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Chaos at Cross-waves in Fluid Free Surface

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Abstract: The phenomenon of chaotic cross-waves generation in fluid free surface in two finite size containers is studied. The waves may be excited by harmonic axisymmetric deformations of the inner shell in the volume between two cylinders and in a rectangular tank when one wall is a flap wavemaker. Experimental observations have revealed that waves are excited in two different resonance regimes. The first type of waves corresponds to forced resonance, in which axisymmetric patterns are realized with eigenfrequencies equal to the frequency of excitation. The second kind of waves is parametric resonance waves and in this case the waves are "transverse", with their crests and troughs aligned perpendicular to the vibrating wall. These so-called cross-waves have frequencies equal to half of that of the wavemaker. The existence of chaotic attractors was established for the dynamical system presenting cross-waves and forced waves interaction at fluid free-surface in a volume between two cylinders of finite length. In the case of one cross-wave in a rectangular tank no chaotic regimes were found.
Keywords: Cross-waves, Wavemaker, Fluid free sureface, Averaged systems, Parametric resonance, Chaotic simulation.

1 Introduction

The phenomenon of cross-waves generation in free-surface waves of a fluid confined in a rectangular tank with the finite depth and one wall as a flap wavemaker is rather known, Faraday, 1831, [3]. The waves may be excited by harmonic oscillations of wavemaker and depending on the vibration frequency both axisymmetric and non-symmetric wave patterns may arise. Experimental observations have revealed that waves are excited in two different resonance regimes. The first type of waves corresponds to forced resonance, in which axisymmetric patterns are realized with eigenfrequencies equal to the frequency of excitation. The second kind of waves is parametric resonance waves and in this case the waves are "transverse", with their crests and troughs aligned perpendicular to the vibrating wall. These so-called cross-waves have frequencies equal to half of that of the wavemaker, Faraday, 1831, [3]. To obtain a lucid picture of energy transmission from the wavemaker motion to the fluid free-surface motion the method of superposition, Lamé, 1852, [8], has been used. This method allows to construct a simple mathematical model, which shows how the cross-waves can be generated directly by the wavemaker. All previous theories have considered cross-waves problem applying the

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Havelock's, 1929, [2], solution of the wavemaker problem for a semi-infinite tank with an infinite depth and a radiation condition instead of zero velocity condition at the finite bottom.
As the second task the phenomenon of deterioration of fluid free-surface waves between two cylindrical shells when the inner wall vibrates radially is considered in the present paper.

2 Approximation of Cross-waves in Rectangular Container

Let us theoretically consider the nonlinear problems of fluid free-surface waves which are excited by a flap wavemaker at one wall of rectangular tank of a finite length and depth. From the experimental observations, Krasnopolskaya, 2013, [6], we may conclude that the pattern formation has a resonance character, every pattern having its "own" frequency. Assuming that the fluid is inviscid and incompressible, and that the induced motion is irrotational, the velocity field can be written as $\mathbf{v} = \nabla\varphi$. Let us consider that patterns can be described in terms of normal modes with characteristic eigenfrequencies, we approximate free surface displacement waves, when the excitation frequency ω is twice as large as one of the eigenfrequencies, i.e. $\omega \approx 2\omega_{nm}$, and also is close to other eigenfrequency $\omega \approx \omega_{l0}$, as a function written in the form

$$\xi \approx \xi_{nm}(t)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{b} + \xi_{lo}(t)\cos\frac{l\pi x}{L} + \xi_{00}. \tag{2.1}$$

When $\xi_{nm}(t) = O(\varepsilon_1^{3/2})$, $\xi_{lo} = O(\varepsilon_1)$, $\xi_{lo} = O(\varepsilon_1)$ and $\varepsilon_1 = \dfrac{a\omega_{nm}^2}{g}$

