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Replication of Discrete Chaos

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Abstract. Discrete dynamics is a significant instrument for chaos investigation, since the results of the theory are rigorously approved. We provide extension of chaos by implementing chaotic perturbations to exponentially stable difference equations with arbitrarily high dimensions. Our analysis is based on the Li-Yorke definition of chaos. The results are supported with the aid of simulations. Extension of intermittency is also investigated in discussion form.
Keywords: Chaos extension, Li-Yorke chaos, Generalized synchronization, Chaotic set of sequences, Intermittency.

1 Introduction

Discrete equations are popular systems to provide a wide range of chaos and important to approve the existence rigorously [1]-[6]. The first mathematical definition of chaos for one dimensional maps is introduced by Li and Yorke [1]. The concept of snap-back repellers for high dimensional discrete equations was introduced by Marotto [2]. According to the results of the paper [2], if a multi-dimensional continuously differentiable map has a snap-back repeller, then it is Li-Yorke chaotic. Li-Yorke sensitivity, which links the Li-Yorke chaos with the notion of sensitivity, is studied in [5], and generalizations of Li-Yorke chaos to mappings in Banach spaces and complete metric spaces are provided in [4]. The Smale Horseshoe map, is first studied by Smale [7] and it is an example of a diffeomorphism which is structurally stable and possesses a chaotic invariant set [6,8–10]. The horseshoe map is prominent due to its usage for the recognition of chaotic dynamics, and can arise both in discrete and continuous cases, for example in the Hénon map [11,12] and in the Duffing equation [13,14], whenever one has transverse homoclinic orbits. If one considers the famous Lorenz [15] or Van der Pol equations [16,17], we do not have mathematically strictly proven chaos of a certain type despite there have been considered significant simplifications [18]. We propose in the present paper a rigorously confirmed

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method for chaos extension from known chaotic discrete equations to arbitrarily high dimensional ones.

To explain the procedure of chaos extension that is mentioned in the present study, let us consider the discrete equation $u_{n+1} = L[u_n] + h_n$, where L is a linear operator with spectra inside the unit circle in the complex plane. If the sequence $\{h_n\}$ is considered as an *input* with a certain property such as boundedness, periodicity or almost periodicity, then the discrete equation produces a solution, *output*, with a similar feature, boundedness/periodicity/almost periodicity [19,20]. Taking support from this fact, it is reasonable to consider the problem that whether chaotic inputs generate chaotic outputs. In the solution of this problem, one encounters with a difficulty such that chaotic sequences are not defined clearly as in the case of former properties. In other words, the chaoticity property can not be characterized through a single function. Instead, whichever chaos type is considered, chaotic properties include *interrelation* of functions. Exclusively, in the description of chaos through period-doubling cascade, this is expressed implicitly. This is true for discrete as well as for continuous-time chaos. Under the circumstances, we are forced to handle the problem by means of a special unfamiliar way of "collection of sequences". However, formally, one can formulate the results of the paper in the old fashion as the generation of a chaotic output from a chaotic input. It will be seen better if we agree to call the chaotic sequence that one which belongs to the chaotic set.

Throughout the paper, $\mathbb{R}$ and $\mathbb{Z}$ will denote the sets of real numbers and integers, respectively.

The problem that is investigated in the present study is as follows. We consider the discrete equations

$$x_{n+1} = F(x_n), \tag{1}$$

and

$$y_{n+1} = Ay_n + f(y_n) + g(x_n), \tag{2}$$

where $n \in \mathbb{Z}$, A is a nonsingular, constant $q \times q$ real valued matrix, and the functions $F : \mathbb{R}^p \to \mathbb{R}^p$, $f : \mathbb{R}^q \to \mathbb{R}^q$ and $g : \mathbb{R}^p \to \mathbb{R}^q$ are continuous in all their arguments. We suppose that the map F admits the chaos and possesses an invariant set $\Lambda \subset \mathbb{R}^p$. Our main goal is to show that equation (2) exhibits chaotic motions.

