

# On a Topological Problem of Strange Attractors

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**Abstract.** In this note, we consider self-affine attractors that are generated by an expanding  $n \times n$  matrix (i.e. all of its eigenvalues have moduli  $> 1$ ) and a finite set of vectors in  $\mathbb{Z}^n$ . Here we concentrate on the problem of connectedness. Although, there has been intensive study on the topic recently, this problem is not settled even in the one-dimensional case. We focus on some basic attractors, which have not been studied fully, and characterize connectedness.

**Keywords:** Self-affine attractors, Self-affine tiles, Connectedness.

## 1 Introduction

Let  $S_1, \dots, S_q$ ,  $q > 1$ , be contractions on  $\mathbb{R}^n$ , i.e.,  $\|S_j(x) - S_j(y)\| \leq c_j \|x - y\|$  for all  $x, y \in \mathbb{R}^n$  with  $0 < c_j < 1$ . Here  $\|\cdot\|$  stands for the usual Euclidean norm, but this norm may be replaced by any other norm on  $\mathbb{R}^n$ . It is well known [4] that there exists a unique non-empty compact set  $F \subset \mathbb{R}^n$  such that

$$F = \bigcup_{j=1}^q S_j(F).$$

Let  $M_n(\mathbb{R})$  denote the set of  $n \times n$  matrices with real entries. We will assume that

$$S_j(x) = T^{-1}(x + d_j), \quad x \in \mathbb{R}^n,$$

where  $d_j \in \mathbb{R}^n$ , called *digits*, and  $T \in M_n(\mathbb{R})$ . Then  $F$  is called a *self-affine set* or a *self-affine fractal*, and can be viewed as the invariant set or the attractor of the (affine) *iterated function system* (IFS)  $\{S_j(x)\}$  (in the terminology of dynamical systems). Let  $M_n(\mathbb{Z})$  be the set of  $n \times n$  integer matrices. Further, if  $D := \{d_1, \dots, d_q\} \subset \mathbb{Z}^n$  and  $T \in M_n(\mathbb{Z})$ , it is called an *integral self-affine set* and we will primarily consider such sets in this paper. If, additionally,  $|\det(T)| = q$  and the integral self-affine set  $F$  has positive Lebesgue measure, then  $F$  is called an *integral self-affine tile*. We sometimes write  $F(T, D)$  for



$F$  to stress the dependence on  $T$  and  $D$ . For such tiles, the positivity of the Lebesgue measure is equivalent to having nonempty interior [2].

There is a demand to develop analysis on fractal spaces, in order to deal with physical phenomena like heat and electricity flow in disordered media, vibrations of fractal materials and turbulence in fluids. Without a better understanding of the topology of fractals, this seems to be a difficult task. There is a growing literature on the formalization and representation of topological questions; see [3] for a survey of the field.

One of the interesting aspects of the self-affine sets is the *connectedness*, which roughly means the attractor cannot be written as a disjoint union of two pieces. This property is important in computer vision and remote sensing [8,20]. We mention that connected self-affine fractals are curves; thus, they are sometimes referred to as self-affine curves [10]. There is some motivation for studying connected self-affine tiles because they are related to number systems, wavelets, torus maps. Recently, there have been intensive investigations on the topic by Kirat and Lau [12,10], Akiyama and Thuswaldner [1,16], Ngai and Tang [18,19] and Luo et al. [16,15].

In this note, we consider planar integral self-affine fractals obtained from  $2 \times 2$  integer matrices with reducible characteristic polynomials, and report our findings on their connectedness. However, our considerations can be generalized to higher dimensions. As for the organization of the paper, in Section 2, we deal with special cases and state some simple, but unconventional techniques to check the connectedness. In literature, most of the papers on the connectedness have some restrictions on the digit set. Here our aim is to remove such restrictions in Section 3.

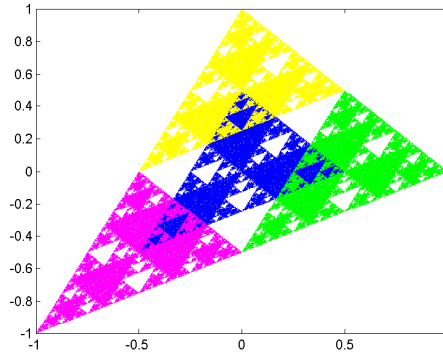
## 2 Some Unconventional Techniques

Usually, connectedness criteria were given by using a “graph” with vertices in  $D$  [6,12]. In this section, we present graph-independent techniques to check the connectedness or disconnectedness. Throughout the paper,  $T^{-1}$  is a contraction. Let  $\#D$  denote the number of elements in  $D$ . We first recall a known result.

