Chaotic Modeling and Simulation (CMSIM) 2: 199-206, 2014

The Presence of Chaos in the GDP Growth Rate Time Series

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Abstract: The goal of this paper is to find chaos in the Gross domestic product (GDP) growth rate of selected European countries. We chose only those European countries where data is available since 1980, because we needed the longest time series possible. These are the following states: Belgium, Finland, France, Norway, Spain, Switzerland and United Kingdom. At first we will estimate the time delay and the embedding dimension, which is needed for the largest Lyapunov exponent estimation. The largest Lyapunov exponent is one of the important indicators of chaos and is generally well-known. Subsequently we will calculate the 0-1 test for chaos. Finally we will compute the Hurst exponent by using the Rescaled Range analysis. The Hurst exponent is a numerical estimate of the predictability of a time series. The results indicated that chaotic behaviors obviously exist in GDP growth rate.
Keywords: Chaos theory, GDP, GDP growth rate, Time series analysis, Phase Space Reconstruction, Hurst exponent, largest Lyapunov exponent.

1. Introduction

Humanity has always been concerned with the question of whether the processes in the real world are deterministic in nature. Determinism can be understood variously. In this paper we assume a mathematical sense of determinism, which is given by equations and initial conditions. Mathematical models that are not deterministic because they involve randomness are called stochastic. Are the processes in the real world deterministic or stochastic in nature? Real processes in nature, according to the expectation of Mandelbrot [16], lie somewhere between pure deterministic process and white noise. This is why we can describe reality either by a stochastic or deterministic model. The Hurst coefficient can give us an answer to this.
An interesting case of determinism is deterministic chaos. The only purely stochastic process is a mathematical model described by mathematical statistics. The statistical model often works and is one of many possible descriptions if we do not know the system. This also applies to economic quantities, including forecasts for GDP. The basic question is therefore the existence of chaotic behavior. If the system behaves chaotically, we are forced to accept only limited predictions. In this paper we will try to show the chaotic behavior of GDP growth rate.

Received: 24 April 2013 / Accepted: 11 January 2014

ISSN 2241-0503

2. Methods of analyzing

In short, we will describe the basic definitions and the basic methods for examining the input data.

2.1 Phase space reconstruction

According to Henry [9], the main goal in nonlinear time series analysis is to determine whether or not a given time series is of a deterministic nature. If it is, then further questions of interest are: What is the dimension of the phase space supporting the data set? Is the data set chaotic?

The key to answering these questions is embodied in the method of phase space reconstruction, which has been rigorously proven by the embedding theorems of Takens [19]. Takens theorem was independently suggested for example Packard [17]. Takens' theorem transforms the prediction problem from time extrapolation to phase space interpolation.

Let there be given a time series $x_1, x_2, \ldots, x_N$ which is embedded into the m-dimensional phase space by the time delay vectors. A point in the phase space is given as:

$$Y_n = x_n, x_{n-\tau}, \ldots, x_{n-(m-1)\tau} \quad n = 1,2,\ldots, N-(m-1)\tau \tag{1}$$

where τ is the time delay and m is the embedding dimension. Different choices of τ and m yield different reconstructed trajectories. How can we determine optimal τ and m?

2.2 Optimal time delay

A one-to-one embedding can be obtained for any value of the time delay $\tau > 0$. However, very small time delays will result in near-linear reconstructions with high correlations between consecutive phase space points and very large delays might obscure the deterministic structure linking points along a single degree of freedom. If the time delay is commensurate with a characteristic time in the underlying dynamics, then this too may result in a distorted reconstruction.

In order to estimate τ, two criteria are important according to Kodba [12]. First, τ has to be large enough so that the information we get from measuring the value of x at time $n + \tau$ is significantly different from the information we already have by knowing the value of x at time n. Only then will it be possible to gather enough information about all other system variables that influence the value of x to reconstruct the whole attractor. Second, τ should not be larger than the typical time in which the system loses memory of its initial state. This is particularly important for chaotic systems, which are intrinsically unpredictable and hence lose memory of the initial state as time progresses. [14]

Following this reasoning, Fraser and Swinney [3] introduced the mutual information between x_n and $x_{n+\tau}$ as a suitable quantity for determining τ. The mutual information between x_n and $x_{n+\tau}$ quantifies the amount of information we have about the state $x_{n+\tau}$ presuming we know the state x_n. Now we can define mutual information function:

$$I(\tau) = -\sum_{h=1}^{j}\sum_{k=1}^{j} P_{h,k}(\tau) \ln \frac{P_{h,k}(\tau)}{P_h P_k} \tag{2}$$

where P_h and P_k denote the probabilities that the variable assumes a value inside the h^{th} and k^{th} bins, respectively, and $P_{h,k}(\tau)$is the joint probability that x_n is in bin h and $x_{n+\tau}$ is in bin k. Hence, the first minimum of $I(\tau)$ marks the optimal choice for the time delay.