A concept which is related to our theory of chaos extension is the generalized synchronization [21]-[24]. According to the results of [23], generalized synchronization occurs in system (1)+(2) if and only if there exist sets $B_x \subset \mathbb{R}^p$, $B_y \subset \mathbb{R}^q$ such that the criterion

(A) $\lim_{n\to\infty} \|y_n - \overline{y}_n\| = 0,$

holds, for all $(x_0, y_0), (x_0, \overline{y}_0) \in B_x \times B_y$, where $\{y_n\}$ and $\{\overline{y}_n\}$ are solutions of equation (2) with the same solution $\{x_n\}$ of (1). Taking advantage of the criterion (A), in the next section, we will show that generalized synchronization occurs in the dynamics of equation (1)+(2).

In this paper, besides the presence of generalized synchronization, we show that if equation (1) is chaotic in the sense of Li-Yorke then the same is true for (2). In other words, equation (2) preserves the chaos type of equation (1). This is the main difference between the papers [23,24] and the present one. Moreover, we will show by examples, the convenience of our method to equations which possess intermittency and Neimark-Sacker bifurcation resulting in a stable closed curve.

2 Preliminaries

Let us describe the ingredients of Li-Yorke chaos [1]-[4]. Consider a set of uniformly bounded sequences

$$\mathscr{B} = \left\{ \{\eta_n\} : \sup_{n\in\mathbb{Z}} \|\eta_n\| \leq M_{\mathscr{B}} \right\},$$

where $M_{\mathscr{B}}$ is a positive real number.

We say that a pair of sequences $\left(\{\eta_n^1\}, \{\eta_n^2\}\right) \in \mathscr{B}\times\mathscr{B}$ is proximal if for an arbitrary small number $\epsilon > 0$ and an arbitrary large natural number E, there exist an integer m_0 and a natural number $E_0 \geq E$ such that $\left\|\eta_n^1 - \eta_n^2\right\| < \epsilon$ for $m_0 \leq n \leq m_0 + E_0$.

It is mentioned in [3,5] that a pair of sequences $\left(\{\eta_n^1\}, \{\eta_n^2\}\right)$ is proximal if $\liminf_{n\to\infty} \left\|\eta_n^1 - \eta_n^2\right\| = 0$. It is worth saying that our definition for proximality, which is adapted to the collection $\mathscr{B}$ and needed for our extension purposes, is, in general, stronger than the one mentioned in the classical sense. Nevertheless, one can achieve the equivalence of both definitions for equations of the form (1), for example, by requesting a Lipschitz condition on the function F.

Another feature of Li-Yorke chaos is the following one. A pair of sequences $\left(\{\eta_n^1\}, \{\eta_n^2\}\right) \in \mathscr{B}\times\mathscr{B}$ is called not asymptotic if $\limsup_{n\to\infty} \left\|\eta_n^1 - \eta_n^2\right\| > 0$.

We call a pair of sequences $\left(\{\eta_n^1\}, \{\eta_n^2\}\right) \in \mathscr{B}\times\mathscr{B}$ as a Li-Yorke pair, if they are proximal and not asymptotic. On the other hand, a subset $\mathscr{C} \subset \mathscr{B}$ is called a scrambled set if it does not contain any periodic sequences and for any distinct sequences $\{\eta_n^1\}, \{\eta_n^2\} \in \mathscr{C}$, the pair $\left(\{\eta_n^1\}, \{\eta_n^2\}\right)$ is a Li-Yorke pair.

The collection $\mathscr{B}$ is called a Li$-$Yorke chaotic set if: (i) $\mathscr{B}$ admits a periodic sequence of period k, for any natural number k; (ii) $\mathscr{B}$ possesses an uncountable scrambled set $\mathscr{C}$; (iii) For any sequence $\{\eta_n\} \in \mathscr{C}$ and any periodic sequence $\{\xi_n\} \in \mathscr{B}$, we have $\limsup_{n\to\infty} \|\eta_n - \xi_n\| > 0$.

In the remaining parts, the uniform norm $\|\Gamma\| = \sup_{\|v\|=1} \|\Gamma v\|$ for matrices will be utilized.