Where a is an amplitude of wavemaker oscillations, L is the length, b is the width and h is the depth of the fluid container. Then a potential of fluid velocity $\varphi = \varphi_1 + \varphi_2 + \varphi_0$ as the solution of the harmonic equation and according to [5] has following components

$$\varphi_1 = \varphi_{nm}(t)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{b}\frac{\mathrm{ch}[k_{nm}(z+h)]}{\mathrm{ch}(k_{nm}h)} + \varphi_{lo}\cos\frac{l\pi x}{L}\frac{\mathrm{ch}[k_{lo}(z+h)]}{\mathrm{ch}(k_{lo}h)}.$$

$$\varphi_2 = -\varepsilon_1\cos\omega t\sum_{j=1}^{\infty}\frac{4gh[1-(-1)^j]}{\omega_{nm}\alpha_{0j}(\mathrm{th}\,\alpha_{0j}L)(j\pi)^2}\cos\frac{j\pi z}{h}\frac{\mathrm{ch}\,\alpha_{0j}(x-L)}{\mathrm{ch}\,\alpha_{0j}} -$$

$$-\varepsilon_1\dot{\xi}_{nm}\sin\omega t\frac{g\left[\dfrac{3}{2}\left(\dfrac{n\pi}{L}\right)^2 + \dfrac{k_{nm}\,\mathrm{th}\,k_{nm}h}{h}\right]}{\omega_{nm}^2 h\alpha_{0j}\alpha_{m0}\ \mathrm{th}\,\alpha_{m0}L(k_{nm}\,\mathrm{th}\,k_{nm}h)}\cos\frac{m\pi y}{b}\frac{\mathrm{ch}\,\alpha_{m0}(x-L)}{\mathrm{ch}\,\alpha_{m0}} -$$

$$-\varepsilon_1 \dot{\xi}_{nm}(t)\sin\omega t \sum_{j=1}^{\infty} \frac{g\left(\frac{n\pi}{L}\right)^2 [1-(-1)^j]}{\omega_{nm}^2 \alpha_{mj} (j\pi)^2 \operatorname{th}\alpha_{mj} L(k_{nm} \operatorname{th} k_{nm} h)} \cos\frac{m\pi y}{b} \cos\frac{j\pi z}{h} \frac{\operatorname{ch}\alpha_{mj}(x-L)}{\operatorname{ch}\alpha_{mj}}.$$

Where $\varphi_{nm}(t) = O(\varepsilon_1^{1/2})$ and $\varphi_{lo} = O(\varepsilon_1)$

Using kinematical free-surface boundary conditions, Krasnopolskaya, 2012, [5],

$$(\varphi_0)_z + (\varphi_1)_z + \xi(\varphi_0)_{zz} + \xi(\varphi_1)_{zz} + \xi^2(\varphi_1)_{zzz} + \xi(\varphi_2)_{zz} =$$
$$= \xi_t + (\varphi_1)_x \xi_x + (\varphi_1)_y \xi_y + (\varphi_0)_x \xi_x + (\varphi_2)_y \xi_y +$$
$$+(\varphi_1)_{xz} \xi \xi_x + (\varphi_1)_{yz} \xi \xi_y ,$$

we may find that the amplitude of the resonant cross-wave mode is

$$\Phi_{nm}(t) = \frac{\dot{\xi}_{nm}}{k_{nm} th k_{nm} h} - \varepsilon_1 \xi_{nm} D \cos\omega t; \qquad (2.2)$$

when

$$D = \frac{1}{k_{nm} \operatorname{th} k_{nm} h}\left[\sum_j \frac{8g[1-(-1)^j]}{\omega_{nm} h^2 \alpha_{0j} (\operatorname{th}\alpha_{0j} L)}\right] \int_0^L \cos^2\frac{n\pi x}{L} \frac{\operatorname{ch}\alpha_{0j}(x-L)}{\operatorname{ch}\alpha_{0j} L} dx +$$

$$+\frac{g}{\omega_{nm} L} - \frac{g}{\omega_{nm} L} \int_0^L \left(\frac{n\pi}{L}\right)(x-L)\sin 2\frac{n\pi x}{L} dx.$$