2.3 Optimal embedding dimension

The embedding dimension m is conventionally chosen using the "false nearest neighbors'" method. This method measures the percentage of close neighboring points in a given dimension that remain so in the next highest dimension. The minimum embedding dimension capable of containing the reconstructed attractor is that for which the percentage of false nearest neighbors drops to zero for a given tolerance level ε. [14]

In order to calculate the fraction of false nearest neighbors the following algorithm is used according to Kennel [11]. Given a point $p(i)$ in the m-dimensional embedding space, one first has to find a neighbour $p(j)$, so that

$$\|p(i)-p(j)\| \le \varepsilon \quad (3)$$

We then calculate the normalized distance R_i between the $(m + 1)th$ embedding coordinate of points $p(i)$ and $p(j)$ according to the equation:

$$R_i = \frac{|x_{i+m\tau} - x_{j+m\tau}|}{\|p(i)-p(j)\|} \quad (4)$$

If R_i is larger than a given threshold R_{tr} , then $p(i)$ is marked as having a false nearest neighbor. Equation (4) has to be applied for the whole time series and for various $m = 1, 2, \ldots$ until the fraction of points for which $R_i > R_{tr}$ is negligible [12].

2.4 The largest Lyapunov exponent

Lyapunov exponent λ of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. Quantitatively, two trajectories in phase space with initial separation δZ_0 diverge.

$$\delta Z(t) \approx e^{\lambda t} |\delta Z_0| \quad (5)$$

The largest Lyapunov exponent (LLE) can be defined as follows:

$$\lambda = \lim_{\substack{\delta Z_0 \to 0 \\ t \to \infty}} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|} \quad (6)$$

The limit $\delta Z0 \to 0$ ensures the validity of the linear approximation at any time. LLE determines a notion of predictability for a dynamical system. A positive LLE is usually taken as an indication that the system is chaotic (provided some other conditions are met, e.g., phase space compactness) [15].

We have used the Rosenstein algorithm, which counts the LLE as follows:

$$\lambda_1(i) = \frac{1}{i\Delta t} . \frac{1}{(M-i)} \sum_{j=1}^{M-i} \ln \frac{d_j(i)}{d_j(0)} \quad (7)$$

Where $d_j(i)$ is distance from the j point to its nearest neighbor after i time steps and M is the number of reconstructed points. For more information see [6, 18].

2.5 The 0-1 test for chaos

New test for the presence of deterministic chaos was developed by Gottwald & Melbourne [7]. Their '0 - 1 test for chaos takes as input a time series of measurements, and returns a single scalar value usually in the range 0 - 1. In contrast the 0 - 1 test does not depend on phase space reconstruction but rather works directly with the time series given. The input is the time-series data and the output is 0 or 1, depending on whether the dynamics is non-chaotic or chaotic.

Briefly, the 0-1 test takes as input a scalar time series of observations $\varphi_1, \ldots, \varphi_N$. We have used the algorithm according to Dawes & Freeland [1]. First, we must fix a real parameter c and construct the Fourier transformed series:

$$z_n = \sum_{j=1}^{n} \phi_j e^{ijc}, \quad n = 1, \ldots, N \tag{8}$$

Then we have computed the smoothed mean square displacement:

$$M_c(n) = \frac{1}{N-p} \sum_{j=1}^{N-p} \left| z_{j+n} - z_j \right|^2 - \left(\sum_{k=1}^{N} \frac{\phi_k}{N} \right)^2 \frac{1-\cos nc}{1-\cos c} \tag{9}$$

Finally we have estimated correlation coefficient to evaluate the strength of the linear growth

$$r_c = \frac{\operatorname{cov}(n, M_c(n))}{\sqrt{\operatorname{cov}(n,n)\operatorname{cov}(M_c(n), M_c(n))}} \tag{10}$$

2.6 Long memory in time series

Hurst exponent *(H)* is widely used to characterize some processes. Hurst exponent is used to evaluate the presence or absence of long-range dependence and its degree in a time-series. For more information see [8, 10]. The Hurst exponent is a measure that has been widely used to evaluate the self-similarity and correlation properties of fractional Brownian noise, the time series produced by a fractional Gaussian process [16]. We can describe self-similarity process following equation:

$$X(at) = a^H X(t) \tag{11}$$

where a is a positive constant, and H is the self-similarity parameter, for $0 < H < 1$.

