Chaotic Modeling and Simulation (CMSIM) 2: 247–255, 2013

Dichotomy and boundary value problems on the whole line

Alexander A. Boichuk[1] and Oleksander A. Pokutnyi[2]

[1] Institute of mathematics of NAS of Ukraine, , 01601 Kiev, Ukraine
(E-mail: boichuk@imath.kiev.ua)
[2] Institute of mathematics of NAS of Ukraine, , 01601 Kiev, Ukraine
(E-mail: lenasas@gmail.com)

Abstract. Necessary and sufficient conditions for normal solvability are obtained for linear differential equations in Banach space. Constructed examples demonstrate that even in the linear case (but certainly not correct) you can select a family of bounded solutions, which tend to an equilibrium positions, so-called homoclinic and heteroclinic trajectories.

Keywords: exponential dichotomy, normally-resolvable operator, pseudoinverse operator.

A lot of papers are devoted to development of constructive methods for the analysis of different classes of boundary value problems. They traditionally occupy one of the central places in the qualitative theory of differential equations. This is due to practical significance of the theory of boundary-value problems for various applications - theory of nonlinear oscillations, theory of stability of motion, control theory and numerous problems in radioengineering, mechanics, biology etc.

Correct and incorrect boundary value problems are studied. Ususally correctness is understood as uniqueness of the solution for arbitrary right-hand side of the equation. Correct boundary value problems for ordinary differential equations, impulsive systems, Noether operator equations became popular relatively recently, they were studied in detail [5]. Analysis of a large class of incorrect boundary value problems was associated with the properties of the generalized inverse operator (which exists for any linear operator in a finite dimensional space).

Efforts aimed to solving problem of the existence of bounded solutions of linear differential equations are mainly devoted to the correct case. Additional boundary conditions can be full filled only in a trivial situations for such problems. After Palmer's work [2] it became clear that in the general case, even a finite set of differential equations can not have one bounded solution, and it

Received: 5 April 2012 / Accepted: 12 October 2012

ISSN 2241-0503

makes sense to study the boundary value problem in the incorrect case. Using the pseudoinverse operators approach one can obtain the conditions under which a family of bounded solutions satisfying the supplementary boundary conditions can be identified.

1 Statement of the Problem

In a Banach space $\mathbf{X}$ we consider a boundary value problem

$$\frac{dx}{dt} = A(t)x(t) + f(t) \ , \tag{1}$$

$$lx(\cdot) = \alpha \ , \tag{2}$$

where the vector - function $f(t)$ acts from R into the Banach space $\mathbf{X}$,

$$f(t) \in BC(R, \mathbf{X}) := \{f(\cdot) : R \to \mathbf{X}, f(\cdot) \in C(R, \mathbf{X}), |||f||| = \sup_{t\in R} \|f(t)\| < \infty\},$$

$BC(R, \mathbf{X})$ is the Banach space of functions continuous and bounded on R; the operator-valued function $A(t)$ is strongly continuous with the norm $|||A||| = \sup_{t\in R} ||A(t)|| < +\infty$; $BC^1(R, \mathbf{X}) := \{x(\cdot) : R \to \mathbf{X},\ x(\cdot) \in C^1(R, \mathbf{X}), |||x||| = \sup_{t\in R}\{\|x(t)\|, \|x^1(t)\|\} < \infty\}$, - the space of functions continuously differentiable on R and bounded together with their derivatives; l - linear and bounded operator acts from the space of $BC^1(R, \mathbf{X})$ into the Banach space $\mathbf{Y}$. We determine the conditions of the existence of solutions $x(\cdot) \in BC^1(R, \mathbf{B})$ of boundary value problem (1), (2) under the assumption that the corresponding homogeneous equation

$$\frac{dx}{dt} = A(t)x(t) \tag{3}$$

admits an exponential dichotomy [1–3] on the semi-axes R_+ and R_- with projectors P and Q, respectively, i.e., there exist projectors $P(P^2 = P)$ and $Q(Q^2 = Q)$ and constants $k_{1,2} \geq 1$ and $\alpha_{1,2} > 0$ such that the estimates