The following assumptions will be needed:

(A1) There exist positive numbers L_1 and L_2 such that

$$L_1 \|x - \overline{x}\| \leq \|g(x) - g(\overline{x})\| \leq L_2 \|x - \overline{x}\|,$$

for all $x, \overline{x} \in \mathbb{R}^p$;

(A2) There exists a positive number L_3 such that

$$\|f(y)-f(\overline{y})\| \le L_3 \|y-\overline{y}\|,$$

for all $y, \overline{y} \in \mathbb{R}^q$;

(A3) There exist positive real numbers M_f and M_g such that $\sup_{y\in\mathbb{R}^q} \|f(y)\| \le M_f$ and $\sup_{x\in\mathbb{R}^p} \|g(x)\| \le M_g$;

(A4) $\|A\| + L_3 < 1$.

For a given solution $x = \{x_n\}$ of equation (1), using the standard technique for maps [20], one can verify that there exists a unique bounded solution $\{\phi_n^x\}$ of equation (2). In the notation $\{\phi_n^x\}$, the symbol "x" is devoted to indicate the dependence of the bounded solution on the chosen solution $x = \{x_n\}$ of equation (1). Moreover, the unique bounded solution $\{\phi_n^x\}$, satisfies the relation

$$\phi_n^x = \sum_{j=-\infty}^{n} A^{n-j}[f(\phi_{j-1}^x) + g(x_{j-1})], \; n \in \mathbb{Z}. \tag{3}$$

Let us denote by $\mathscr{A}_x$ the set of all uniformly bounded solutions of equation (1) with initial data from the set Λ. Set $\mathscr{A}_y = \{\{\phi_n^x\} : x = \{x_n\} \in \mathscr{A}_x\}$. Equation (3) implies that for any $\{y_n\} \in \mathscr{A}_y$ the inequality $\sup_{n\in\mathbb{Z}} \|y_n\| \le H$ holds, where $H = \dfrac{M_f + M_g}{1-\|A\|}$.

We say that $\mathscr{A}_y$ is an attractor if for each solution $\{y_n\}$ of equation (2), there exists a solution $\{\widetilde{y}_n\} \in \mathscr{A}_y$ such that $\|y_n - \widetilde{y}_n\| \to 0$ as $n \to \infty$. We will verify the attractiveness of the set $\mathscr{A}_y$ in the next assertion.

Lemma 1. *$\mathscr{A}_y$ is an attractor.*

Proof. Consider an arbitrary solution $\{y_n\}$ of equation (2) with a fixed solution $\{x_n\}$ of equation (1). The relations

$$y_n = A^n y_0 + \sum_{j=1}^{n} A^{n-j}[f(y_{j-1}) + g(x_{j-1})],$$

$$\phi_n^x = A^n \phi_0^x + \sum_{j=1}^{n} A^{n-j}[f(\phi_{j-1}^x) + g(x_{j-1})],$$

imply for each $n \ge 1$ the inequality

$$\|A\|^{-n} \|y_n - \phi_n^x\| \le \|y_0 - \phi_0^x\| + \frac{L_3}{\|A\|} \sum_{j=0}^{n-1} \|A\|^{-j} \left\|y_j - \phi_j^x\right\|.$$

Applying Gronwall inequality, one can obtain that

$$\|y_n - \phi_n^x\| \le \|y_0 - \phi_0^x\| (\|A\| + L_3)^n.$$

According to condition $(A4)$, we have $\|y_n - \phi_n^x\| \to 0$ as $n \to \infty$. □

One can verify using Lemma 1 that any two solutions $\{y_n\}, \{\overline{y}_n\}$ of equation (2) with the same $\{x_n\}$ satisfy the criterion (A). Therefore, generalized synchronization occurs in equation (1)+(2).

Extension of chaos in the sense of Li-Yorke will be handled in the next section.

3 Extension of Li-Yorke chaos

In Lemma 2 and Lemma 3, we will consider the replication of the ingredients of Li-Yorke chaos, and the main result will be presented in Theorem 1.

Lemma 2. *If a pair of sequences* $(\{x_n\}, \{\overline{x}_n\}) \in \mathscr{A}_x \times \mathscr{A}_x$ *is proximal, then the pair* $\left(\{\phi_n^x\}, \{\phi_n^{\overline{x}}\}\right) \in \mathscr{A}_y \times \mathscr{A}_y$ *is also proximal.*

Proof. Fix an arbitrary small positive number ϵ and an arbitrary large natural number E. Let $\gamma = 1/\left(\frac{L_2}{1-\|A\|-L_3} + \frac{2(M_f+M_g)}{1-\|A\|}\right)$.