We have used a methodology known as Rescaled Range analysis or R/S analysis. To calculate the Hurst exponent, one must estimate the dependence of the rescaled range on the time span n of observation. The Hurst exponent is defined in terms of the asymptotic behavior of the rescaled range as a function of the time span of a time series as follows:

$$E\left[\frac{R(n)}{S(n)}\right] = Cn^H \quad \text{as} \quad n \to \infty \tag{12}$$

Where *[R(n)/S(n)]* is the rescaled range; *E[y]* is expected value; *n* is number of data points in a time series, *C* is a constant. For more information see [13].

3. Analysis of GDP growth rate time series

3.1 Input data

The GDP in current prices in millions of national currency (including 'euro fixed' series for euro area countries) is used in this paper. We have used data (quarterly, seasonally adjusted and adjusted data by working days) from the Eurostat between the years 1980 - 2012. According to Eurostat [2], seasonal adjustment is a treatment of infra-annual time series to remove the spurious effect of seasonal patterns from the series' trend and cycle. These patterns can be caused by weather, public holidays such as Christmas, the timing of school vacations or of dividend payments and a number of other reasons.

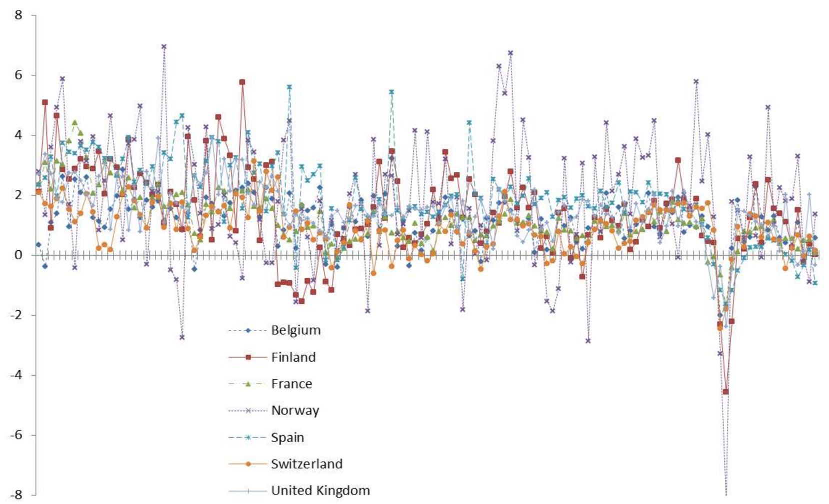

Fig. 1. GDP growth rate time series.

Generally, the main problem in analyzing the GDP time series is the lack of data. That is why we chose only those European countries where data is available since 1980. These are the following states: Belgium, Finland, France, Norway, Spain, Switzerland and United Kingdom. So, we have 132 values from these countries. The analysis of such short time series in the context of nonlinear dynamics or in the presence of chaos can be questionable. We know, according to Horák [4] or Galka [5], that for this kind of method results are provable for at least 10^3 data-points. Analysis of short time series (order of 10^1) may lead to a spurious estimation of the invariants e.g. LLE. Despite the above, we have no choice but to analyze GDP time series in the context of nonlinear dynamics and try to find chaotic behavior of GDP growth rate time series. Therefore, all results are only estimates. The second problem can be the presence of trends in time series. Trended data are not suitable for future analysis to study chaos dynamics. There is no universal way to remove the trend from the data set. The results often depend strongly on how the data are detrended. This is solved using the GDP growth rate (cf. Figure 1).