Applying the dynamical boundary condition

$$(\varphi_0)_t + (\varphi_1)_t + \xi(\varphi_1)_{tz} + \xi^2(\varphi_1)_{tzz} + (\varphi_2)_t + g\xi +$$
$$+\left[(\varphi_1)_x^2 + (\varphi_1)_y^2 + (\varphi_1)_z^2\right] + (\varphi_1)_x(\varphi_2)_x + (\varphi_1)_y(\varphi_2)_y + (\varphi_1)_z(\varphi_2)_z +$$
$$+(\varphi_1)_x(\varphi_0)_x + (\varphi_1)_z(\varphi_0)_z + (\varphi_1)_z \xi(\varphi_1)_{zz} +$$
$$+(\varphi_1)_x \xi(\varphi_1)_{xz} + (\varphi_1)_y \xi(\varphi_1)_{yz} = F_0(t),$$

we can get for the resonant amplitude an equation of parametric oscillations

$$\ddot{\xi}_{nm} + \omega_{nm}^2 \xi_{nm} - \frac{9}{16}\omega_{nm}^2 k_{nm}^2 \xi_{nm}^3 + \frac{3}{4}k_{nm}^2 \xi_{nm} \dot{\xi}_{nm}^2 +$$
$$+\varepsilon_1 D_2 \xi_{nm} \sin\omega t - \varepsilon_1 D_5 \dot{\xi}_{nm} \cos\omega t = 0. \qquad (2.3)$$

We can write it for the rectangular tank with $L = 50$ m, h= 2.5 m, $b = 6.8$ m and for the wave numbers $n = 40$, $m = 10$ in the form

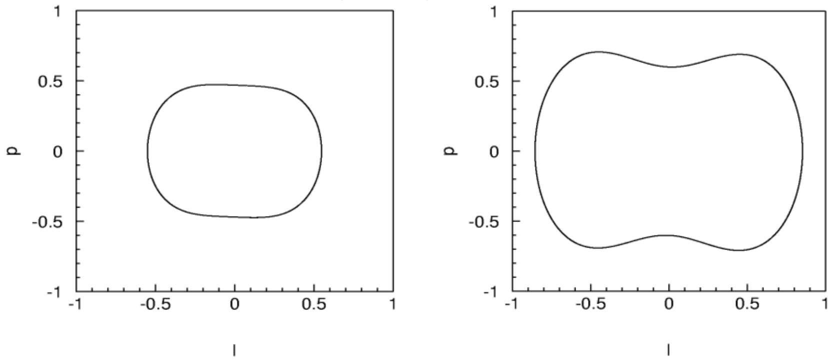
$$\ddot{\xi}_{nm} + \omega_{nm}^2 \xi_{nm} - \frac{9}{16}\omega_{nm}^2 k_{nm}^2 \xi_{nm}^3 + \frac{3}{4}k_{nm}^2 \xi_{nm} \dot{\xi}_{nm}^2 +$$
$$+0.0478\omega_{nm}^2 \xi_{nm} \sin\omega t - 0.0299\omega_{nm} \dot{\xi}_{nm} \cos\omega t = 0. \qquad (2.4)$$

Where $k_{nm}^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{b}\right)^2$, the frequency is $\omega_{nm} = 2\pi 1.143\,\text{Hz}$. We may use the transformation to the dimensionless variables $l = \xi_{nm} / \mu$, p, $\tau = \omega_{nm} t$, and finally get a dynamical system (when $\omega = 2\pi 2.27$ Hz and $\mu = 0.26$ m) in the following form

$$\dot{l} = p$$

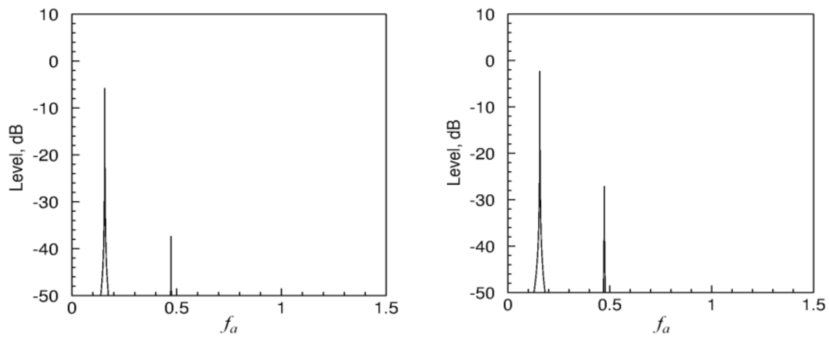
$$\dot{p} = -l - \alpha p + 1.0504 l^3 - 1.4003 l p^2 - 0.0478 A l \sin(2\tau - \beta\tau) + 0.0299 A p \cos(2\tau - \beta\tau)$$

