sizes change periodically every two years. In the second model the structure of the population is stabilized and varies every three years along with the population size and age group sizes. Examining the dynamics of the population, one can see the mechanism of regulation or harvestable surplus of the population size without affecting long term stability, or average population size. Indeed, according to the second model all individuals of the third stage may be taken away after the childbearing period ($a_{33} = 0$) every year. In spite of the structure of the population, its size and age group sizes vary periodically, in this case the population remains persistent.

Stabilized periodic structure of macro system is determined by its initial structure and not its initial size. Indeed, by iterating the map F of the form (1) m times, we write out $F^m y = \Phi^{(m)}(y) A^m y$ where

$$\Phi^{(m)}(y) = \prod_{i=0}^{m-1} \Phi(F^i y),$$

$y \in$ X. Hence it follows that the directions of $F^m y$ and $A^m y$ coincide, $m = 1, 2, \ldots$ i.e., the directions of nonzero vectors of the trajectory $\{F^m y\}$ as $m \to \infty$ are defined by the linear part of the map F and are independent of the form of $\Phi(y)$.

Let all nonzero entries of the matrix A be equal to 3.2 in the examples 3-4. Then $\mu_1 = \ldots = \mu_p = \mu = 3.2$, $\psi_{\mu_p} \circ \ldots \circ \psi_{\mu_1} = \psi^p_\mu$. and $\omega_{\psi^p_\mu} x$ is a cycle of period 2 for any $x \in$ I [2, p. 26]. According to [20] there are more than one periodic attractors and therefore more than one different dynamics of the map $\psi_{\mu_p} \circ \ldots \circ \psi_{\mu_1}$, $p \geq 1$ at the fixed parameter values. So, if there is only one asymptotic regime of the map $\psi_{\mu_p} \circ \ldots \circ \psi_{\mu_1}$ in the interval I at the fixed $\mu_1, \ldots, \mu_p$, then macro systems with the same initial structure will have the same final state. If there are more than one asymptotic regimes of the map $\psi_{\mu_p} \circ \ldots \circ \psi_{\mu_1}$ in the interval I at the fixed $\mu_1, \ldots, \mu_p$, then macro systems with the same initial structure will have the same stabilized periodic structure and may have different sizes changed periodically (or not).

8 Conclusion

In this article we describe an approach we have developed to study multiparameter nonlinear dynamics. The advantages of applying the results of the qualitative theory and using the method of one-dimensional superpositions in a simulation of the dynamics are as follows:

1. The dynamical systems considered are nonlinear and thus very sensitive to the data entry errors. The proposed method simplifies computations of the ω- limit set of the system trajectory. Firstly, a stable cycle of rays of period p, which contains the ω- limit set, is identified using the system matrix of the n-th order. Secondly, the ω- limit set is obtained using the non-linear one-dimensional map. As a result, this leads to markedly reduced computing times, especially when the order n and the periods p are large.

2. In an n- dimensional case, $n > 3$ it is impossible to obtain a graphic image of ω- limit sets of the dynamical system, e.g. to realize their types. However, we can get graphic visualization of the part of ω- limit sets consisting of vectors along the first unit vector of the stable invariant set containing the ω- limit set. In an XY coordinate system, vectors along the unit vector of the bisector of the first coordinate angle which have the same coordinates can be easily plotted.

3. Theoretical results of the qualitative theory help us to correctly interpret the numerical results as well as to conduct an accurate computer simulation of the system dynamics. We specify the number of iterations to detect a stable cycle of rays containing the ω- limit set of the system as well as the number of iterations to compute the ω- limit set.

4. The determination of the dynamical parameters and the calculation of their number and values by the formulas provides the description of the system dynamics in stable cycles of rays containing ω- limit sets of the system and therefore, the identification of the regions with different dynamics. Their number may be very large, e.g. greater than the number of the system parameters.

However, one can see that this number cannot be reduced without the loss of accuracy of the system behavior description.

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