$$\begin{cases} \left\|U(t)PU^{-1}(s)\right\| \leq k_1 e^{-\alpha_1(t-s)}, \quad t \geq s, \\ \left\|U(t)(E-P)U^{-1}(s)\right\| \leq k_1 e^{\alpha_1(t-s)}, \quad s \geq t, \quad \text{for all} \quad t, s \in R_+, \end{cases}$$

and

$$\begin{cases} \left\|U(t)QU^{-1}(s)\right\| \leq k_2 e^{-\alpha_2(t-s)}, \quad t \geq s, \\ \left\|U(t)(E-Q)U^{-1}(s)\right\| \leq k_2 e^{\alpha_2(t-s)}, \quad s \geq t, \quad \text{for all} \quad t, s \in R_- \end{cases}$$

hold, where $U(t) = U(t, 0)$ is the evolution operator of Eq. (3) such that

$$\frac{dU(t)}{dt} = A(t)U(t), \quad U(0) = E \text{ is the identity operator [1, p.145]} \ .$$

2 Preliminaries

Now we formulate the following result, which is proved in [4] for the nonhomogeneous equation (1).

Theorem 1. *Suppose that the homogeneous equation (3) admits an exponential dichotomy on the semi-axes R_+ and R_- with projectors P and Q, respectively. If the operator*

$$D = P - (E - Q) : \mathbf{X} \to \mathbf{X} \tag{4}$$

acting from the Banach space $\mathbf{X}$ onto itself is invertible in the generalized sense [5, p.26], then

(i) in order that solutions of Eq. (1) bounded on the entire real axis exist, it is necessary and sufficient that the function $f(t) \in BC(R, \mathbf{X})$ satisfies the condition

$$\int_{-\infty}^{+\infty} H(t)\, f(t)\, dt = 0; \tag{5}$$

where

$$H(t) = \mathcal{P}_{N(D^*)} Q U^{-1}(t) = \mathcal{P}_{N(D^*)}(E - P) U^{-1}(t),$$

(ii) under condition (5), solutions bounded on the entire axis of Eq. (1) have the form

$$x(t, c) = U(t) P \mathcal{P}_{N(D)} c + (G[f])(t), \qquad \forall \ c \in \mathbf{X}, \tag{6}$$

where

$$(G[f])(t) = U(t) \begin{cases} \int_0^t PU^{-1}(s) f(s)\, ds - \int_t^\infty (E-P)U^{-1}(s) f(s)\, ds + \\ +PD^- \left[\int_0^\infty (E-P)U^{-1}(s) f(s)\, ds + \int_{-\infty}^0 QU^{-1}(s) f(s) ds \right], \quad t \geq 0, \\ \int_{-\infty}^t QU^{-1}(s) f(s)\, ds - \int_t^0 (E-Q)U^{-1}(s) f(s)\, ds + \\ +(E-Q)D^- \left[\int_0^\infty (E-P)U^{-1}(s) f(s)\, ds + \int_{-\infty}^0 QU^{-1}(s) f(s)\, ds \right], \ t \leq 0 \end{cases} \tag{7}$$

is the generalized Green operator of the problem for solutions bounded on the entire axis, D^- - is the generalized inverse of D , $mathcalP_{N(D)} = E - D^- D$ and $\mathcal{P}_{N(D^*)} = E - DD^-$, c is an arbitrary constant element of the Banach space $\mathbf{X}$.

3 Main result

We now show that under condition from the theorem 1, the boundary value problem can be solved using the operator $B_0 = lU(\cdot) P \mathcal{P}_{N(D)} : \mathbf{X} \to \mathbf{Y}$.