According to our assumption that the pair $(\{x_n\}, \{\overline{x}_n\}) \in \mathscr{A}_x \times \mathscr{A}_x$ is proximal, there exist an integer m_0 and a natural number $E_0 \geq E$ such that $\|x_n - \overline{x}_n\| < \gamma\epsilon$ for $m_0 \leq n \leq m_0 + E_0$. Throughout the proof, let us denote $y_n = \phi_n^x$ and $\overline{y}_n = \phi_n^{\overline{x}}$, $n \in \mathbb{Z}$.

Making use of the equations

$$y_n = \sum_{j=-\infty}^{n} A^{n-j} \left[f(y_{j-1}) + g(x_{j-1})\right]$$

and

$$\overline{y}_n = \sum_{j=-\infty}^{n} A^{n-j} \left[f(\overline{y}_{j-1}) + g(\overline{x}_{j-1})\right],$$

we obtain for $n \geq m_0 + 1$ that

$$y_n - \overline{y}_n = \sum_{j=-\infty}^{m_0} A^{n-j} \left[f(y_{j-1}) + g(x_{j-1}) - f(\overline{y}_{j-1}) - g(\overline{x}_{j-1})\right]$$
$$+ \sum_{j=m_0+1}^{n} A^{n-j} \left[f(y_{j-1}) - f(\overline{y}_{j-1})\right] + \sum_{j=m_0+1}^{n} A^{n-j} \left[g(x_{j-1}) - g(\overline{x}_{j-1})\right].$$

Therefore, the inequality

$$\begin{aligned}
\|y_n - \overline{y}_n\| &\leq \sum_{j=-\infty}^{m_0} 2(M_f + M_g) \|A\|^{n-j} + \sum_{j=m_0+1}^{n} L_2\gamma\epsilon \|A\|^{n-j} \\
&+ \sum_{j=m_0+1}^{n} L_3 \|A\|^{n-j} \left\|y_{j-1} - \overline{y}_{j-1}\right\| \\
&= \frac{2(M_f + M_g)}{1 - \|A\|} \|A\|^{n-m_0} + \frac{L_2\gamma\epsilon}{1 - \|A\|} \left(1 - \|A\|^{n-m_0}\right) \\
&+ \sum_{j=m_0}^{n-1} L_3 \|A\|^{n+1-j} \left\|y_j - \overline{y}_j\right\|
\end{aligned}$$

holds for $m_0 + 1 \leq n \leq m_0 + E_0 + 1$. Multiplication of the both sides of the last inequality by the term $\|A\|^{-n}$ gives us

$$\|A\|^{-n} \|y_n - \overline{y}_n\| \leq \left(\frac{2(M_f + M_g) - L_2\gamma\epsilon}{1 - \|A\|}\right) \|A\|^{-m_0} + \frac{L_2\gamma\epsilon}{1 - \|A\|} \|A\|^{-n}$$

$$+\sum_{j=m_0}^{n-1}\frac{L_3}{\|A\|}\|A\|^{-j}\left\|y_j-\overline{y}_j\right\|.$$

By the help of the Gronwall inequality, one can attain that

$$\|A\|^{-n}\|y_n-\overline{y}_n\| \le \left(\frac{2(M_f+M_g)-L_2\gamma\epsilon}{1-\|A\|}\right)\|A\|^{-m_0}+\frac{L_2\gamma\epsilon}{1-\|A\|}\|A\|^{-n}$$
$$+\sum_{j=m_0}^{n-1}\frac{L_3}{\|A\|}\Big[\left(\frac{2(M_f+M_g)-L_2\gamma\epsilon}{1-\|A\|}\right)\|A\|^{-m_0}$$
$$+\frac{L_2\gamma\epsilon}{1-\|A\|}\|A\|^{-j}\Big]\left(1+\frac{L_3}{\|A\|}\right)^{n-1-j}$$
$$=\frac{L_2\gamma\epsilon}{1-\|A\|}\|A\|^{-n}+\left(\frac{2(M_f+M_g)-L_2\gamma\epsilon}{1-\|A\|}\right)\|A\|^{-m_0}\left(\frac{\|A\|}{\|A\|+L_3}\right)^{m_0-n}$$
$$+\frac{L_2L_3\gamma\epsilon}{(1-\|A\|)(1-\|A\|-L_3)}\|A\|^{-n}\left(1-(\|A\|+L_3)^{n-m_0}\right).$$