**Proposition 1.** [12] *Suppose  $T = [\pm q]$  with  $q \in \mathbb{N}$ , and  $D \subseteq \mathbb{R}$  with  $\#D = q$ . Then  $F(T, D)$  is a connected tile if and only if, up to a translation,  $D = \{0, a, 2a, \dots, (q-1)a\}$  for some  $a > 0$ .*

As one may notice  $q$  and  $D$  are not arbitrary in Proposition 1 since  $q \in \mathbb{N}$  and  $\#D = q$ . By using the approach in [9,11], we can remove such restrictions. For that purpose, we consider the convex hull of  $F$  and denote it by  $K$ . Also let  $K_1 = \bigcup_{j=1}^q S_j(K)$ . Then we have the following.

**Proposition 2.** *Let  $D = \{0, d_2v, \dots, d_qv\} \subset \mathbb{R}^n$  with  $v \in \mathbb{R}^n \setminus \{0\}$  and  $T = pI$ , where  $p \in \mathbb{R}$  and  $I$  is the identity matrix. Then  $F(T, D)$  is connected if and only if  $K = K_1$ .*



**Fig. 1.** The Sierpiński tile

*Remark 1.* A digit set  $D$  as in Proposition 2 is called a *collinear* digit set. It is easy to check the condition  $K = K_1$  in the proposition because  $K$  is a closed interval. If  $T = \pm 2I$ , then  $F(T, D)$  is connected for any digit set. A famous example of this type is the Sierpiński tile (see Figure 1), for which  $T = 2I$  and  $D = \{d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, d_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, d_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ .

The disconnectedness of  $F(T, D)$  was studied in [13]. Here we want to mention another unconventional sufficient condition for disconnectedness. In the rest of the paper, we study attractors  $F(T, D)$  in the plane such that  $T \in M_2(\mathbb{Z})$  has a reducible characteristic polynomial. From [10], we know that such matrices are conjugate to one of the following lower triangular matrices

$$\begin{bmatrix} n & 0 \\ t & m \end{bmatrix}, \quad \text{where } |n| \geq |m|, \text{ and } t = 0 \text{ or } t = 1. \tag{1}$$

We also let

$$S = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} : 0 \leq i \leq |n| - 1, 0 \leq j \leq |m| - 1 \right\}.$$

The attractors of the next proposition can be considered as a generalization of Sierpiński carpets [17]. Let  $\dim_S(F)$  be the *singular value dimension* of  $F$  (see [5]). We call a collinear digit set  $D$  with  $v$  is an eigenvector of  $T$  *eigen-collinear*. In that case,  $F$  is a subset of a line segment. By using Corollary 5 in [5], we obtain the following.

**Proposition 3.** *Assume that  $T$  is as in (1),  $D \subset S$ , and  $D$  is not eigen-collinear. Then  $F(T, D)$  is disconnected if  $\log_{|m|} r + \log_{|n|} \left(\frac{q}{r}\right) \neq \dim_S(F)$ , where  $q = \#D$  and  $r$  is the number of  $j$  so that  $\begin{bmatrix} i \\ j \end{bmatrix}$  for some  $i$ .*

*Remark 2.* It is easy to check the sufficient condition for the attractors  $F(T, D)$  in Proposition 3 because, in that case,

$$\dim_S(F) = \begin{cases} 1 + \log_{|n|} \left(\frac{q}{|m|}\right) & \text{if } |m| < q \leq |mn|, \\ \log_{|m|} q & \text{if } q \leq |m|. \end{cases}$$

### 3 General Digit Sets

In this section, we will present a practical way of checking the connectedness of  $F(T, D)$  with  $T$  as in (1) and  $D \subset \mathbb{Z}^2$ . Note that it is enough to consider the case  $n, m > 0$ , since  $F(T, D) = F(T^2, D + TD)$ . By translating  $D$ , we will assume that  $D$  has nonnegative entries. Let  $\mathcal{N} = (F - F) \cap \mathbb{Z}^2$ . Set

$$\Delta D = D - D, \quad a_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ m-1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} n-1 \\ 0 \end{bmatrix},$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

First, we begin with the special class of fractals  $F$  in Proposition 3, where  $D \subset S$ .