Theorem 2. *Let's conditions from the theorem 1 are satisfied. If the operator*

$$B_0 : \mathbf{X} \longrightarrow \mathbf{Y}$$

acting from the Banach space $\mathbf{X}$ *into the Banach space* $\mathbf{Y}$ *is invertible in the generalized sense, then*

(i) in order that solutions of boundary value problem (1), (2) exist, it is necessary and sufficient that

$$\mathcal{P}_{N(B_0^*)}(\alpha - l((G[f])(\cdot))) = 0 \; ; \tag{8}$$

(ii) under condition (8) solutions of boundary value problem (1), (2) have the form

$$x(t,\overline{c}) = U(t)P\mathcal{P}_{N(D)}\mathcal{P}_{N(B_0)}\overline{c} + U(t)P\mathcal{P}_{N(D)}B_0^-(\alpha - l(G[f])(\cdot)) + (G[f])(t), \forall \overline{c} \in \mathbf{X},$$

where $(G[f])(\cdot)$ *- is generalized Green operator defined below;* B_0^- *- is generalized inverse of* B_0*,* $\mathcal{P}_{N(B_0^*)}$ *- projector, which project* $\mathbf{X}$ *onto the kernel of adjoint operator* B_0^**.*

Proof. *From the theorem 1, we have that the family of bounded solutions of the equation (1) has the form* $x(t,c) = U(t)P\mathcal{P}_{N(D)}c + (G[f])(t)$*. We substitute this solutions to the equation (2):*

$$l(U(\cdot)P\mathcal{P}_{N(D)}c + (G[f])(\cdot)) = \alpha.$$

Since the operator l *is linear we have :*

$$l(U(\cdot)P\mathcal{P}_{N(D)})c + l((G[f])(\cdot)) = \alpha,$$

and we have finally the operator equation :

$$B_0 c = \alpha - l((G[f])(\cdot)).$$

Since operator B_0 *is invertible in the generalized sence , then in order that solutions of the boundary value problem (1),(2) exist it is necessary and sufficient [5] that*

$$\mathcal{P}_{N(B_0^*)}(\alpha - l((G[f])(\cdot))) = 0.$$

If this condition is satisfied, then

$$c = \mathcal{P}_{N(B_0)}\overline{c} + B_0^-(\alpha - l((G[f])(\cdot))), \quad \forall \overline{c} \in \mathbf{X}.$$

Then the family of bounded solutions of the boundary value problem (1), (2) has the form:

$$x(t,\overline{c}) = U(t)P\mathcal{P}_{N(D)}\mathcal{P}_{N(B_0)}\overline{c} + U(t)P\mathcal{P}_{N(D)}B_0^-(\alpha - l((G[f])(\cdot))) + (G[f])(t)$$

Remark. *If* $\mathbf{Y} = \mathbf{X} \times \mathbf{X}$*,* $lx = (x(+\infty), x(-\infty)) = (\alpha, \alpha) \in \mathbf{X} \times \mathbf{X}$*, where* α *- equilibrium point of (1), then all bounded solutions of boundary value problem (1), (2) are homoclinic paths [6].*

4 Examples

1. We now illustrate the assertions proved above. Consider the next boundary value problem

$$\frac{dx}{dt} = A(t)x(t) + f(t), \tag{9}$$

$$lx(\cdot) = x(b) - x(a) = \alpha, \tag{10}$$

where $A(t)$ *- is operator in the form of a countably-dimensional matrix that, for every real value* t*, acts on the Banach space* $\mathbf{B} = l_p,\ p \in [1; +\infty)$ *and*

$$x(t) = col\{x_1(t), x_2(t), \dots x_k(t), \dots\} \ \in \ BC^1(R, l_p),$$

$$f(t) = col\{f_1(t), f_2(t), \dots, f_k(t), \dots\} \ \in \ BC(R, l_p)$$

- are countable vector - columns; $a, b \in R, b > 0, a < 0;$

$$\alpha = col\{\alpha_1, \alpha_2, \dots, \alpha_k, \dots\} \in l_p$$

- constant vector ($\alpha_i \in R, i \in N$*).*

Consider boundary value problem (9), (10) with the operator

$$A(t) = \begin{pmatrix} \overbrace{th\ t \quad 0 \quad 0}^{k} & \dots & \dots \\ 0 \quad th\ t \quad 0 & \dots & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots \\ 0 \quad 0 \quad th\ t & \dots & \dots \\ 0 \quad 0 \quad 0 & -\,th\ t & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots \end{pmatrix} : l_p \to l_p. \tag{11}$$