3.2 Calculation of the largest Lyapunov exponents

At first we will estimate the time delay and the embedding dimension, which is needed for the largest Lyapunov exponent estimation. We will use the mutual information approach to determine the time delay. The first minimum of the mutual information function $I(\tau)$ (2) marks the optimal choice for the time delay. The embedding dimension *m* is chosen using the "false nearest neighbors'" method. The minimum embedding dimension capable of containing the reconstructed attractor is that for which the percentage of false nearest neighbors drops to zero for a given tolerance level ε. Then, we calculated the LLE using the Rosenstein algorithm. All computed values are positive (cf. Figure 2). A positive LLE is usually taken as an indication that the system is chaotic.

	Tau	ED	LLE	H	test 0- 1
Belgium	2	3	0,032	0,76	0,99
Finland	3	3	0,840	0,88	0,99
France	3	3	0,084	0,96	0,82
Norway	1	3	0,011	0,54	0,99
Spain	2	3	0,078	0,99	0,95
Switzerland	3	3	0,585	0,82	0,99
United Kingdom	4	3	0,071	0,96	0,98
Average	2,571	3,000	0,243	0,844	0,959
SD	0,904	0,000	0,306	0,146	0,058

Fig. 2. The optimal time delay, The optimal embedding dimension, The Largest Lyapunov exponent, The Hurst exponent, Value of chaos test 0-1 for selected countries, average and standard deviation

3.3 Results of the 0-1 Test for Chaos

In this chapter we calculate the correlation coefficient as was shown above. The correlation coefficient is near to 0 for non-chaotic data and near 1 for chaotic data. All computed values are very close to 1 (cf. Figure 2). Hence, we can convincingly assume there to be chaotic behavior in the GDP growth rate time series.

3.4 Calculation of the Hurst exponent

The Rescaled Range analysis gave us values of the Hurst exponent between 0,54 (Norway) and 0,99 (Spain) (cf. Figure 2). Most values indicate the presence of long memory in GDP growth rate time series except the value of the Hurst exponent for the Norway GDP growth rate, which indicates random walk. Those values are in accordance with our expectations. We know that the value of H is between 0 and 1, whilst real time series are usually higher than 0,5. If the exponent value is close to 0 or 1, it means that the time-series has long-range dependence. We can assume that the true value lies somewhere between those values. We think that those values are sufficient for a credible prediction. Now we also know that the fractal dimension DF =2-H. We have estimated the values of the fractal dimension selected time series between 1,01 and 1,46.

4. Conclusions

Chaos theory has changed the thinking of scientists and the methodology of science. Making a theoretical prediction and then matching it to the experiment is not possible in chaotic processes. Long term forecasts are, in principle, also impossible according to chaos theory. The main problem is in the quantity and quality of data. Some improvement of measurement cannot help us adequately, because it is a fight against power of exponential rate. Nonlinear dynamics and chaos theory have also corrected the old reductionist tendency in science. Now it is known that real processes are nonlinear and a linear view can be wrong. The basic question is therefore - the existence of chaotic behavior. If the system behaves chaotically, we are forced to accept only limited predictions. But it is much better than random processes.

Although we analyzed various GDP growth rate time series, the results came out very similar. We have shown in this paper that the GDP growth rate time series are chaotic and contain long memory. First, we computed the values of the time delays and the embedding dimensions. The average value of computed time delays is 2,6. In all 7 cases we chose the value 3 as the optimal embedding dimension. Subsequently, we calculated the LLE and all computed values were positive. A positive LLE is usually taken as an indication that the system is chaotic. If the fractal dimension is low, the LLE is positive and the Kolmogorov entropy has a finite positive value, chaos is probably present. Then we conducted the 0-1 test for chaos according to which chaos was present. All computed values were very close to 1. Hence, we can convincingly assume there to be chaotic behavior in the GDP growth rate time series. From these estimations it can be concluded that the GDP growth rate time series is chaotic. Finally we have computed the Hurst exponent by Rescaled Range analysis. Most values indicate the presence of long memory in GDP growth rate time series, except the value for the Norway GDP growth rate.

We know that the main problem when analyzing GDP time series is the lack of data. As mentioned above, we chose only those European countries where data is available since 1980. Although these time series are not ideal in length, they are acceptable for analysis. The results came out mostly very similar. The presence of chaos in selected GDP growth time series is not only a coincidence.

In the future we would like to focus on the proper statistical significance for nonlinearity and on predicting the GDP. In particular, the surrogate data approach (e.g. Theiler et al. [20]) is a powerful tool for detecting actual nonlinear behavior, and distinguishing it from other phenomena.

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