This system (at $\beta = 2 - \omega / \omega_{nm}$ =0.014 and additional damping forces with $\alpha = 0.01$) has for any initial conditions only regular solutions. As an example in the fig.1 the phase portraits for different values of parameter *A* (which is proportional to the amplitude of wavemaker oscillations) are shown. Power spectra are presented in fig.2. They are discrete for different values of *A*.



a) $A = 12$ b) $A = 27$

Fig. 1. Phase portraits for different values of wavemaker oscillations *A*.



a) $A = 12$ b) $A = 27$

Fig. 2. Power spectra computed for *l* time realization for different *A*.

3 Two Mode Model of Cross-waves in a Cylindrical Tank

Now we theoretically consider the nonlinear problems of fluid free-surface waves which are excited by inner shell vibrations in a volume between two cylinders of finite length. It is useful to relate the fluid motion to the cylindrical coordinate system (r,θ,x). The fluid has an average depth d; the average position of the free surface is taken as $x=0$, so that the solid tank bottom is at $x=-d$. The fluid is confined between a solid outer cylinder at $r=R_2$ and a deformable inner cylinder (which acts as the wavemaker) at average radius $R_1=r_1+a_0(d)^{-1}\int_{-d}^{0}\cos(\eta x)dx=r_1+2a_0/\pi$. This inner cylinder vibrates harmonically in such a way that the position of the wall of the inner cylinder is $r=R_1+\chi_1(x,t)=R_1-(a_0+a_1\cos\omega t)\cos\eta x-2a_0/\pi$, where $\eta=\pi/(2d)$. The potential ϕ can be written as the sum of three harmonic functions $\phi=\phi_0+\phi_1+\phi_2$, Lamé, 1852, [8]. The solution of the linear problem for ϕ_1 can be written in the form, Krasnopolskaya, 1996, [4]

$$\phi_1=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\phi_{ij}^{c,s}(t)\frac{\cosh k_{ij}(x+d)}{N_{ij}\cosh k_{ij}d}\psi_{ij}^{c,s}(r,\theta), \tag{3.1}$$

on the complete systems of azimuthal ($\cos i\theta$, $\sin i\theta$), and radial eigenfunctions $\chi_{ij}(k_{ij}r)=J_i(k_{ij}r)-\dfrac{J_{i'}(k_{ij}R_1)}{Y_{i'}(k_{ij}R_1)}Y_i(k_{ij}r)$, with some arbitrary amplitudes $\phi_{ij}^{c,s}(t)$. In the solution (3.1) the notations $\psi_{ij}^{c,s}(r,\theta)=\chi_{ij}(k_{ij}r)(\cos i\theta,\sin i\theta)$ are used, where J_i and Y_i are the i-th order Bessel functions of the first and the second kind, respectively, and N_{ij} is a normalization constant, where the index c (or s) indicates that the eigenfunction $\cos i\theta$ (or $\sin i\theta$) is chosen as the circumferential component; k_{ij} represents eigen wave numbers. The system of functions $\psi_{ij}(r,\theta)$, with $i=0,1,2,...$ and $j=1,2,3,...$, is a complete orthogonal system, so any function of the variables r and θ can be represented using the usual procedure of Fourier series expansion. Thus, the free surface displacement $\zeta(r,\theta,t)-\zeta_0(t)$ can be written as ($\zeta_0(t)$ is the mean level of fluid free surface oscillations)

$$\zeta(r,\theta,t)-\zeta_0(t)=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\zeta_{ij}^{c,s}(t)\frac{\psi_{ij}^{c,s}(r,\theta)}{N_{ij}}. \tag{3.2}$$