The evolution operator of system (9), (11) has the form:

$$U(t) = \begin{pmatrix} \overbrace{(e^t + e^{-t})/2 \quad 0 \quad 0}^{k} & \dots & \dots \\ 0 \quad (e^t + e^{-t})/2 \quad 0 & \dots & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots \\ 0 \quad 0 \quad (e^t + e^{-t})/2 & \dots & \dots \\ 0 \quad 0 \quad 0 & 2/(e^t + e^{-t}) & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots \end{pmatrix};$$

The operator inverse to $U(t)$ *has the form*

$$U^{-1}(t) = \begin{pmatrix} \overbrace{2/(e^t + e^{-t}) \quad 0 \quad 0}^{k} & \dots & \dots \\ 0 \quad 2/(e^t + e^{-t}) \quad 0 & \dots & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots \\ 0 \quad 0 \quad 2/(e^t + e^{-t}) & \dots & \dots \\ 0 \quad 0 \quad 0 & (e^t + e^{-t})/2 & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots \end{pmatrix};$$

and the corresponding homogeneous system is exponentially - dichotmous on both semi-axes R_+ and R_- with the projectors

$$P = \begin{pmatrix} \overbrace{0 \quad\quad 0}^{k} & \dots & \dots \\ \dots \quad \dots & \dots & \dots \\ 0 \quad\quad 0 & \dots & \dots \\ 0 \quad\quad 0 & 1 & \dots \\ 0 \quad\quad 0 & 0 & \dots \\ \dots \quad \dots & \dots & \dots \end{pmatrix} \quad and \quad Q = \begin{pmatrix} \overbrace{1 \quad\quad 0 \quad\quad \dots}^{k} & \dots \\ \dots \quad \dots \quad \dots & \dots \\ 0 \quad\quad \dots \quad\quad 1 & \dots \\ 0 \quad\quad 0 \quad\quad 0 & \dots \\ 0 \quad\quad 0 \quad\quad 0 & \dots \\ \dots \quad \dots \quad \dots & \dots \end{pmatrix}$$

, respectively. Thus, we have

$$D = P - (E - Q) = 0, \qquad \mathcal{P}_{N(D)} = \mathcal{P}_{N(D^*)} = E.$$

Since $dimR[\mathcal{P}_{N(D^*)}Q] = k$, *then operator* $\mathcal{P}_{N(D^*)}Q$ *is finite-dimensional:*

$$H(t) = [\mathcal{P}_{N(D^*)}Q]U^{-1}(t) = \begin{pmatrix} \overbrace{1 \quad\quad 0 \quad\quad \dots}^{k} & \dots \\ \dots \quad \dots \quad \dots & \dots \\ 0 \quad\quad \dots \quad\quad 1 & \dots \\ 0 \quad\quad 0 \quad\quad 0 & \dots \\ 0 \quad\quad 0 \quad\quad 0 & \dots \\ \dots \quad \dots \quad \dots & \dots \end{pmatrix} U^{-1}(t) = diag\{H_k(t), 0\},$$

where

$$H_k(t) = \begin{pmatrix} 2/(e^t + e^{-t}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 2/(e^t + e^{-t}) \end{pmatrix} \text{ is a } k \times k - \text{ dimensional matrix .}$$

According theorem 1, for the existence of solutions of system (9), (11) bounded on the entire axis, it is necessary and sufficient that following conditions be satisfied:

$$\int_{-\infty}^{+\infty} H_k(t)f(t)dt = 0 \quad \Leftrightarrow \quad \begin{cases} \int_{-\infty}^{+\infty} \frac{f_1(t)}{e^t + e^{-t}} dt = 0 \\ \dots \\ \int_{-\infty}^{+\infty} \frac{f_k(t)}{e^t + e^{-t}} dt = 0. \end{cases} \tag{12}$$

Thus, in order that system (3), (11) have solutions bounded on the entire axis, it is necessary and sufficient that exactly k conditions be satisfied; the other functions $f_i(t)$ for all $i \geq k+1$ can be taken arbitrary from the class $BC(R, l_p)$. Moreover, system (3), (11) has countably many linearly independent bounded solutions. For example, as a vector function f from the class $BC(R, l_p)$, one can take an arbitrary vector function whose first k components are odd functions.