**Proposition 4.** *Assume that  $F$  is as in Proposition 3,  $t = 1$  and  $n, m > 0$ . Then*

- (i) if  $a_2 \notin \Delta D$ , then  $F$  is disconnected,
- (ii) otherwise,

$$\mathcal{N} = \{\pm e_i \mid i \in \{1, 2\} \text{ and } a_i \in \Delta D\} \cup \{\pm e_4 \mid a_1 - a_2 \in \Delta D\}.$$

**Proposition 5.** *Assume that  $F$  is as in Proposition 3,  $t = 0$  and  $n, m > 0$ . Let  $b_2 = a_2, b_3 = a_3$ . Then*

- (i) if  $b_1, b_2, b_3, b_1 - b_2 \notin \Delta D$ , then  $F$  is disconnected,
- (ii) otherwise,

$$\mathcal{N} = \{\pm e_i \mid i \in \{1, 2, 3\} \text{ and } b_i \in \Delta D\} \cup \{\pm e_4 \mid b_1 - b_2 \in \Delta D\}.$$

For a digit set  $D$ , an  $s$ -chain (in  $D$ ) is a finite sequence  $\{d_1, \dots, d_s\}$  of  $s$  vectors in  $D$  such that  $d_i - d_{i+1} \in \mathcal{N}$  for  $i = 1, \dots, s - 1$ . Then we can put the connectedness criterion in [12] into the following form.

**Proposition 6.**  *$F$  is connected if and only if, by re-indexing  $D$  (if necessary),  $D$  forms a  $q$ -chain.*

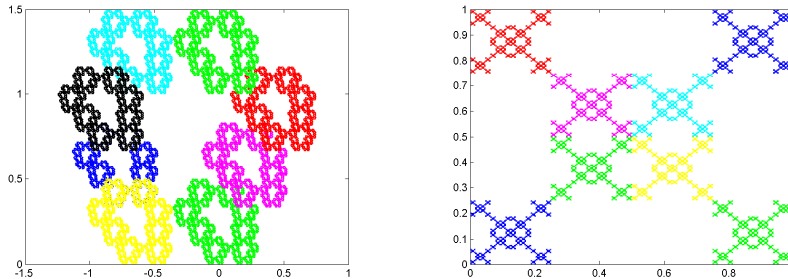


Fig. 2.

*Remark 3.* Note that if  $D = \{d_1, \dots, d_q\}$  after re-indexing, it is possible that  $d_i = d_j$  for  $i \neq j$ . In view of Proposition 4 and Proposition 5, Proposition 6 is quite feasible. That is, the connectedness can be decided by a simple inspection of  $D$  using  $\mathcal{N}$  in Propositions 4-5. That is, we get a graph-independent way of checking the connectedness. Two examples are given in Figure 2, for which  $T = \begin{bmatrix} -3 & -1 \\ 0 & 3 \end{bmatrix}$ , and  $D = \{d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, d_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, d_4 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, d_5 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, d_6 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, d_7 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, d_8 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}\}$ , and for the second fractal on the right  $T = 4I$  and  $D = \{d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, d_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, d_4 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, d_5 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, d_6 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, d_7 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, d_8 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}\}$ .

We now consider the general case  $D \subset \mathbb{Z}^2$ . Let

$$\mathcal{M}_1 = \{\pm(ke_1 \pm le_2) \mid k, l \in \mathbb{N} \text{ and } ka_1 \pm la_2 \in \Delta D\},$$

$$\mathcal{M}_0 = \{\pm(ke_1 \pm le_2) \mid k, l \in \mathbb{N} \text{ and } kb_1 \pm lb_2 \in \Delta D\}.$$

Note that it is possible that  $\mathcal{M}_1 = \emptyset$  or  $\mathcal{M}_0 = \emptyset$ .

**Proposition 7.** *Assume that  $T$  is as in (1) with  $t = 1, n, m > 0$  and  $D \subset \mathbb{Z}^2$ . Then*

- (i) *if  $ka_2, ka_1 \pm la_2 \notin \Delta D$  for all  $k, l \in \mathbb{N}$ , then  $F$  is disconnected,*
- (ii) *otherwise, then  $\mathcal{N} = \{\pm ke_i \mid k \in \mathbb{N}, i \in \{1, 2\} \text{ and } ka_i \in \Delta D\} \cup \mathcal{M}_1$ .*

**Proposition 8.** *Assume that  $T$  is as in (1) with  $t = 0, n, m > 0$  and  $D \subset \mathbb{Z}^2$ . Let  $b_2 = a_2, b_3 = a_3$ . Then*

- (i) *if  $kb_1, kb_2, kb_3, kb_1 \pm lb_2 \notin \Delta D$  for all  $k, l \in \mathbb{N}$ , then  $F$  is disconnected,*
- (ii) *otherwise, then  $\mathcal{N} = \{\pm ke_i \mid k \in \mathbb{N}, i \in \{1, 2, 3\} \text{ and } kb_i \in \Delta D\} \cup \mathcal{M}_0$ .*

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