Under a parametric resonance, when the excitation frequency is twice as large as one of the eigenfrequencies, i.e. $\omega \approx 2\omega_{nm}$, and according the experimental observations we may assume that the free-surface displacement can be approximated by two resonant modes. So that we may write [4]

$$\zeta \approx \frac{1}{N_{nm}} \zeta_{nm} \psi^{c}_{nm}(r,\theta) + \frac{1}{N_{0l}} \zeta_{0l} \psi_{0l}(r) + \zeta_0 \tag{3.3}$$

where ψ_{0l} is the axisymmetric mode which has the eigenfrequency by a value very close to ω, i.e. $\omega_{0l} \approx \omega$. From the experimental observations follows that cross-waves has ampliteds much bigger than the amplitudes of the forced waves with the frequency ω of the wavemaker vibrations. So that we can seek the unknown functions in the form

$$\zeta_{nm}(t) = \varepsilon_1^{1/2} \lambda_1 \left[p_1(\tau_1) \cos\frac{\omega t}{2} + q_1(\tau_1) \sin\frac{\omega t}{2} \right];$$
$$\zeta_{0l}(t) = \varepsilon_1 \lambda_0 \left[p_2(\tau_1) \cos\omega t + q_2(\tau_1) \sin\omega t \right], \tag{3.4}$$

where $\lambda_1 = k_{nm}^{-1} \,\mathrm{th}(k_{nm}h)$, $\varepsilon_1 = \dfrac{a\omega_{nm}^2}{g}$ is a small parameter, $\tau_1 = \dfrac{1}{4}\varepsilon_1 \omega t$ is a dimensionless slow time, $\lambda_0 = k_{0l}^{-1}\,\mathrm{th}(k_{0l}h)$. By substitution of the expressions (3.4) into boundary conditions, Krasnopolskaya, 1996, [4] and averaging over the fast time ωt we finally obtain the dynamical system in the form, Krasnopolskaya, 2013, [7],

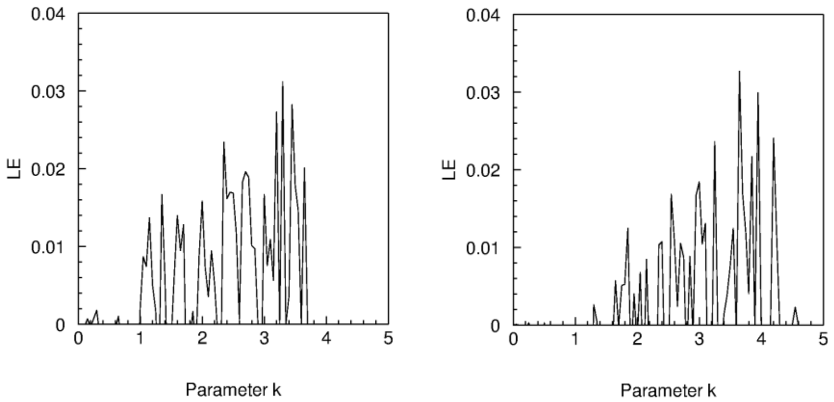
$$\begin{aligned}
\frac{dp_1}{d\tau_1} &= -\alpha p_1 - \vartheta q_1 + \beta_3 q_1 + \beta(q_1 p_2 - p_1 q_2);\\
\frac{dq_1}{d\tau_1} &= -\alpha q_1 + \vartheta p_1 + \beta_3 p_1 + \beta(p_1 p_2 + q_1 q_2);\\
\frac{dp_2}{d\tau_1} &= -\alpha p_2 - \beta_2 q_2 - 2\beta_4 p_1 q_1;\\
\frac{dq_2}{d\tau_1} &= -\alpha q_2 + \beta_2 p_2 + \beta_4 (p_1^2 - q_1^2) + \beta_5,
\end{aligned} \tag{3.5}$$

where $\vartheta = \left[\beta_1 + \frac{\beta_6}{2}(p_1^2 + q_1^2) \right]$, $\alpha = \frac{\delta}{\omega_{nm}}$, δ is the ratio of actual to critical damping of the mode, β_i (*i=1,2,…6*) are constant coefficients. The dynamical system (3.5) is nonlinear, so numerical solutions were obtained. We used the following coefficients (Becker, 1991, [1], Krasnopolskaya, 1996, [4]) and data:

$\alpha = 0.01$; β_3 =1.3k; β_4 =0.25; β_5 =0.235k; β_6 =1.12; β = -1.531;

$p_1(0) = q_1(0) = p_2(0) = q_2(0) = 0.5$.