For solving boundary value problem we find the matrix B_0 :

$$B_0 = lU(\cdot)P\mathcal{P}_{N(D)} = U(b)P\mathcal{P}_{N(D)} - U(a)P\mathcal{P}_{N(D)},$$

and finally

$$B_0 = \begin{pmatrix} \overbrace{0 \quad 0 \quad \dots}^{k} & \dots & \dots & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots & \dots \\ 0 \quad \dots \quad 0 & \dots & \dots & \dots \\ 0 \quad \dots \quad 0 & \frac{cha-chb}{cha \cdot chb} & \dots & \dots \\ 0 \quad \dots \quad 0 & \dots & \frac{cha-chb}{cha \cdot chb} & \dots \\ \dots \quad \dots \quad \dots & \dots & \dots & \dots \end{pmatrix} : l_p \to l_p.$$

Since $a \neq b$ then operator $\mathcal{P}_{N(B_0^*)}$ have the form :

$$\mathcal{P}_{N(B_0^*)} = \begin{pmatrix} \overbrace{1 \quad 0 \quad \dots}^{k} \dots \dots \dots \\ \dots \quad \dots \quad \dots \dots \dots \dots \\ 0 \quad \dots \quad 1 \quad \dots \dots \dots \\ 0 \quad \dots \quad 0 \quad 0 \quad \dots \dots \\ 0 \quad \dots \quad 0 \quad \dots \quad 0 \quad \dots \\ \dots \quad \dots \quad \dots \dots \dots \dots \end{pmatrix} : l_q \to l_q \ (1/p + 1/q = 1),$$

and

$$G[f](b) - G[f](a) = \begin{pmatrix} -\int_{-\infty}^{a} \frac{2f_1(s)}{e^s+e^{-s}} ds - \int_{b}^{+\infty} \frac{2f_1(s)}{e^s+e^{-s}} ds \\ \dots \\ -\int_{-\infty}^{a} \frac{2f_k(s)}{e^s+e^{-s}} ds - \int_{b}^{+\infty} \frac{2f_k(s)}{e^s+e^{-s}} ds \\ \\ \frac{1}{2}\int_{a}^{b} (e^s + e^{-s}) f_{k+1}(s) ds \\ \dots \end{pmatrix}.$$

$$\mathcal{P}_{N(B_0^*)}(\alpha - l(G[f])(\cdot)) = 0 \quad \Leftrightarrow \quad \begin{cases} \int_{-\infty}^{a} \frac{2f_1(s)}{e^s+e^{-s}} ds + \int_{b}^{+\infty} \frac{2f_1(s)}{e^s+e^{-s}} ds = -\alpha_1 \\ \dots \\ \int_{-\infty}^{a} \frac{2f_k(s)}{e^s+e^{-s}} ds + \int_{b}^{+\infty} \frac{2f_k(s)}{e^s+e^{-s}} ds = -\alpha_k. \end{cases} \tag{13}$$

Thus, according to Theorem 2, boundary value problem (9), (10), (11) possesses at least one solution bounded on R if and only if the vector-function f satisfies conditions (12), (13).

2. Consider one-dimensional boundary value problem

$$\frac{dx(t)}{dt} = -tht\ x(t) + f(t),$$

$$lx = (x(+\infty), x(-\infty)) = (\alpha_1, \alpha_2) \in R^2. \tag{14}$$

a) let $f(t) = \frac{2e^{-t}}{e^t+e^{-t}}$ and $(\alpha_1, \alpha_2) = (0, -2)$. The set of bounded solutions which satisfy boundary condition (14) have the form:

$$x(t, c) = \frac{2}{e^t + e^{-t}}\ c - \frac{2e^{-t}}{e^t + e^{-t}} + \frac{2}{e^t + e^{-t}},\ for\ all\ c\ \in R.$$

Integral curves for different values of the parameter c are shown in Figure 1.