Thus, we have

$$\|y_n-\overline{y}_n\| \le \frac{L_2\gamma\epsilon}{1-\|A\|}+\left(\frac{2(M_f+M_g)-L_2\gamma\epsilon}{1-\|A\|}\right)(\|A\|+L_3)^{n-m_0}$$
$$+\frac{L_2L_3\gamma\epsilon}{(1-\|A\|)(1-\|A\|-L_3)}\left(1-(\|A\|+L_3)^{n-m_0}\right)$$
$$=\frac{2(M_f+M_g)}{1-\|A\|}(\|A\|+L_3)^{n-m_0}+\frac{L_2\gamma\epsilon}{1-\|A\|-L_3}\left(1-(\|A\|+L_3)^{n-m_0}\right)$$
$$\le\frac{2(M_f+M_g)}{1-\|A\|}(\|A\|+L_3)^{n-m_0}+\frac{L_2\gamma\epsilon}{1-\|A\|-L_3}.$$

Suppose that the natural number E is sufficiently large such that

$$\left\lfloor\frac{E}{2}\right\rfloor>\frac{\ln(\gamma\epsilon)}{\ln(\|A\|+L_3)}-1,$$

where $\left\lfloor\frac{E}{2}\right\rfloor$ denotes the greatest integer which is not larger than $E/2$.

Under the circumstances, if $m_0+\left\lfloor\frac{E}{2}\right\rfloor+1\le n\le m_0+E_0+1$, then

$$(\|A\|+L_3)^{n-m_0}<\gamma\epsilon,$$

and hence the inequality

$$\|y_n-\overline{y}_n\|<\left(\frac{L_2}{1-\|A\|-L_3}+\frac{2(M_f+M_g)}{1-\|A\|}\right)\gamma\epsilon=\epsilon$$

is obtained. As a consequence, the pair $\left(\{\phi_n^x\},\{\phi_n^{\overline{x}}\}\right)\in\mathscr{A}_y\times\mathscr{A}_y$ is proximal. □

Lemma 3. *If a pair of sequences $(\{x_n\},\{\overline{x}_n\})\in\mathscr{A}_x\times\mathscr{A}_x$ is not asymptotic, then the same is true for the pair $\left(\{\phi_n^x\},\{\phi_n^{\overline{x}}\}\right)\in\mathscr{A}_y\times\mathscr{A}_y$.*

Proof. By means of equation (2) one can show for a pair $(\{x_n\}, \{\overline{x}_n\}) \in \mathscr{A}_x \times \mathscr{A}_x$ that

$$\left\|\phi^x_{k+1} - \phi^{\overline{x}}_{k+1}\right\| \geq \|x_k - \overline{x}_k\| - (\|A\| + L_3) \left\|\phi^x_k - \phi^{\overline{x}}_k\right\|, \ k \in \mathbb{Z}.$$

Therefore, we have

$$\sup_{k \geq n} \left\|\phi^x_{k+1} - \phi^{\overline{x}}_{k+1}\right\| \geq \sup_{k \geq n} \|x_n - \overline{x}_k\| - (\|A\| + L_3) \sup_{k \geq n} \left\|\phi^x_k - \phi^{\overline{x}}_k\right\|$$

for any $n \in \mathbb{Z}$. The last inequality implies that

$$\sup_{k \geq n} \left\|\phi^x_{k+1} - \phi^{\overline{x}}_{k+1}\right\| \geq \frac{1}{1 + \|A\| + L_3} \sup_{k \geq n} \|x_k - \overline{x}_k\|. \tag{4}$$

Since $(\{x_n\}, \{\overline{x}_n\}) \in \mathscr{A}_x \times \mathscr{A}_x$ is assumed to be not asymptotic, we have that

$$\limsup_{n \to \infty} \|x_n - \overline{x}_n\| = \lim_{n \to \infty} \left(\sup_{k \geq n} \|x_k - \overline{x}_k\| \right) > 0.$$