For these parameters and for different values of *k* (which is dimensionless amplitude of the wavemaker vibrations) extensive numerical calculations were carried out in order to find all steady state regimes. In Figure 3 dependences of the maximum Lyapunov exponents on value *k* are shown for the different values of the detuning parameters β_1 and β_2 .

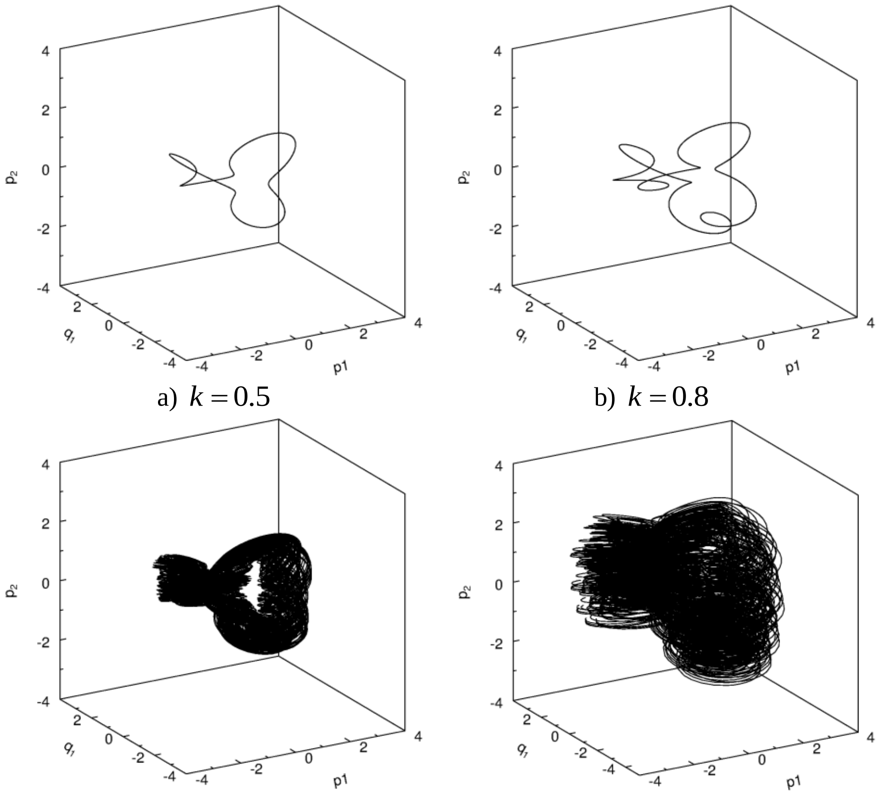


a) at $\beta_1 = 0$, $\beta_2 = 0.2$ b) at $\beta_1 = 0.2$, $\beta_2 = 0$

Fig. 3. The dependence of the maximum Lyapunov exponent on value k.

Comparing these dependencies we may conclude that the dynamical system, which corresponds to the case when there is no detuning between the half of the frequency of excitation ω and the eigenfrequency of the cross-waves ω_{nm}, i.e. β_1 =0, and there is the detuning of frequencies for the axisymmetric mode β_2 =0.2, has chaotic regimes in the wider area of the parameter k changing.

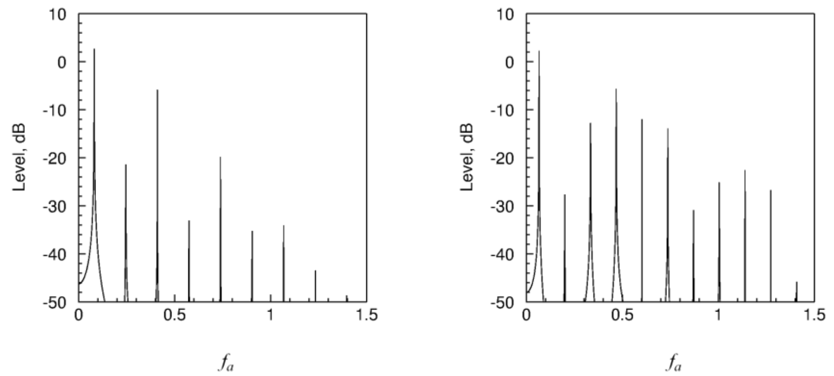
To demonstrate this we show in Figure 4 and 6 the phase portraits of solutions for the first case and the second when $\beta_1 = 0.2$ and there is no detuning for the axisymmetric mode, i.e. β_2 =0. In Figure 4 c) we have the chaotic attractor and in Figure 6 c) the regular cycle. And in Figure 4 attractors occupy bigger areas. Power spectra for considered cases are show in Figures 5 and 7 correspondently.