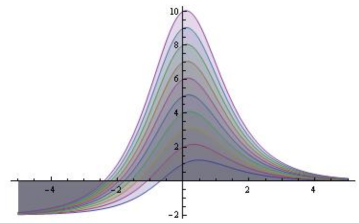

Fig. 1. Integral curves for different values of the parameter c

b) let $f(t) = 2\ tht$ *and* $(\alpha_1, \alpha_2) = (2, 2)$. *In this case equation (1) has equilibrium solution* $x_0(t) = 2$ *and a set of homoclinic paths have the next form:*

$$x(t, c) = \frac{2}{e^t + e^{-t}}\ c + 2 - \frac{4}{e^t + e^{-t}},\ for\ all\ c \in R.$$

Integral curves for different values of the parameter c are shown in Figure 2.

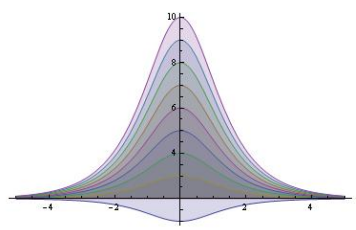

Fig. 2. Integral curves for different values of the parameter c

3. Consider two-dimensional boundary value problem

$$\frac{dx_1(t)}{dt} = -tht\ x_1(t) + f_1(t),$$

$$\frac{dx_2(t)}{dt} = -tht\ x_2(t) + f_2(t),$$

$$l(x_1, x_2) = (x_1(+\infty), x_1(-\infty), x_2(+\infty), x_2(-\infty)) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, -2, 2, 2) \in R^4,$$

where $f_1(t) = \frac{2e^{-t}}{e^t + e^{-t}}$, $f_2(t) = 2\ tht$ *(direct product of examples 2a, 2b). This problem has a two-parametric family of bounded solutions*

$$\begin{aligned} x_1(t, c_1) &= \tfrac{2}{e^t+e^{-t}}\ c_1 - \tfrac{2e^{-t}}{e^t+e^{-t}} + \tfrac{2}{e^t+e^{-t}}, \\ x_2(t, c_2) &= \tfrac{2}{e^t+e^{-t}}\ c_2 + 2 - \tfrac{4}{e^t+e^{-t}}, \end{aligned} \quad for\ all\ c_1, c_2 \in R.$$

The phase portrait of this system is shown for different parameters in Figure 3 (in plane x_1, x_2*).*

We see that the portrait resembles a horseshoe.

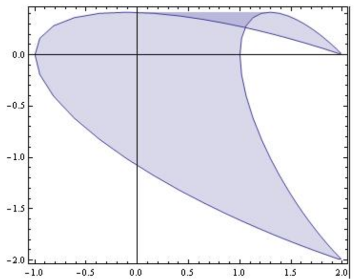

Fig. 3. The phase portrait of system

References

1. *Yu. M. Daletskii and M. G. Krein, Stability of Solutions of Differential Equations in Banach Space. Nauka, Moscow. 1970 (in Russian).*
2. *K. J. Palmer. Exponential dichotomies and transversal homoclinic points. J. Different. Equat.* **55***.-1984. P. 225–256.*
3. *Sacker R.J., Sell G.R. Dichotomies for Linear Evolutionary Equaitons in Banach Spaces. Journal of Differential Equations. 113, 17-67, 1994.*
4. *A. A. Boichuk and A.A. Pokutnij. Bounded solutions of linear differential equations in Banach space. Nonlinear Oscillations,* **9***, no 1. - 2006. - P. 3-14; http://www.springer.com/.*
5. *A. A. Boichuk and A. M. Samoilenko, Generalized Inverse Operators and Fredholm Boundary-Value Problems. VSP, Utrecht - Boston. 2004.*
6. *Guckenheimer J., Holmes Ph. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. 2002. - 235p.*