Thus, making use of inequality (4), one can verify that

$$\limsup_{n \to \infty} \left\|\phi^x_{n+1} - \phi^{\overline{x}}_{n+1}\right\| = \lim_{n \to \infty} \left(\sup_{k \geq n} \left\|\phi^x_k - \phi^{\overline{x}}_k\right\| \right) > 0.$$

Consequently, the pair $\left(\{\phi^x_n\}, \{\phi^{\overline{x}}_n\}\right) \in \mathscr{A}_y \times \mathscr{A}_y$ is not asymptotic. $\square$

The main theorem of the present paper is the following one.

Theorem 1. *If $\mathscr{A}_x$ is a Li-Yorke chaotic set, then the same is true for $\mathscr{A}_y$.*

Proof. Assume that the set $\mathscr{A}_x$ is Li-Yorke chaotic. One can show that for any natural number k, the sequence $x = \{x_n\} \in \mathscr{P}_x$ is $k-$periodic if and only if $\{\phi^x_n\}$ is $k-$periodic. Therefore, the set $\mathscr{A}_y$ admits a $k-$periodic sequence for any natural number k. Denote by $\mathscr{P}_x$ the set of periodic solutions of (1), and let

$$\mathscr{P}_y = \{\{\phi^x_n\} : \ x = \{x_n\} \in \mathscr{P}_x\}.$$

Suppose that the set $\mathscr{C}_x$ is an uncountable scrambled set inside $\mathscr{A}_x$. Define the set $\mathscr{C}_y = \{\{\phi^x_n\} : \ x = \{x_n\} \in \mathscr{C}_x\}$. Condition $(A1)$ implies that there is a one-to-one correspondence between the elements of $\mathscr{C}_x$ and $\mathscr{C}_y$. Therefore, $\mathscr{C}_y$ is uncountable. Moreover, using the same condition one can show that no periodic sequences exist inside $\mathscr{C}_y$, since no such sequences take place inside $\mathscr{C}_x$.

Since the collection $\mathscr{A}_x$ is assumed to be chaotic in the sense of Li-Yorke, each pair of sequences inside $\mathscr{C}_x \times \mathscr{C}_x$ is proximal. Lemma 2 implies that the same feature is valid for each pair inside $\mathscr{C}_y \times \mathscr{C}_y$. On the other hand, according to Lemma 3, any couple $(\{y_n\}, \{\overline{y}_n\}) \in \mathscr{C}_y \times \mathscr{C}_y$ satisfies the property that $\limsup_{n \to \infty} \|y_n - \overline{y}_n\| > 0$. Hence, the set $\mathscr{C}_y$ is an uncountable scrambled set inside $\mathscr{A}_y$. Moreover, each pair inside $\mathscr{C}_y \times \mathscr{P}_y$ is also not asymptotic, since the same is true for each pair inside $\mathscr{C}_x \times \mathscr{P}_x$. Consequently, $\mathscr{A}_y$ is Li-Yorke chaotic. $\square$

4 An example

In this part, as the source of chaotic perturbations, we will use the logistic map

$$x_{n+1} = \mu x_n(1 - x_n), \tag{5}$$

where μ is a parameter and $x_0 \in \Lambda = [0, 1]$. If $0 < \mu \leq 4$ then the set Λ is invariant under the iterations of equation (5) [14]. For the parameter value $\mu = 3.9$, Li-Yorke chaos takes place in the dynamics of the logistic map [1].

Let us consider the map

$$\begin{aligned} y_{n+1} &= -\frac{1}{4}y_n + \frac{1}{6}z_n + \frac{1}{3}y_n^3, \\ z_{n+1} &= \frac{1}{5}y_n + \frac{1}{10}z_n. \end{aligned} \tag{6}$$

Equation (6) possesses a stable equilibrium point, and does not admit chaos.