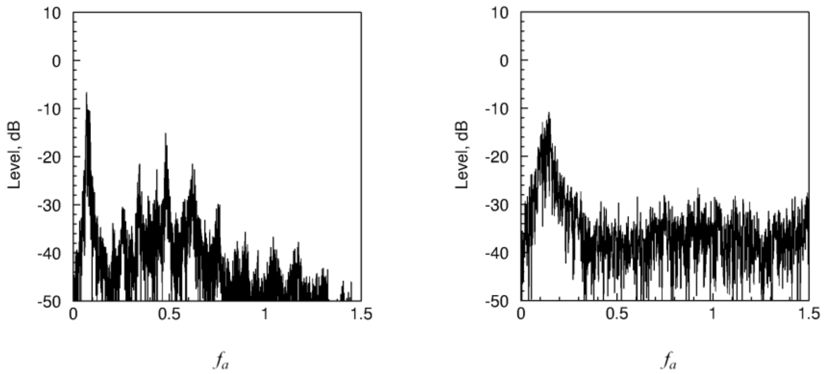
a) $k = 0.5$ b) $k = 0.8$

c) $k = 1$ d) $k = 3$

Fig.4. Phase portraits for regular (cases a, b) and chaotic regimes (cases c, d) when $\beta_1 = 0,\ \beta_2 = 0.2$.