We perturb equation (6) by the solutions of (5) with the parameter value $\mu = 3.9$, and set up the equation

$$\begin{aligned} y_{n+1} &= -\frac{1}{4}y_n + \frac{1}{6}z_n + \frac{1}{3}y_n^3 + \tan\left(\frac{x_n}{4}\right), \\ z_{n+1} &= \frac{1}{5}y_n + \frac{1}{10}z_n + \frac{1}{2}e^{x_n}. \end{aligned} \tag{7}$$

Equation (7) is in the form of (2), where $A = \begin{pmatrix} -1/4 & 1/6 \\ 1/5 & 1/10 \end{pmatrix}$. The conditions $(A1), (A2)$ are satisfied with $L_1 = 3\sqrt{2}/8$, $L_2 = (e+1)/2$ and $L_3 = 0.16$. One can verify that condition $(A4)$ holds for equation (7).

In compliance with Theorem 1, the chaos of the logistic map (5) is extended through equation (7). Moreover, the dynamics of equation (5)+(7) exhibits generalized synchronization.

Let us consider a solution of equation (5)+(7) with $x_0 = 0.46$, $y_0 = 0.35$ and $z_0 = 1.23$. Figure 1 depicts the y and z coordinates of the solution. Figure 1 supports our theoretical results such that the solution behaves chaotically.

5 Discussions

In this part of the paper, we will discuss through simulations the extension of chaos around closed invariant curves of discrete maps and replication of intermittency.

5.1 Chaos extension around closed invariant curves

Consider the delayed logistic map [14,25], which is represented by the equation

$$\begin{aligned} y_{n+1} &= z_n, \\ z_{n+1} &= \lambda z_n(1 - y_n), \end{aligned} \tag{8}$$

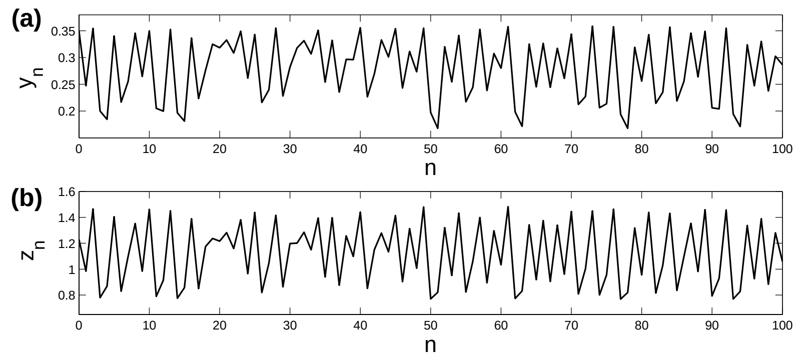

Fig. 1. The extension of chaos in equation (7). (a) The graph of y-coordinate; (b) The graph of z-coordinate. The parameter value $\mu = 3.9$ is used in the map (5) such that Li-Yorke chaos takes place. The presented pictures support the theoretical results such that the chaos of the logistic map is extended.

where λ is a positive real parameter.

Equation (8) describes a population dynamics model, where z_n is the density of a population at time n, and λ is the growth rate. In this model, the growth is determined not only by the current population but also by its density in the past [25].

According to the results mentioned in [14,25], for the parameter value $\lambda = \lambda_0 \equiv 2$, the fixed point $(1/2, 1/2)$ of equation (8) undergoes a supercritical Neimark-Sacker bifurcation. In other words, for $\lambda > 2$ and $\lambda - 2$ sufficiently small, the delayed logistic map has a unique attracting closed invariant curve encircling the fixed point $(1 - 1/\lambda, 1 - 1/\lambda)$.

We use the value $\lambda = 2.01$ from the book [14], and perturb equation (8) by the solutions of the logistic map (5) with the parameter value $\mu = 3.9$ to set up the equation

$$\begin{aligned} y_{n+1} &= z_n + 0.0045x_n, \\ z_{n+1} &= 2.01z_n(1 - y_n). \end{aligned} \tag{9}$$

Let us consider the trajectory of equation (5)+(9) with $x_0 = 0.4209$, $y_0 = 0.4316$ and $z_0 = 0.4717$. Figure 2 depicts the projection of this trajectory for $0 \le n \le 10000$ on the $y - z$ plane. One can see in the figure that the solution behaves chaotically around the stable invariant curve of equation (8). This picture reveals that our theoretical results can be used not only for systems with stable equilibrium points but also with attracting closed curves.

5.2 Replication of intermittency

We will discuss the extension of intermittency [26] in this subsection through simulations.

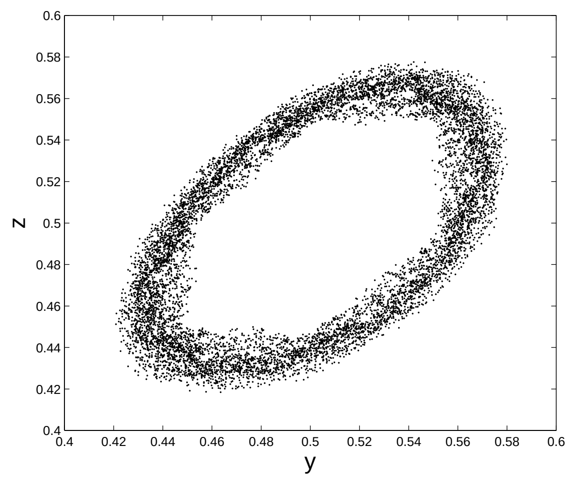

Fig. 2. Chaotic behavior in the perturbed delayed logistic equation (9).

The logistic map (5) with the parameter value $\mu = 3.828$ is known to exhibit chaos through intermittency [27]. We perturb the equation

$$y_{n+1} = \frac{1}{2}y_n + 0.002y_n^3, \tag{10}$$

by solutions of the logistic map (5) with the parameter value $\mu = 3.828$, and constitute the equation

$$y_{n+1} = \frac{1}{2}y_n + 0.002y_n^3 + 3x_n. \tag{11}$$

One can verify that the conditions $(A1) - (A4)$ are valid for equation (11).

To illustrate the replication of intermittency, in Figure 3, we visualize the graphs of both coordinates of the solution of equation (5) + (11) with $x_0 = 0.1608$, $y_0 = 4.3641$. Figure 12 reveals the extension of intermittency such that the pictures in (a) and (b) indicate the interruption of regular motions by sporadic bursts of chaotic behavior, in the logistic map and equation (11), respectively. It is also seen that equation (11) mimics the regular/irregular behavior of the logistic map with a delay, which is a expectable behavior due to the construction of the equation (1) + (2).

6 Conclusion

In the present paper, we provide a theoretical method for replication of chaos in discrete-time dynamics in such a way that the ingredients of chaos from a prior discrete equation are mimicked by a secondary one. We make use of the ingredients of chaos in the sense of Li-Yorke and investigate numerically the replication of intermittency. Chaos extension around closed invariant curves

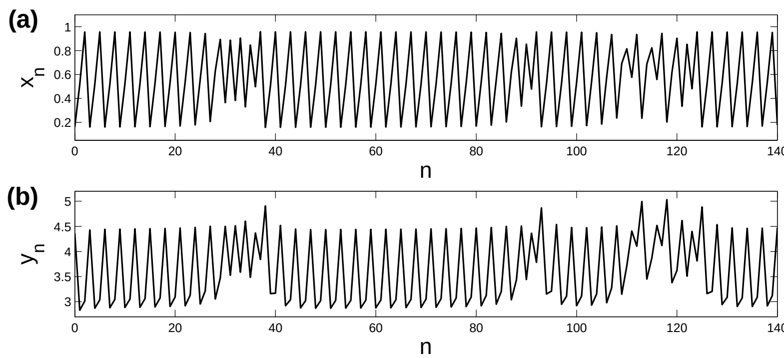

Fig. 3. Extension of intermittency in equation (11). The picture in (a) represents the intermittent behavior in the logistic map, while the picture in (b) indicate its replication in equation (11). It is observable in the figure that the $y-$coordinate mimics the regular/irregular behavior of the $x-$coordinate with a delay.

is also discussed through an example. It is worth noting that the extension procedure is valid without any constraints on the dimensions of the considered equations. As a result, a chaotic attractor in a high dimensional phase space takes place.

In the classical input/output problems, generally, one considers single functions as the input and the output. However, in the solution of the chaos extension problem, we have taken into account the concept of collection of sequences, since there is no definition for a solely chaotic sequence in discrete-time as well as in continuous-time dynamics. From the input/output problem point of view, our results emphasize that both the input and the output are chaos of the same type for the discussed equation.

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