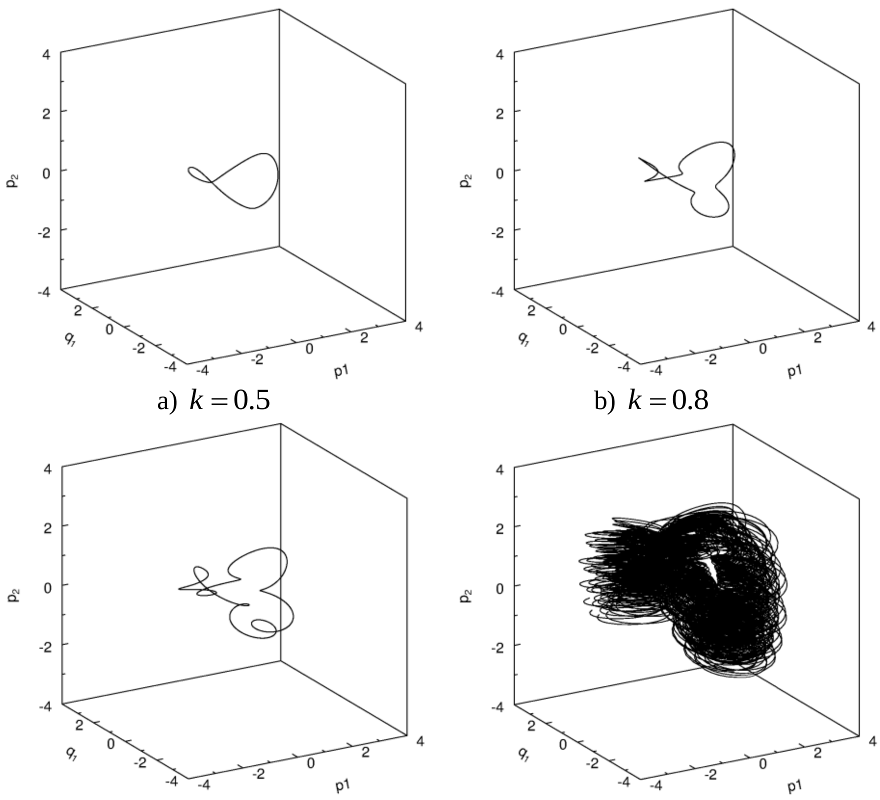


a) $k = 0.5$ b) $k = 0.8$



c) $k=1$ d) $k=3$

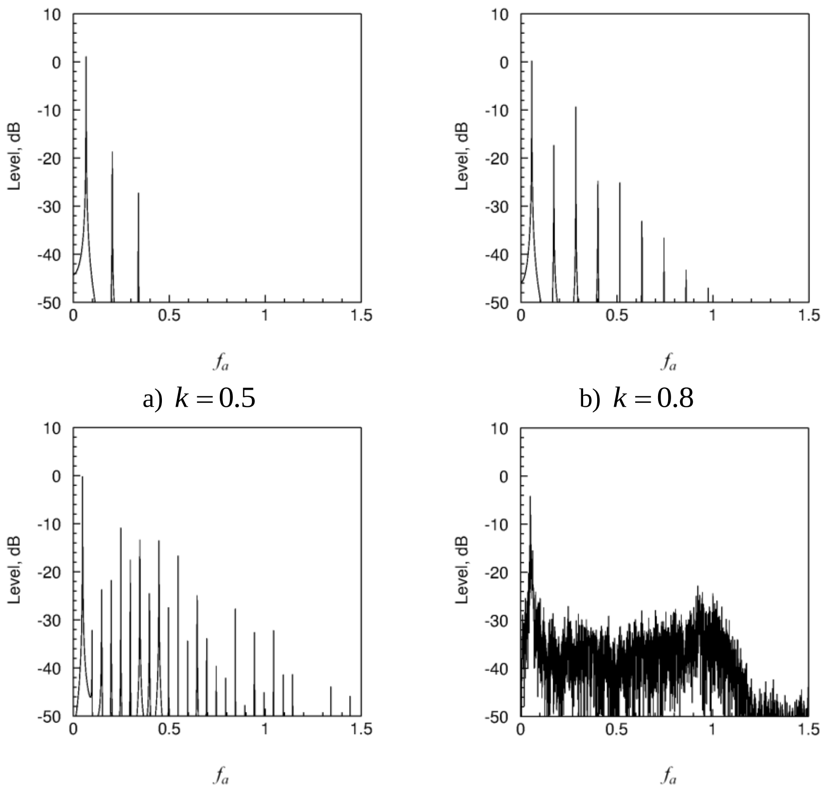
Fig. 5. Power spectra computed for p_1 data (cases a, b, c and d) when $\beta_1=0$, $\beta_2=0.2$.



a) $k=0.5$ b) $k=0.8$

c) $k=1$ d) $k=3$

Fig.6. Phase portraits for regular (cases a, b) and chaotic regimes (cases c, d) when $\beta_1=0.2$, $\beta_2=0$.



a) $k = 0.5$ b) $k = 0.8$

c) $k = 1$ d) $k = 3$

Fig. 7. Power spectra computed for p_1 data (cases a, b, c and d) for the case when $\beta_1 = 0.2,\ \beta_2 = 0$.

As we may conclude from numerical data and graphs in Figures 3-7 the dynamical system (3.5) has both regular and chaotic regimes. The chaotic regimes could be realized when $k \geq 1$ for the first case and $k \geq 1.6$ for the second considered case. For such values of corresponding amplitudes of wavemaker oscillations the largest Lyapunov exponents are positive, phase portraits have complicated structures of trajectory sets and power spectra are continuous ones.

4 Conclusions

Two new models expressing interaction of two eigenmodes at the condition of parametric resonances for the cross-waves of fluid free surface oscillations are developed. Models are simulated. The existence of chaotic attractors was established for the dynamical system presenting cross-waves and forced waves interaction at fluid free-surface in a volume between two cylinders of finite length. For the system describing resonant cross-waves in the rectangular tank no chaotic regimes were found because the connection coefficients of cross-waves with the axisymmetric waves under the forced resonance are values on much smaller order then considered here. So that there are less factors to destabilize the system.

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